Stellar evolution

R. J. TAYLER

Astronomy Centre, School of Mathematical and Physical Sciences,
University of Sussex, Falmer, Sussex

Contents

Abstract. Stars obviously must evolve, since they continually radiate energy into space, but direct observation of the evolution of most stars seems impossible. Typically evolutionary time scales are between 10^6 and 10^{10} years. Despite this, studies of stellar evolution have progressed considerably.

On the observational side, it has been realized that, although the classification of stars into groups with similar properties is very useful, these groups are not

exclusive and any star in its full life history may be a member of several. In addition, physical groupings of stars exist in which the main difference between stars is almost certainly their mass, while there are less important differences in age and chemical composition. Massive stars evolve more rapidly than low-mass stars and study of star clusters gives information about stellar evolution.

Theoretical studies of stellar evolution have been stimulated by the development of large computers capable of solving the full equations of stellar structure, with no important physical processes neglected. These calculations have now followed stars in a wide mass range through a large fraction of their life history. The results have been compared with observations and there is at least a good qualitative agreement. There are important detailed discrepancies between theory and observation and there are stages of stellar evolution which have not yet been reached by the calculations. One very important uncertainty in present theoretical studies is the role of large-scale instabilities such as mass loss.

Most of this article is devoted to a discussion of the results of recent theoretical calculations and to the comparison of these results with observation. A brief description is given of the equations of stellar structure and the physics of stellar interiors but there is no account of the way in which these equations are solved. Because of limitations on space most of the discussion is qualitative rather than quantitative, with the emphasis on presenting the main ideas of the subject rather than the precise results of the latest calculations.

1. Introduction

It may at first sight seem strange that the evolution of the stars can be a subject of serious study. It has for some time been well established that the Sun's properties can only be changing significantly in a characteristic time scale of thousands of millions of years so that direct observation of its present evolution is ruled out. There are indeed observable variations, both regular and catastrophic, in the properties of some stars (Cepheid variables, novae and supernovae for example) but it is by no means apparent that we are observing normal stellar evolution rather than accident or disease. We now know that some stars do pass through their normal evolution at a considerably faster rate than the Sun, but even so the probability of direct observation of genuine evolutionary changes in the properties of any individual star is very slight.

How then has the subject developed without the stimulus of direct observations? The main factor has been the recognition of regularities in the properties of stars and of the existence of natural groupings of stars which are a better subject of study than the general field. The first discovery was the Hertzsprung–Russell diagram (HR diagram). When astronomers first studied the spectra of stars they found that there were many groups of stars with very similar spectra and they were able to introduce a one-dimensional classification of spectral types to which almost all stars belonged. Originally in the Harvard classification the classes labelled A, B, C, ... were principally characterized by decreasing strength of the hydrogen absorption lines; subsequently Saha showed that there was a more natural ordering in which the succession of spectral types was mainly characterized by variation in stellar surface temperature†.

† The present sequence is OBAFGKMRNS with the hottest class the O stars. The one-dimensional sequence really ends at M; RNS stars have similar surface temperatures but in cool stars small composition differences have a large effect on the spectra. OB stars are called early type stars, KM late type.

Hertzsprung and Russell showed that there was a correlation between spectral type and total output of light in the wavelength range studied (luminosity). When the luminosity of nearby stars, for which distances were known so that apparent luminosity could be converted into true luminosity, was plotted against their spectral type the stars did not fill the whole diagram but were rather concentrated in two main regions (see figure 1). The great majority were in the main sequence

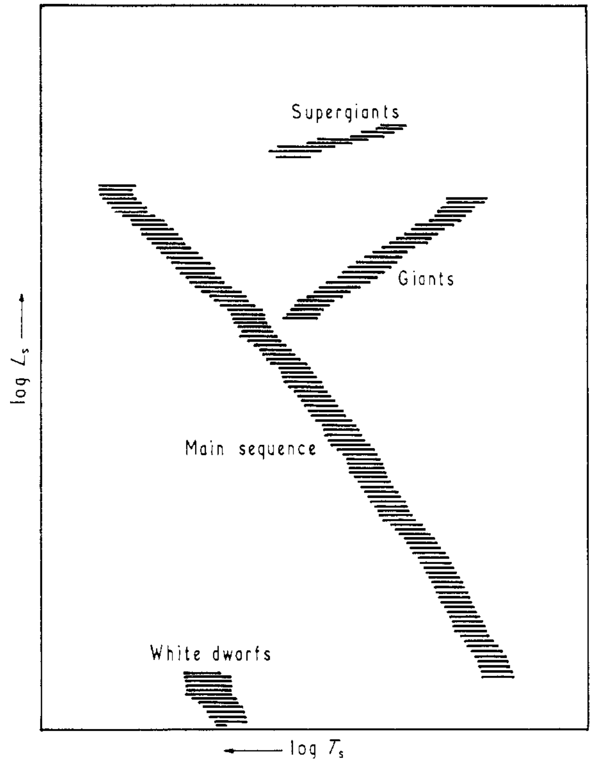

Figure 1. A schematic Hertzsprung–Russell diagram for nearby stars.

with the other substantial number in the giant branch, so called because the radii of giants were large when known. Two further groups were the supergiants and white dwarfs. The existence of this HR diagram immediately poses two problems for theories of stellar structure and evolution, and these are the main problems to be discussed in this article. Why are most stars placed in these two main regions, and are 90% of all stars main-sequence stars throughout their life or do stars spend about 90% of their life in a main-sequence state?

Russell soon proposed an evolutionary answer to the second question. He suggested that stars evolved by contraction, with the result that stars started their lives at the top of the giant branch and eventually became red dwarfs at the bottom of the main sequence. This particular theory was discarded long ago; it is still believed that stars lose gravitational energy as they evolve so that the main mass of the star does contract, but the contraction of the core can be accompanied by expansion of the outer layers.

