

337 (1964).

<sup>11</sup>This argument has been used by D. C. Mattis and O. Penrose (private communication) to make rigorous some of the magnetic applications of the Bogoliubov inequality.

<sup>12</sup>The angular brackets now denote full canonical ensemble averages, and the vector operators in the denominator of (36) are combined as in a scalar product. The proof of (36) is the obvious generalization of that

given by N. D. Mermin and H. Wagner, *Phys. Rev. Letters* **17**, 1133 (1966). The Bogoliubov inequality was first applied to crystals by H. Wagner, *Z. Physik* **195**, 273 (1966).

<sup>13</sup>This is easily proved to be bounded in the thermodynamic limit for any potentials  $\Phi$  for which the free energy per particle is bounded.

<sup>14</sup>R. E. Peierls, *Phys. Rev.* **54**, 918 (1938).

## Superfluid Phase Transitions in One and Two Dimensions

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A recently proved theorem of statistical physics states that a one- or two-dimensional Bose liquid in homogeneous thermodynamic equilibrium cannot undergo Bose-Einstein condensation. The theorem does not apply to inhomogeneous liquids. This is shown by considering the ideal Bose liquid (i) in the presence of a gravitational field and (ii) in rotational motion. The existence of well-defined one- and two-dimensional phase transitions with Bose-Einstein condensation is proved by these explicit examples.

### 1. INTRODUCTION

The problem of one- and two-dimensional superfluidity is of considerable importance. Since three-dimensional superfluidity is closely connected to Bose-Einstein condensation,<sup>1</sup> it is important to know whether this connection exists in one and two dimensions. An inequality of Bogoliubov was used by Hohenberg<sup>2</sup> to rule out Bose-Einstein condensation in one- and two-dimensional liquids in homogeneous thermodynamic equilibrium. The question of Bose-Einstein condensation in inhomogeneous liquids was left unresolved. It turns out that an external disturbance which breaks translational invariance may cause Bose-Einstein condensation in one and two dimensions. The purpose of this paper is to give explicit examples of this fact. The ideal Bose liquid is considered (i) in the presence of an external gravitational force and (ii) in rotational motion. In each case, a well-defined one- or two-dimensional phase transition occurs accompanied by macroscopic occupation of a single-particle state.

### 2. METHOD OF CALCULATION

In the grand canonical ensemble the equations of state are computed from the partition function

$$Z_G = \text{tr}[\exp - \beta(H - \mu N)], \quad \beta = (kT)^{-1}. \quad (1)$$

For the ideal Bose liquid, the total Hamiltonian is the sum of single-particle Hamiltonians,

$$H = \sum_{i=1}^N h(i), \quad (2)$$

and the energy spectrum is determined by the single-particle eigenvalue problem

$$h\psi_j(\vec{r}) = E_j\psi_j(\vec{r}). \quad (3)$$

The partition function is given by

$$Z_G = \sum_{\{n_j\}} \exp - \beta[\sum_j n_j(E_j - \mu)], \quad (4)$$

and the summation over the occupation numbers is easily performed,

$$\ln Z_G = - \sum_j \ln[1 - \exp - \beta(E_j - \mu)]. \quad (5)$$

Expanding the right-hand side of the above equation in powers of the activity

$$\xi = \exp(\beta\mu) \quad (6)$$

$$\text{yields } \ln Z_G = \sum_{s=1}^{\infty} s^{-1} \gamma(s\beta) \xi^s, \quad (7)$$

$$\text{where } \gamma(t) = \sum_j \exp(-E_j t). \quad (8)$$

The crux of the problem is to evaluate the right-hand side of Eq. (8) in the thermodynamic limit. This is accomplished by Green's-function methods.

The Green's function for the operator  $h$  may be defined by

$$G(\vec{r}, \vec{r}', t) = \theta(t) \sum_j \exp(-E_j t) \psi_j^*(\vec{r}') \psi_j(\vec{r}), \quad (9)$$

where  $\theta(t)$  is unity if  $t > 0$  and zero if  $t < 0$ . It follows from Eq. (9) and the completeness of the eigenfunctions [Eq. (3)] that

$$[(\partial/\partial t) + h]G(\vec{r}, \vec{r}', t) = \delta(t) \delta(\vec{r} - \vec{r}'). \quad (10)$$

Furthermore, it follows from Eqs. (8) and (9) that

$$\gamma(t) = \int G(\vec{r}, \vec{r}, t) d^k r, \quad (11)$$

where  $k$  is the number of dimensions.

The rules of calculating the thermodynamic equations of state for the inhomogeneous ideal Bose liquid are now evident: (i) The cause of the inhomogeneity is described by a single-particle Hamiltonian  $h$ ; (ii) the Green's-function problem [Eq. (10)] is solved; and (iii) the grand canonical partition function  $Z_G$  is calculated via Eqs. (7) and (11). In three (two, one) dimensions the thermodynamic potential per unit volume (area, length) is given by

$$\varphi = kT \lim[\Omega^{-1} \ln Z_G], \quad (12)$$

where  $\Omega$  is the "volume" of the system and  $\lim$  refers to the thermodynamic limit of infinite size. If  $\varphi(T, \mu)$  is known, then the thermodynamic properties follow from the Gibbs-Duhem relation

$$d\varphi = n d\mu + s dT, \quad (13)$$

where  $n$  and  $s$  represent, respectively, the particle number and entropy per unit volume.

The rules for taking the thermodynamic limit are more complicated for inhomogeneous system than for homogeneous systems. One complication is that the equations of state for inhomogeneous systems often depend on the geometrical shape of the substance. Furthermore, the free energy per particle may diverge in the thermodynamic limit if the strength of the inhomogeneity is not properly taken into account. In a previous paper<sup>3</sup> the author discussed these points with reference to rotating liquids. The details will not be reviewed here, but the closely analogous situation of a liquid in a uniform gravitational field will be briefly explained. The point is that a precise thermodynamic limiting process is essential for the existence of sharp singularities (i. e., phase transitions) in the thermodynamic equations of state.

Consider a liquid in a cylindrical container of height  $l$  located in a uniform gravitational field  $g$  parallel to the axis of the cylinder. In the thermodynamic limit the volume grows ever larger. If the height of the cylinder were to remain finite, then the ratio of the boundary surface area to the enclosed volume would also remain finite. This is unacceptable<sup>4</sup> even for the special case  $g = 0$ . The correct procedure is to let  $l \rightarrow \infty$ . However, a new problem arises. The particles near the top of the container have  $mg l$  more gravitational potential energy than the particles near the bottom of the

container. The energy per particle may therefore diverge unless  $g \rightarrow 0$  with the product  $g l$  remaining finite.<sup>5</sup> This condition is needed to complete the rules for taking the thermodynamic limit in the presence of a uniform gravitational field.

### 3. GRAVITATIONAL FORCE

The thermodynamic properties of the one-dimensional ideal Bose liquid in a box of length  $l$  (along the vertical axis) in a uniform gravitational field  $g$  can be computed by the methods described above. It is shown in Appendix A that

$$\lim[l^{-1} \gamma(t)] = (m/2\pi \hbar^2 t)^{1/2} (mgl t)^{-1} (1 - e^{-mgl t}). \quad (14)$$

From Eqs. (7), (12), and (14), it follows that

$$\varphi = kT \chi_1(e^{\beta\mu}), \quad (15)$$

where

$$\chi_1(\xi) = (\lambda\eta)^{-1} \sum_{s=1}^{\infty} s^{-5/2} (1 - e^{-s\eta}) \xi^s. \quad (16)$$