The second significant development on the observational side was the realization that clusters of stars which form physically homogeneous groups have much better defined HR diagrams than an arbitrary group of field stars. Furthermore, since the dimensions of the clusters are usually small compared with their distances from the Earth it is a good approximation to suppose that all members of the cluster are equidistant from us and to use HR diagrams in terms of apparent luminosities instead of absolute luminosities. Finally, the stars in a gravitationally bound cluster were probably formed close together at about the same time and should probably have approximately the same age and chemical composition. In this case the principal characteristic differentiating members of the cluster should be the stellar mass. The attempt to understand the HR diagrams of individual star clusters and to understand why not all clusters have similar diagrams has been the greatest spur to the development of the theory of stellar evolution.

On the theoretical side the chief barriers to an understanding of how a star evolves (as it must do as it is continually losing energy into space) were, before 1938, the lack of a true idea of the mechanism of energy release in stellar interiors and the belief that, even if nuclear reactions changed the chemical composition of the interior of a star, there would be mixing currents efficient enough to keep the star chemically homogeneous. A reasonable understanding of the properties of main-sequence stars can be obtained without a very precise knowledge of the source of energy, but this no longer remains true when evolution is studied.

Calculations of models of stars of uniform chemical composition soon showed that they could not explain the existence of the giant branch. Although it was soon realized that stars with an inhomogeneity of chemical composition could possess giant structure, it was difficult to see how they arose naturally. Hoyle and Lyttleton suggested that they were formed by accretion of the, as yet undetected, interstellar hydrogen to give an exterior of lower molecular weight than the interior. However, it was subsequently found that the process was not efficient enough to produce the observed number of giants, and it was also difficult to see how such a process would lead to the sharply defined giant branches of clusters.

The discovery in 1938 of nuclear reaction chains releasing energy, the realization in 1950 that mixing in stellar interiors was much less important than was previously believed and the recognition at about the same time of the role that star clusters could play in the comparison of theory with observation meant that only one more thing was needed for rapid progress in the study of stellar evolution. This was the development of powerful rapid computers capable of solving the full set of non-linear differential equations of stellar evolution with all physical factors taken into account. The existence of such computers has led to very extensive developments in the studies of stellar evolution, particularly in the past five years.

Although the evolution of no star has yet been followed theoretically from birth to death, for stars in the mass range $0{\cdot}5\ M_\odot$ to $15\ M_\odot$† a considerable fraction of the life history has been followed. There is still no very good account of the formation stage but there now seems to be quite a good theory of the immediate pre-main-sequence contraction. The main-sequence stage, in which conversion of

† Throughout this article the standard abbreviation $M_\odot$ is used for the solar mass, 2×10^{33} g.

hydrogen to helium supplies the star's energy loss, is well understood and post-main-sequence evolution has been followed to the phase of helium burning or carbon burning at the centre of the star. It is clear that it will only be a matter of time before these calculations are carried to a later stage of nuclear evolution. There are of course still some uncertainties in the basic physics of stellar interiors and these will be mentioned later.

As soon as it was realized that nuclear reactions in an unmixed star could lead to the formation of a red giant, the predictions of the evolutionary theory were compared with the HR diagrams of star clusters and it was found that some of the qualitative features of these diagrams could readily be explained. In particular, the idea that an individual cluster diagram contained stars of the same age and chemical composition but different mass, while the differences between cluster diagrams were accounted for by differing cluster age and chemical composition, seemed generally satisfactory. Today much more precise correlations are demanded between theory and observation but, although some types of star have not yet been placed in the evolutionary picture and there are still some important discrepancies between theory and observation, there is no reason to doubt the foundations of stellar evolution theory.

There is one important qualification to the discussion above. Almost all of the stellar evolution calculations refer to the mathematically simplest problem, the evolution of an isolated, spherically symmetrical star. This means that the theory is not directly applicable to rapidly rotating stars, stars with strong magnetic fields or to members of close binary systems. Clearly, slowly rotating stars, stars with weak magnetic fields and partners in wide binaries are likely to behave in a similar manner to spherical stars and their structure has been studied by perturbation methods. It is clear that the importance of these perturbing effects may vary during the star's evolution. At present the study of highly perturbed stars is in its early stages and is mainly concerned with main-sequence structure. There are both mathematical and physical difficulties involved.

The remainder of this article is arranged as follows. Section 2 contains a discussion of the observations and § 3 is concerned with the equations of stellar structure, the physics of stellar interiors and the structure of main-sequence stars. Immediate pre- and post-main-sequence evolution is studied in §§ 4 and 5, while ideas on some later stages of evolution are discussed in § 6. Section 7 is devoted to variable stars and § 8 to the possible importance of perturbing forces such as rotation and magnetic fields.

The subject is very large and the bibliography is not intended to be exhaustive; a recent list of post-1958 papers on stellar interiors by Langer, Hertz and Cox (1966) contains over 800 entries. This is an elementary review article on stellar evolution and the theoretical bias of the author will also be apparent. Not much space is devoted to the fine details of the subject. The reader will find these in the more detailed treatments by authors such as Schwarzschild (1958), Burbidge and Burbidge (1958), Hayashi, Hoshi and Sugimoto (1962), Sears and Brownlee (1965), Hayashi (1966) and Iben (1967 a). There are also two recent conference proceedings (Gratton 1963, Stein and Cameron 1966). An observational astronomer's viewpoint can be found in books by Struve (1950) and Baade (1963) and in a recent review article by Eggen (1965). The book by Aller and McLaughlin (1965) is a review volume on stellar structure and most of its chapters are referred to individually below. The most comprehensive and detailed discussion of stellar evolution theory is that by Hayashi, Hoshi and Sugimoto (1962).

2. Observations bearing on stellar evolution

2.1. *Types of observational information*

First we have the observations which can be made for all stars that can be observed. They are concerned with the quantity and quality of the radiation which we receive. They are (i) apparent luminosity and (ii) spectral distribution of light. Both of these observations will be incomplete as they can only refer to the limited wavelength range accessible to observation; however, observations from rockets and satellites will gradually fill in some of the gaps in our knowledge. In order to convert apparent luminosity to absolute luminosity two quantities are required: (iii) stellar distance and (iv) an estimate of interstellar obscuration. The distance can be measured directly by trigonometric methods for the most nearby stars; otherwise indirect methods must be used including those which depend on recognizing specific types of stars whose properties are well known from a study of stars in the solar neighbourhood. An estimate of obscuration rests on similar methods. (ii) contains far more information than can usually be usefully employed but from it is extracted information about (v) stellar surface temperature and (vi) composition of surface layers. If a star is a partner in a binary system whose distance is known and whose apparent orbital elements can be observed, it is possible to find a value for (vii) stellar mass. If rather less information is available it may be possible to find one relation between the masses of the two components and if the stars are similar an estimate of the individual masses is then available. In practice there are only a very limited number of really reliable mass determinations and almost all of these are for main-sequence stars. For only a few nearby giant stars has (viii) stellar radius been observed either by interferometric techniques or by occultation observations in an eclipsing binary system; i.e. observation of the time taken for a star to disappear during an eclipse. Recently Hanbury Brown *et al.* (1967 a, b) have developed a stellar interferometer which promises to increase knowledge of stellar radii considerably.

For particular types of stars more information may be available. There are several classes of intrinsic variable stars; that is stars which vary in luminosity in a regular or semi-regular manner without being partners in multiple systems. For these stars much information is available, such as the light curve and the period. More detailed studies of stellar spectra enable us to see that some stars rotate rapidly while the existence of Zeeman splitting shows that other stars have strong magnetic fields.

There is clearly a large amount of detailed observation which needs eventually to be explained by theory. In particular it is important to try to understand to what extent the present observed properties of a star are due to (*a*) birth conditions and (*b*) life history, and it is particularly with the latter that this article is concerned.

Up till now, because observations of luminosities and colours are much more extensive than observations of masses and radii, most emphasis has been placed on trying to understand the positions of stars in the Hertzsprung–Russell diagram. In fact this article could reasonably be subtitled the interpretation of the Hertzsprung–Russell diagram. The phrase HR diagram is used to cover a variety of different diagrams. The original diagram consisted of a plot of spectral type, determined by the broad features of the spectrum, against luminosity in some wavelength range, determined by the sensitivity of a photographic plate. Subsequently, when it was shown that the sequence of spectral types was essentially a temperature sequence, logarithm of surface temperature replaced spectral type. However, both of these are rather subjective measures of the quality of the star's light output and today it is more usual to use colour index.

Colour index is usually defined in terms of the photoelectric measurement of the radiation emitted in a few well-chosen spectral bands. The most commonly used system involves three bands in the ultra-violet, blue and yellow regions of the spectrum and the magnitudes† corresponding to these measurements are denoted by U, B and V. The difference between any two magnitudes is called a colour index; if stars radiated like black bodies, colour index would be directly related to surface temperature. The HR diagram is then a plot of magnitude against colour, most frequently V against $B-V$.

The theoretician also plots HR diagrams but they refer to the bolometric luminosity L_s, the total output of the star in all frequencies, and effective temperature T_e, defined so that

$$L_s = \pi a c r_s^2 T_e^4 \tag{2.1}$$

where r_s is the stellar radius. T_e is then the temperature of a black body with the same radius and luminosity as the star. In order to compare theoretical and observational HR diagrams three things are required. The observations, which are directly apparent magnitudes, must be converted to absolute magnitudes by making allowance for distance and obscuration. Then a bolometric correction is required to convert V to bolometric magnitude and a relation is required between $\log T_e$ and $B-V$. The bolometric correction is established for certain standard stars by observations in a large number of photoelectric bands; the number is being increased by ability to work above the Earth's atmosphere. If a star's apparent luminosity at all wavelengths and its angular diameter were known, T_e could be determined without a knowledge of stellar distance. In practice determinations of T_e are usually more indirect. In any detailed comparison between theory and observations uncertainties in bolometric corrections and the $(\log T_e, B-V)$ relation must be borne in mind.

2.2. *General character of observations*

Figure 2 shows a composite HR diagram for all types of stars. As well as the main sequence and giant branch mentioned earlier there are other fairly well-defined regions populated by special groups of stars. A short description of some of these groups is as follows.

White dwarfs. Compared with main-sequence stars, these are very under-luminous for their colour. It is clear that, for any reasonable relation between

† Magnitude is a logarithmic measure of luminosity with a factor of 100 in luminosity corresponding to 5 magnitudes and with the largest magnitudes corresponding to lowest luminosities.

colour and effective temperature, they must have very small radii and be very dense; a typical white-dwarf density is apparently 10^6 g cm^{-3}.

Intrinsic variable stars. There are several groups of intrinsic variable stars of which the best known are the RR Lyrae variables, Cepheid variables and long-period (Mira) variables. These stars all show periodic variations in their luminosity, and radial velocities inferred from their spectra show that the stellar radius is changing with time (pulsation). Although the variations are periodic and period

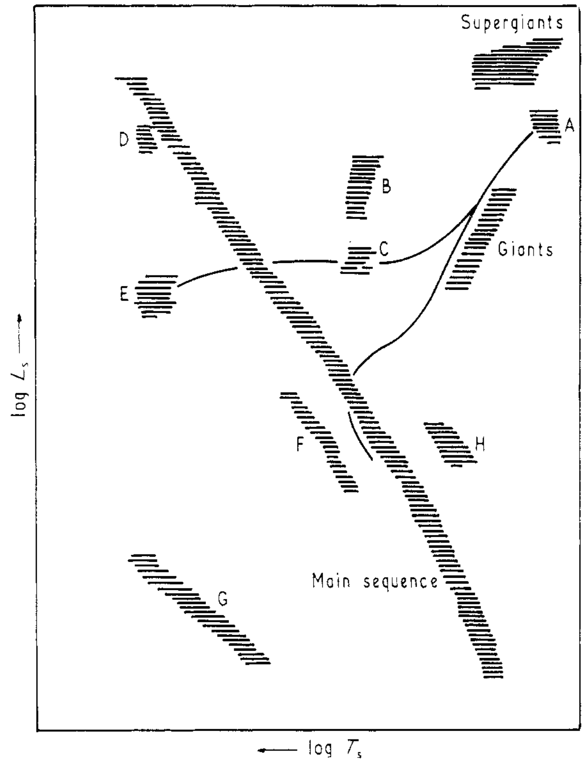

Figure 2. A composite Hertzsprung–Russell diagram. Special groups of stars include the following: A, long-period variables; B, Cepheid variables; C, RR Lyrae stars; D, Wolf–Rayet stars; E, old novae and nuclei of planetary nebulae; F, sub-dwarfs; G, white dwarfs; H, T Tauri stars.

changes when observed are very slow, the light and velocity curves are not usually symmetrical or particularly regular. The rise to luminosity maximum is usually more rapid than the decline to minimum and there can be considerable irregularities in the curve including secondary maxima. The periods vary from a few hours for RR Lyrae variables to several hundred days for long-period variables.