In the above equation

$$\lambda = (2\pi \hbar^2 \beta/m)^{1/2} \quad (17)$$

is the thermal wavelength and

$$\eta = \beta mgl. \quad (18)$$

The density follows from the thermodynamic relation

$$n = (\partial\varphi/\partial\mu)_T = \xi \chi_1'(\xi). \quad (19)$$

It is

$$n = (\lambda\eta)^{-1} \sum_{s=1}^{\infty} s^{-3/2} (1 - e^{-s\eta}) \xi^s. \quad (20)$$

As the activity varies through its physical values ( $0 \leq \xi \leq 1$ ), the density varies from zero to the critical value

$$n_c(g, T) = (\lambda\eta)^{-1} \sum_{s=1}^{\infty} s^{-3/2} (1 - e^{-s\eta}). \quad (21)$$

The finite critical density signals the onset of Bose-Einstein condensation. For  $n > n_c$ , a fraction

$$\alpha(g, T) = (n - n_c)/n \quad (22)$$

of the particles are in the condensate. Note that as  $g \rightarrow 0$ ,  $n_c \rightarrow \infty$  which implies the well-known result that Bose-Einstein condensation is absent in the one-dimensional homogeneous ideal Bose liquid. However, when  $\eta \neq 0$  one-dimensional Bose-Einstein condensation is clearly possible. Furthermore, Eqs. (13), (15), and (16) completely determine the thermodynamic properties of all phases.

The two-dimensional ideal Bose liquid in a uniform gravitational field is qualitatively similar to the one-dimensional case. Instead of working out the details of this model, we shall consider a different example of two-dimensional Bose-Einstein condensation.

## 4. ROTATIONAL MOTION

Consider the two-dimensional ideal Bose liquid in a circular box of radius  $R$  which is rotating at angular velocity  $\omega$ . It is shown in Appendix B that

$$\lim[(\pi R^2)^{-1}\gamma(t)] = (m/2\pi\hbar^2 t)(2/m\omega^2 R^2 t)(e^{m\omega^2 R^2 t/2} - 1). \quad (23)$$

From Eqs. (7), (12), and (23) it follows that

$$\varphi = k T \chi_2(e^{\beta\mu}), \quad (24)$$

where

$$\chi_2(\xi) = (\lambda^2 \xi)^{-1} \sum_{s=1}^{\infty} s^{-3} (e^{s\xi} - 1) \xi^s. \quad (25)$$

In the above equation

$$\xi = \frac{1}{2} \beta m \omega^2 R^2. \quad (26)$$

The thermodynamic properties of this system are completely determined by Eqs. (24), (25), and the two-dimensional Gibbs-Duhem relation for rotating systems

$$d\varphi = n d\mu + s dT + \Lambda d\omega, \quad (27)$$

where  $n$ ,  $s$ , and  $\Lambda$  represent, respectively, the particle number, entropy, and angular momentum per unit area.

As the chemical potential varies through its physical values from  $(-\infty)$  to  $(-kT\xi)$ , the density

$$n = \zeta \chi_2'(\xi) \quad (28)$$

varies from zero to the critical value

$$n_c(\omega, T) = (\lambda^2 \xi)^{-1} \sum_{s=1}^{\infty} s^{-2} (1 - e^{-s\xi}). \quad (29)$$

For  $n > n_c$ , a fraction

$$\alpha(\omega, T) = (n - n_c)/n \quad (30)$$

of the particles are in the condensate. If rotation is absent ( $\omega \rightarrow 0$ ), then  $n_c \rightarrow \infty$  which implies that Bose-Einstein condensation is absent. However, when  $\xi \neq 0$ , two-dimensional Bose-Einstein condensation is clearly possible.

## 5. CONCLUSION

We shall conclude this paper with a brief discussion of what has been established about the role of Bose-Einstein condensation in one- and two-dimensional superfluidity. First of all, Hohenberg proved that Bose-Einstein condensation is non-existent in homogeneous one- and two-dimensional liquids in thermodynamic equilibrium. We have shown that inhomogeneities in the ideal Bose liquid can be accompanied by Bose-Einstein condensation, but the situation when particle interactions are included is still unclear. Finally, the most important question of whether or not Bose-Einstein condensation plays an important role in dynamic processes is completely unresolved.