There are also other variables whose properties are far from regular. An example of these are the T Tauri variables which lie close to and above the lower main sequence. They show quite large but irregular variations in luminosity and evidence of outflow of matter from their surfaces.

Planetary nebulae. The ideal theoretician's planetary nebula consists of a hot blue star surrounded by a sphere of gas in which the ultra-violet light of the hot star is absorbed and re-emitted largely in the visible. It seems likely that the gas has been expelled by the star. Real planetary nebulae are usually far from being spherically symmetric. If the properties of the exciting star can be inferred from those of the nebula, it appears that they should lie a little to the left of the blue end of the main sequence.

Sub-dwarfs. This name tends to be used for all stars which lie below the main sequence as defined by nearby stars. Some of these stars have additional peculiarities but the ordinary sub-dwarf shows weak metal lines in its spectrum indicating that it has a lower abundance of heavy elements than, for example, the Sun.

Wolf–Rayet stars. These differ from other stars in the same region of the HR diagram by the possession of spectra with bright emission lines. They have extended atmospheres and these are apparently produced by the continuous ejection of material from the stars with velocities of up to 1000 km s^{-1}. There are several other classes of star with extended envelopes.

Novae and supernovae. These are not shown in figure 2 but some description of them must be given. Both of these types of star show a sudden increase in their luminosity followed by a decline. It is now thought that novae and supernovae are very distinct types of star.

The energy liberated in a typical nova outburst is 10^{44} to 10^{45} erg compared with a solar luminosity of 4×10^{33} erg s^{-1}. The brightest novae have a maximum luminosity about $10^6 L_\odot$†. The nova phenomenon can be recurrent and, although mass is lost from the star in the outburst, the amount is small, being typically about $10^{-3} M_\odot$. There appears to be no great difference between a pre-nova and a post-nova.

A supernova outburst is on a much greater scale. The energy release in visible radiation alone can be about 10^{49} erg and all supernovae have a maximum luminosity much greater than $10^8 L_\odot$. There is no observational information about pre-supernovae and not very much about post-supernovae, although some radio sources in our Galaxy are associated with supernova remnants (e.g. Crab nebula). The mass loss is believed to be substantial. On the basis of their light curves supernovae were divided into two groups, type 1 and type 2. Subsequently the type 2 supernova was identified with explosion of a massive star and type 1 with explosion of a low-mass star. Recently Zwicky (1965) has suggested that there are at least five types of supernova.

Baade (1944 a, b) introduced the concept that our Galaxy (and other galaxies) was composed of stars of two populations, population I and population II, and this has played an important role in all subsequent discussions of galactic structure and evolution and stellar evolution. In studying the nearby Andromeda galaxy M31 and its two companions he showed that the composite HR diagram of the central regions of M31 and of the companions resembled that of a globular cluster (figure 3), while the HR diagram of the outer regions of M31 resembled that of a galactic cluster (figure 4). The brightest stars of the central regions of M31 were red supergiants while the brightest stars in the outer regions were blue supergiants. He called the galactic-cluster-type stars population I and the globular-cluster-type stars population II‡. He found also that, from their position in the Galaxy, other

† $L_\odot$ denotes the solar luminosity.

‡ For a fascinating description of how he developed this concept see Baade (1963).

groups of stars discussed above could be classified as population I or II. Thus population I included blue supergiants, Cepheid variables, T Tauri stars, Wolf–Rayet stars, supernovae of type 2 and also the gas and dust of the Galaxy, while population II included RR Lyrae stars, planetary nebulae, sub-dwarfs, novae and supernovae of type 1. In Baade's original classification it was the position of a star in the Galaxy which primarily determined its population but since then it has become apparent that its place of origin is important; thus high-velocity stars in the solar neighbourhood, which is primarily populated by stars of population I, are classified as population II stars. Their place of origin is in the halo region of the Galaxy where the globular clusters are also situated.

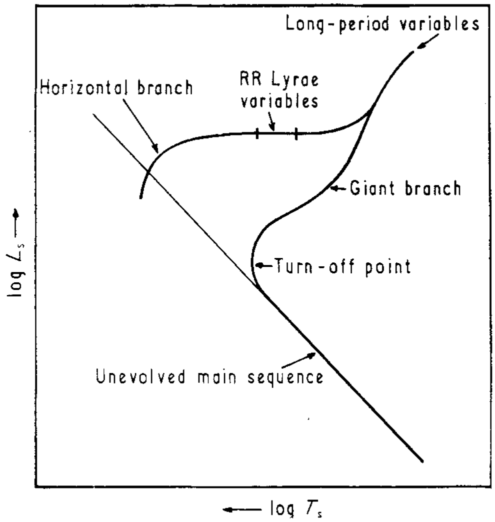

Figure 3. A schematic globular cluster HR diagram.

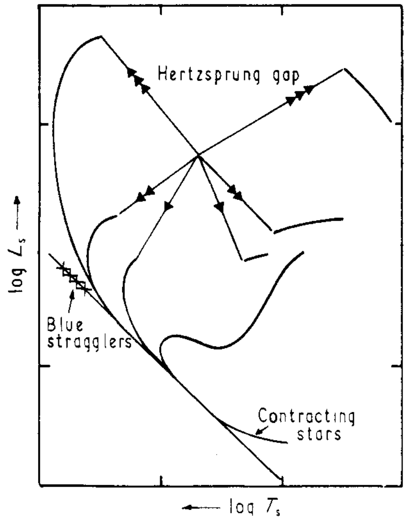

Figure 4. Schematic HR diagrams for galactic clusters.

It soon became apparent that there were two principal factors which distinguished stars of the two populations, age and chemical composition; it also became clear that there were intermediate populations between two extremes. In broadest outline stars of population II are old and have low metal content while stars of population I are young and have higher metal content. It is the task of galactic evolution theory to explain why there are correlations between age, kinematics and chemical composition of stars, although stellar evolution calculations have a role in discussing how rapidly the chemical composition of the Galaxy can be changed by nuclear reactions in stars. However, we are alerted to the possibility that differing chemical composition may be important in distinguishing the evolution of one type of star from another.

Although most of this article will be concerned with single stars there are large numbers of binary and multiple systems; possibly even a majority of stars in the solar neighbourhood are binary. If the binaries are well spaced they will behave to a large extent like single stars but unresolved binaries will lead to a composite point in the HR diagram. It should be mentioned here that some binaries are ill-matched pairs with the more massive star apparently being the least evolved.

This is contrary to expectations as the mass–luminosity relation for nearby stars of known mass and the theoretical calculations both predict that the luminosity of a star is proportional to a fairly high power of its mass ($L \propto M^4$, say) while its energy supply is proportional to the mass. Thus the lifetime should decrease quite rapidly with increasing mass. Ill-matched binaries will be mentioned again briefly in § 8.

Stars are also observed to rotate and to possess magnetic fields. Inasmuch as rotation is concerned there is one fact which should presumably eventually be explained by theories of star formation or pre-main-sequence evolution. Main-sequence stars of spectral type earlier than F are on the average rapid rotators while stars of later spectral type rotate slowly; the break in rotational velocity is very abrupt. Stars with strong magnetic fields (e.g. peculiar A stars) are found to have spectral peculiarities and, by inference, abundance anomalies and this also requires explanation. Neither of these problems will, however, be discussed in the present article.

2.3. *Star clusters*

It has already been mentioned in the introduction that the idea that clusters were fairly homogeneous groups of stars differentiated mainly by their masses served as a great spur to studies of stellar evolution. Clusters are of two main types, globular and galactic, although there are intermediate types. A typical globular cluster is a large aggregate of stars with characteristic spherical or spheroidal shape. It may contain well over 10^5 stars. Galactic clusters are much smaller and do not have the compact regular appearance of globular clusters. A large galactic cluster perhaps contains 10^3 stars. Galactic clusters are concentrated in the plane of the Galaxy while globular clusters have a much more spherical distribution in space.

A schematic HR diagram for a globular cluster is shown in figure 3. The brightest stars are red supergiants at the top of the giant branch. Special types of stars which are found in globular clusters are RR Lyrae variables which are situated on part of the horizontal branch and long-period variables which may be found at the tip of the giant branch. A planetary nebula has also been found in a globular cluster (M15). A set of schematic galactic-cluster diagrams is shown in figure 4. Most of these have blue supergiants as their brightest stars. The diagrams differ considerably amongst themselves particularly in the position of the turn-off point from the main sequence. They differ from the globular clusters by the occurrence of the Hertzsprung gap between the main sequence and the giant branch and in the absence of a well-developed horizontal branch. There are few stars in the Hertzsprung gap but if Cepheid variables are members of a galactic cluster they are found there. In some clusters a number of blue stragglers are found, stars on the main sequence above the turn-off point. In some clusters there are stars above the lower end of the main sequence and these include T Tauri variables. It should be noted that even if a galactic cluster has a turn-off point in a similar position to a globular cluster the detailed character of the HR diagram is quite different. The amount of detail that can be seen in a cluster depends on its distance from us. The ideal galactic cluster for study is one whose turn-off point is at a high absolute magnitude and which is simultaneously near enough to us for the stars which are to the right of the lower end of the main sequence also to be visible.

As well as globular and galactic clusters there is a third important type of stellar group, the expanding stellar association. These are groups of stars which are expanding from a point and they typically contain bright O main-sequence stars and T Tauri stars. O stars being of very high luminosity can only have a limited life as main-sequence stars, because of the high power dependence of luminosity on mass mentioned earlier. The present expansion velocity of an association enables an estimate to be made of its age and this gives a figure lower than the maximum possible main-sequence age of the blue supergiants. It thus appears that T Tauri stars are young stars and that in an expanding association we are observing a direct consequence of a multiple star birth process.

2.4. *Questions posed for theory*

We have first the question asked at the start of this article: Are all stars main-sequence stars for most of their lives or are 90% of stars main-sequence stars for their whole life? We are now confident that the former possibility is correct and that the main-sequence phase is the one in which a star is burning its most plentiful supply of nuclear fuel, hydrogen into helium, while most of the more exotic phases of stellar evolution follow after the completion of the main-sequence phase. Initial post-main-sequence evolution takes stars into the giant branch and this explains the broad outlines of the HR diagrams of both galactic and globular clusters. The position of the turn-off point in a cluster diagram is an indication of cluster age. Bright stars evolve more rapidly than faint stars and main-sequence stars from above the turn-off point have already evolved into the giant region. Some other factor, probably chemical composition, is required to account for the difference in the shape of globular- and galactic-cluster diagrams with essentially the same turn-off point.

Today increasingly the questions being asked are much more specific. Thus we can ask whether variable stars (for example) are intrinsically peculiar stars, whether their variability has been caused by an accident in their life history or whether most stars might have a variable phase. Today many fewer stars than were once thought are believed to be abnormal. Thus variability may be an inevitable event in the history of most stars, and most stars in a particular mass range may become planetary nebulae. We feel that the two main quantities governing the evolution of a given star are its mass and chemical composition at birth, while recognizing that important effects may be caused by binary nature, rotation and magnetic fields and on occasions these perturbing forces might lead to really abnormal stars. Thus the type of question that is asked today is: Why do variable stars in globular clusters appear in two main regions in the HR diagram, why do T Tauri stars occur in the lower main-sequence region of young galactic clusters, why if Cepheid variables occur in galactic clusters are they found in the Hertzsprung gap and which types of stars become planetary nebulae and what happens to them afterwards? Answers to questions like these are now beginning to be available.

We thus hope to follow individual stars in their evolution around the HR diagram. We do not of course expect that every star will go everywhere; thus some stars will become Cepheid variables while others will become RR Lyrae variables. It should also be mentioned that, when a region of the HR diagram is observed to be empty of stars, it does not mean that there are no stars there. It may be a region crossed by stars in periods of rapid evolution so that the probability of observing a star there is remote. The Hertzsprung gap in galactic clusters is an example of such a region. In comparing theories of stellar evolution with observational HR diagrams we can expect these diagrams to be loci of stars of different masses and the same age. However, since evolution times decrease so rapidly with increasing mass, the cluster diagrams should not differ too radically from the evolutionary track of a single star.