## APPENDIX A

The one-dimensional Hamiltonian for a particle in a uniform gravitational field is given by

$$h = -(\hbar^2/2m)(\partial/\partial y)^2 + mgy. \quad (A1)$$

The Green's function in the absence of a gravitational field is well known,<sup>6</sup>

$$g_0(y, y', t) = \theta(t) \left( \frac{m}{2\pi\hbar^2 t} \right)^{1/2} \exp\left( \frac{-m(y-y')^2}{2\hbar^2 t} \right). \quad (A2)$$

The modification necessary to include the effects of the gravitational field is given by

$$G(y, y', t) = g_0(y, y', t) \exp[-K(y, y', t)], \quad (A3)$$

where

$$K(y, y', t) = \frac{1}{2} mg(y+y')t - \hbar^2 mg^2 t^3 / 24. \quad (A4)$$

Equation (A3) may be verified by substituting Eqs. (A2)–(A4) into the equation of motion

$$[(\partial/\partial t) + h] G(y, y', t) = \delta(t) \delta(y - y'). \quad (A5)$$

It follows from Eq. (11) that

$$\lim[l^{-1}\gamma(t)] = \lim[l^{-1} \int_0^l G(y, y, t) dy]. \quad (A6)$$

In the thermodynamic limit,  $l \rightarrow \infty$  and  $g \rightarrow 0$  with  $gl$  finite. Therefore Eq. (14) follows from Eqs. (A2)–(A4) and (A6).

## APPENDIX B

The appropriate single-particle Hamiltonian for the rotating two-dimensional ideal Bose liquid<sup>3</sup> is

$$h = h_0 - \omega l, \quad (B1)$$

where

$$h_0 = -(\hbar^2/2m)[(\partial/\partial x)^2 + (\partial/\partial y)^2], \quad (B2)$$

and

$$l = -i\hbar[x(\partial/\partial y) - y(\partial/\partial x)]. \quad (B3)$$

For the special case,  $\omega = 0$ , the Green's function is a simple product<sup>6</sup>

$$G_0(\vec{r}, \vec{r}', t) = g_0(x, x', t) g_0(y, y', t), \quad (B4)$$

where  $g_0$  is defined in Eq. (A2). For  $\omega \neq 0$ , it follows from the equation of motion

$$[(\partial/\partial t) + h_0 - \omega l] G(\vec{r}, \vec{r}', t) = \delta(t) \delta(\vec{r} - \vec{r}') \quad (B5)$$

that<sup>3</sup>

$$G(\vec{r}, \vec{r}', t) = \exp(\omega t l) G_0(\vec{r}, \vec{r}', t). \quad (B6)$$

The exponential operator in Eq. (B6) is most easily examined if the position  $\vec{r}$  is expressed in polar coordinates  $(\rho, \varphi)$ . The operator  $\exp(\omega t l)$  replaces the real angle  $\varphi$  with the complex angle  $(\varphi - i\hbar\omega t)$ . By writing out Eq. (B4) in polar coordinates, it

then follows that

$$G = G_0 \exp \Xi, \quad (\text{B7})$$

where

$$\begin{aligned} \Xi = & (m\rho\rho'/\hbar^2 t) \\ & \times [\cos(\varphi - \varphi' - i\hbar\omega t) - \cos(\varphi - \varphi')]. \end{aligned} \quad (\text{B8})$$

In particular,

$$G(r, r, t) = \theta(t)(m/2\pi\hbar^2 t) \exp(\sigma\rho^2/R^2), \quad (\text{B9})$$

$$\sigma = (mR^2/\hbar^2 t) [\cosh(\hbar\omega t) - 1]. \quad (\text{B10})$$

From Eqs. (11), (B9), and (B10), it follows that

$$\gamma(t) \rightarrow (\pi R^2)(m/2\pi\hbar^2 t)(1/\sigma)(e^\sigma - 1), \quad (\text{B11})$$

in the limit of large  $R$ . The complete thermodynamic limit involves the rule  $\omega \rightarrow 0, R \rightarrow \infty$  with  $\omega R$  finite,<sup>3</sup> hence

$$\lim \sigma = \frac{1}{2} m\omega^2 R^2 t. \quad (\text{B12})$$

Equation (23) follows from Eqs. (B11) and (B12).