3. Introduction to stellar structure and evolution

3.1. *The equations of stellar structure*

A star is a self-gravitating body in which the attractive force of gravitation is resisted by the pressure gradient of the stellar material and to a lesser extent by magnetic and rotational forces. The most obvious property of a star is that it radiates energy and this immediately tells us that a star must evolve. What is the source of this energy? It was originally suggested by Kelvin and Helmholtz that the energy radiated at the surface was provided by a slow contraction of the star with release of gravitational energy. However, it was soon realized that the consequent Kelvin–Helmholtz time scale was too short, at any rate for the recent history of the Sun. With no other energy sources the Sun's properties would change significantly in 10^7 years, whereas geological evidence suggests that the Sun has not changed significantly in the last 10^9 years.

It is now known that in normal stars the main energy release is from exothermic nuclear reactions in stellar interiors; of these the main one is the conversion of hydrogen into helium which can adequately supply the Sun's radiation for an order of magnitude longer than its present supposed age. This nuclear energy is usually released in the hottest part of the star at, or near, its centre and it must then be carried to the surface. In most stars thermal conduction is much less efficient than radiation as a transporter of energy and, unless the temperature gradient required to carry the outward-flowing energy by radiation becomes too large, this is the main mechanism of energy transport. If the temperature gradient exceeds a critical value, first discussed by K. Schwarzschild, the star becomes unstable to convection and this may then become the main mechanism of energy transport.

One time scale associated with the structure of stars, the Kelvin–Helmholtz or thermal time scale, has already been mentioned. Another important time, as in any hydrodynamic problem, is the time taken for sound to cross the star. This is very much shorter than the contraction time and is typically measured in hours. It is found that in very few cases (variable stars, novae, supernovae) do significant changes in the star's properties occur in times comparable with this hydrodynamic time. This means that frequently most time derivatives can be omitted from the equations of stellar structure, a very important simplification in the calculation of stellar models.

In the deep interior of a star departures from complete thermodynamic equilibrium are very slight and, with particle distributions assumed close to Maxwellian (or in some cases the corresponding quantum-mechanical expressions) and the radiation field almost Planckian, hydrodynamic equations can be used for all the physical variables. The observed radiation comes from the stellar photosphere; here there are certainly important departures from thermodynamic equilibrium in the radiation field but probably not as far as the particles are concerned. If a complete description of the stellar spectrum is to be obtained a kinetic equation must be solved for the radiation field, but in practice this is not usually attempted in studies of stellar structure and evolution. In the earliest calculations it was supposed that, because the photospheric densities and temperatures are very much less than typical values in the interior, the overall properties of the star could be obtained by supposing that the density and temperature both vanished at the surface and that the hydrodynamic equations could be used throughout. In many cases this is a good approximation but there are stars in which a more sophisticated boundary condition is required; this will be discussed further in §4 of this article.

The form of the equations of stellar structure will be illustrated by a discussion of the simplest possible case: a spherically symmetrical star in a quasi-static state. In this case the equations of structure are effectively ordinary differential equations and they have the form

$$\frac{\partial P}{\partial M} = -\frac{GM}{4\pi r^4} \tag{3.1}$$

$$\frac{\partial r}{\partial M} = \frac{1}{4\pi r^2 \rho} \tag{3.2}$$

$$\frac{\partial T}{\partial M} = -\frac{3\kappa L}{64\pi^2 acr^4 T^3} \tag{3.3}$$

$$\frac{\partial L}{\partial M} = \epsilon \tag{3.4}$$

where it is assumed that all energy is transported by radiation, a Lagrangian mass variable M has been used, P is the pressure, ρ density, T temperature, L luminosity and r radius at mass M, κ, the opacity, measures the resistance offered to the passage of radiation, ϵ is the rate of release of energy per gramme, G is the gravitational constant, a the Stefan–Boltzmann constant and c the velocity of light. The derivatives are partial to indicate that, although quantities are slowly varying with time and time derivatives have been dropped from the equations, the physical quantities are functions of M and t.

In equations (3.1) to (3.4) there are three functions, P, κ and ϵ, which require further specification. In a state of thermodynamic equilibrium all of these quantities can be expressed in terms of the two state variables ρ, T and the chemical composition of the star. Thus schematically

$$P = P(\rho,\ T,\ \text{composition}) \tag{3.5}$$

$$\kappa = \kappa(\rho,\ T,\ \text{composition}) \tag{3.6}$$

$$\epsilon = \epsilon(\rho,\ T,\ \text{composition}). \tag{3.7}$$

The structure of the star can then be determined if its total mass M_s and its composition as a function of M are specified.

We now have a fourth-order system of differential equations and four boundary conditions are required. In the simplest case these are taken to be

$$\left.\begin{aligned} L = r = 0 \quad &\text{at} \quad M = 0 \\ \rho = T = 0 \quad &\text{at} \quad M = M_s \end{aligned}\right\}. \tag{3.8}$$

In the past it has been stated that, with the mass and composition given and the boundary conditions (3.8) assumed, the structure of the star is completely determined, the Vogt–Russell theorem. Although this may often be true it is really no more than a statement that there appear to be enough boundary conditions and it has been demonstrated that there can sometimes be more than one solution to the equations. In this case which solution is required is probably determined by the previous history of the system or by stability arguments.

Let us suppose we now have the structure of a star at one stage in its life. How do we discuss its evolution? In the simplest case there is one more equation. If the energy release is due to nuclear reactions leading to change of chemical composition and if there are no mixing currents, we have an equation of the form

$$\frac{\partial}{\partial t}\,(\text{composition}) = f(\rho,\ T,\ \text{composition}). \tag{3.9}$$

This then enables the composition of the next quasi-static model to be determined.

In the calculation of an evolutionary sequence of models the equations used may be more complicated than those given above for several reasons. There may be energy and matter transport by convection; this means that an expression is required for the convective heat transport and equations (3.3) and (3.9) must be modified. When the release of gravitational energy is significant some time-dependent terms must be introduced into equation (3.4). However, although the system of equations may be much more complicated, the essential character of the problem is contained in the discussion above. Discussion of the full set of equations can be found in books and review articles on stellar structure and evolution (Eddington 1926, Chandrasekhar 1939, Schwarzschild 1958, Menzel *et al.* 1963, Hayashi, Hoshi and Sugimoto 1962). For a historical survey of the development of the theory of stellar structure see Cowling (1966).

3.2. *The physics of stellar interiors*

It has been mentioned earlier that expressions are required for the three functions P, κ and ϵ. In most stars a good approximation to the equation of state is the perfect gas law with allowance for the contribution due to radiation pressure,

$$P = \frac{R\rho T}{\mu} + \tfrac{1}{3}aT^4 \tag{3.10}$$

where R is the gas constant and μ is the mean molecular weight of stellar material†. μ must in principle be calculated by considering the ionization equilibrium of all the elements but, as hydrogen and helium are much the most abundant elements and are fully ionized throughout most of the star, a simple approximation to the

† $1/\mu$ is the number of particles per proton mass in the stellar material.

value of μ is usually sufficient. In high-density regions the Pauli exclusion principle causes Fermi–Dirac statistics to be used for the electrons (a degenerate electron gas); the gas pressure is then to a first approximation a function of density alone (see e.g. Chandrasekhar 1939).

The opacity κ (or equivalently the radiative conductivity $4acT^3/3\kappa\rho$) requires a study of all those processes leading to the absorption (bound–bound, bound–free, free–free transitions) and scattering of radiation. In the stellar interior κ is a harmonic mean of the frequency-dependent absorption coefficient

$$\frac{1}{\kappa} = \int \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu \Big/ \int \frac{\partial B_\nu}{\partial T} d\nu \tag{3.11}$$

where κ_ν contains contributions from both true absorption and scattering and B_ν is the Planck function:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}. \tag{3.12}$$

A full account of the method of calculating opacities and some recent results are given by Cox (1965).

Nuclear energy can be released either by the fusion of light nuclei or by the fission of heavy nuclei. Although it has in the past been suggested that radioactive decay might be an important source of energy in astrophysics, indications that hydrogen and helium are overwhelmingly the most abundant elements in the Universe lead to the belief that fusion reactions are the main source of stellar energy. A sequence of exothermic nuclear reactions has been proposed:

(i) hydrogen burning (2×10^7 °K)
(ii) helium burning (2×10^8 °K)
(iii) carbon and oxygen burning (5×10^8 °K)
(iv) α process (10^9 °K)
(v) e process (4×10^9 °K).

The temperatures are typical ones at which the reactions might proceed in normal stars.

Hydrogen burning converts hydrogen into helium through one of two processes: the direct conversion through reactions of the type

$$\mathrm{P(P, e^+ + \nu)D(P, \gamma)^3_2He(^3_2He, P + P)^4_2He} \tag{3.13}$$

and the process using carbon and nitrogen as catalysts

$$\mathrm{^{12}_6C(P, \gamma)^{13}_7N(e^+ + \nu)^{13}_6C(P, \gamma)^{14}_7N(P, \gamma)^{15}_8O(e^+ + \nu)^{15}_7N(P, ^4_2He)^{12}_6C}. \tag{3.14}$$

Helium burning starts with the 3α process

$$\mathrm{3\,^4_2He \rightarrow ^8_4Be + ^4_2He \rightarrow ^{12}_6C^* \rightarrow ^{12}_6C} + \gamma \tag{3.15}$$

which is effectively a three-particle reaction as $^8_4$Be is unstable with a very short half-life. Originally it was thought that further addition of α particles would lead to substantial production of $^{16}_8$O and $^{20}_{10}$Ne together with some $^{24}_{12}$Mg. However, a supposed resonance in $^{20}_{10}$Ne is now known not to exist and this means that the main result of helium burning is $^{12}_6$C and $^{16}_8$O.

Helium burning was first thought to be followed by the α process; after a rise in temperature photons in the tail of the Planck distribution would be energetic enough to eject α particles from $^{20}_{10}\mathrm{Ne}$ to lead to the build-up of heavier nuclei. It is now realized that a rather complicated mixture of carbon burning

$$^{12}_{6}\mathrm{C}(^{12}_{6}\mathrm{C}, \gamma)^{24}_{12}\mathrm{Mg} \tag{3.16}$$

(amongst other reactions), oxygen burning and the α process will occur. As the temperature rises still further, the number of competing reactions becomes very large as the most strongly bound nuclei in the iron region are produced. The e process describes all of these reactions in which it is thought that some approximation to nuclear statistical equilibrium might arise.

An important early discussion of nuclear astrophysics is given by Burbidge *et al.* (1957). Nuclear reactions in stars have recently been surveyed by Reeves (1965) and Fowler *et al.* (1967), while the related problem of the origin of the chemical elements has been discussed by Bashkin (1965) and Tayler (1966).

Recently it has become apparent that there are many endothermic neutrino-emitting reactions which are very important in advanced stages of stellar evolution. In almost all circumstances neutrino mean free paths are so long that any neutrino produced in a stellar interior escapes from the star without further interaction. Neutrinos already occur in reactions (3.13) and (3.14) but they only carry away a small fraction of the energy released in hydrogen burning. At higher temperatures there are several neutrino-emitting reactions which are capable of carrying away more energy than the photon luminosity of the star. This is particularly true if the conserved vector current (CVC) theory of weak interactions is correct (Feynman and Gell-Mann 1958, Sudarshan and Marshak 1958). Three of the most important reactions proposed are

$$\text{photoneutrino process} \qquad \gamma + \mathrm{e}^- \rightarrow \mathrm{e}^- + \bar{\nu} + \nu \tag{3.17}$$

$$\text{pair annihilation} \qquad \mathrm{e}^- + \mathrm{e}^+ \rightarrow \nu + \bar{\nu} \tag{3.18}$$

$$\text{plasma neutrino process} \qquad \Gamma \rightarrow \nu + \bar{\nu} \tag{3.19}$$

where Γ represents a plasmon. Recent articles on neutrino astrophysics include those by Fowler and Hoyle (1964), Chiu (1965, 1966), Ruderman (1965) and Masevich *et al.* (1965).