<sup>1</sup>F. London, *Superfluids* (Dover Publications, Inc., New York, 1960).

<sup>2</sup>P. C. Hohenberg, *Phys. Rev.* **158**, 383 (1967).

<sup>3</sup>A. Widom, *Phys. Rev.* **168**, 150 (1968).

<sup>4</sup>M. E. Fisher, *Arch. Ration. Mech. Anal.* **17**, 377 (1964).

<sup>5</sup>Actually, the energy per particle will diverge only

for particular forms of the interparticle interactions. However, the limiting process  $g \rightarrow 0, l \rightarrow \infty$  with  $gl$  finite gives sensible results for all of the interactions which are known (Ref. 4) to yield proper free energies in the homogeneous case  $g=0$ .

<sup>6</sup>R. Kubo, *Statistical Mechanics* (North-Holland Publishing Co., Amsterdam, 1965), p. 178.

## Renormalization of Critical Exponents by Hidden Variables

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The thermodynamics of the critical point is discussed generally for a system in which some "hidden variable"  $x$ , e.g., a density of impurities, fluctuates in equilibrium but is subject to a constraint (e.g., fixed number of impurity atoms). Under rather general assumptions it is shown that the "ideal" critical exponents  $\alpha, \alpha', \beta, \gamma, \Delta, \dots$  governing the temperature variation at the critical point when  $x \equiv 0$  become renormalized to the values  $\alpha_X = -\alpha/(1-\alpha)$ ,  $\beta_X = \beta/(1-\alpha')$ ,  $\gamma_X = \gamma/(1-\alpha)$ ,  $\Delta_X = \Delta/(1-\alpha)$ ,  $\dots$ . A variety of exactly soluble models are discussed, including a "mobile-electron Ising ferromagnet," for which this renormalization can be checked explicitly. The relevance of these results for the interpretation of theoretical and experimentally observed critical exponents is considered briefly.

### 1. INTRODUCTION

The mathematical content of this article consists chiefly in the analysis of some thermodynamic manipulations appropriate to the imposition of "constraints" on certain previously "free" thermodynamic variables and is quite elementary. To motivate our analysis and reveal its theoretical significance we ask the following fundamental question: "To what extent are the observed values of the critical-point exponents universal?"<sup>1</sup>

A preliminary answer to this question is embodied in the first-three columns of Table I.<sup>2</sup> The two leading columns summarize the experimental situation regarding a number of exponents which describe the temperature dependence of magnetic and fluid systems near their critical points. The best determined exponent is  $\beta$  with values typically

in the range 0.33 to 0.36 *both* for magnets (where  $\beta$  describes the spontaneous magnetization) and for fluids (where  $\beta$  describes the coexistence curve). In practice the specific-heat exponents  $\alpha$  and  $\alpha'$  are less well determined, since it is usually rather hard to know whether the specific heat "really" diverges to infinity ( $\alpha, \alpha' \geq 0$ ) or whether it has only a sharp but finite cusp at  $T_c$  ( $\alpha, \alpha' \leq 0$ ).<sup>3</sup> Nevertheless, the fairly close concurrence of all the exponent values for fluids and magnets does suggest the existence of a "universal" set of exponents.

Theoretically, however, the most precise calculations for the standard three-dimensional Ising models (or, equivalently, lattice fluids)<sup>1</sup> yield exponent values which, despite the numerical uncertainties (indicated by the alternative entries for  $\alpha', \gamma'$ , and  $\Delta'$  in the third column of Table I, seem