There is one further factor which is probably the main source of uncertainty in the structure of main-sequence and slightly evolved stars. It has been mentioned above that in some stars there are zones in which convection occurs. Although conditions for the onset of convection are well understood, there is at present no fully convincing theory of the rate of energy transport by fully developed convection. This is even true in the case of laboratory convection in liquids. It is found that when convection occurs in the deep interior of a star the structure of the star is little affected by this uncertainty in convective energy transport but in the outer layers important uncertainties can arise. There are several theories of the transport of energy by convection which either implicitly or explicitly contain a free parameter. The best known is the mixing-length theory (Biermann 1948, Böhm-Vitense 1958)† in which the free parameter is the mixing length, the distance that elements move before losing their identity. The appropriate mixing length is

† For an alternative convection theory see Faulkner *et al.* (1965).

believed to be of the order of the pressure scale height $P/(\partial P/\partial r)$ and in the absence of a better theory calculations are usually performed for several values of the mixing length.

A recent book on the physics of stellar interiors is that by Frank-Kamenetskii (1962).

3.3. *Schematic life history of a star*

It is now believed that a considerable portion of the evolution of a single spherically symmetrical star is fairly well understood. A schematic life history of such a star is as follows. The star first condenses out of its surroundings to such an extent that its evolution is primarily determined by self-gravitation or other internal processes. It contracts rapidly until it becomes opaque and internal pressure gradients build up to balance the self-gravitation. Subsequently the contraction proceeds more slowly but it cannot be halted unless either the stellar material becomes degenerate and cold throughout or an energy source becomes available which supplies the energy loss from the surface. Moreover the virial theorem (see e.g. Chandrasekhar 1939) for non-degenerate stellar matter states that the internal temperature of a star must rise as it contracts. Nuclear energy sources are 'tapped' in succession as the temperature rises.

If we assume that instability, other than radial collapse, does not occur there are, broadly speaking, four possible life histories for a star.

(i) The star contracts and the central density becomes high enough for electron degeneracy before the central temperature is high enough for significant nuclear reactions to occur. If it is sufficiently degenerate the temperature ceases to rise and the star cools down to a black-dwarf state. Kumar (1963 a, b) estimates that stars with masses less than approximately $0{\cdot}09\,M_\odot$ are of this type and they can have formed and evolved in the lifetime of the Galaxy.

(ii) The collapse is halted at least once by the ignition of a nuclear fuel which can amply supply the energy losses from the star's surface. Finally, the star's central regions become degenerate and it cools through the white-dwarf state. It is known from the work of Chandrasekhar (1939) that this can only happen if the star's mass is less than the Chandrasekhar limiting mass. This mass depends on the composition of the star but is of the order of $1{\cdot}4\,M_\odot$ (see e.g. Hamada and Salpeter 1961, Mestel 1965 c).

(iii) Collapse is halted one or more times by nuclear processes but eventually there remain no more exothermic nuclear reactions capable of halting the collapse, while at the same time the mass still exceeds the Chandrasekhar limit. Then without instability, according to presently established physical theories, the gravitational collapse proceeds indefinitely (see e.g. Harrison *et al.* 1965).

(iv) No nuclear sources are able to halt the collapse and the star enters directly into the state of gravitational collapse with nuclear reactions occurring as a side process.

Unless instabilities upset the order of the masses of the objects, the progression from (i) to (iv) is from least massive to most massive. Moreover, objects of type (iv) (of whose existence we have at present no definite evidence) are too massive to be normally thought of as stars while objects of type (i) have no really interesting evolution. Thus in this article we are essentially concerned with types (ii) and (iii).

3.4. *Main-sequence stellar structure*

For our present study the most important property of the exothermic nuclear reactions that can occur in stars is that hydrogen burning releases more than 80% of the maximum energy which can be obtained by conversion of hydrogen into iron. This suggests that, for any star, the first stage of nuclear burning will last much longer than the subsequent ones. It then only needs the identification of the hydrogen-burning phase with the main-sequence position in the HR diagram for it to become plausible that most stars are main-sequence stars, because stars spend the longest part of their life in hydrogen burning.

These remarks need some qualification. Although for simplicity above we have talked about a succession of nuclear processes, in a fairly evolved star there will be several zones in which different nuclear reactions take place. In addition the actual lifetime on the main sequence depends on how much hydrogen can be burnt before the structure of the star alters radically. Stellar evolution calculations, to be discussed later, show that a star should remain in the region of the main sequence until hydrogen is exhausted in its central regions. How much hydrogen is burnt before central exhaustion occurs depends on whether there are mixing currents in the star's interior. All thermonuclear reactions are very temperature dependent, and in the absence of mixing the initial hydrogen exhaustion only occurs in a small central region. If, however, there are convection currents the region in which hydrogen is depleted may be much larger.

A crucial property of main-sequence stars is that, provided rotation and magnetic fields and other perturbing factors are unimportant at the main-sequence stage, knowledge of their previous life history is not required to determine the main-sequence structure. There are very few nuclear reactions before the star reaches the main sequence and its chemical composition should be essentially uniform before hydrogen burning starts. In addition, the main-sequence phase is so long lived that all time derivatives are effectively zero so that the structure of the star is governed by equations (3.1) to (3.8) which make no reference to past life history. This has been vital to the development of theories of stellar evolution because until quite recently the pre-main-sequence stage of stellar evolution has been far from well understood and even now ideas on star formation are rudimentary.

Main-sequence stars obtain their luminosity by converting hydrogen into helium either through the proton–proton reaction (3.13) or the C–N cycle (3.14), while the main source of opacity is Thomson scattering or bound–free absorption. In the more massive stars the C–N cycle and Thomson scattering are more important, while in low-mass stars the proton–proton chain and bound–free absorption dominate. A high temperature dependence in the nuclear reaction rates encourages convection but the precise condition for the existence of a central convective core also depends on the opacity law and a core is more probable when Thomson scattering is important. As the C–N cycle is highly temperature dependent, massive main-sequence stars have convective cores while low-mass stars do not. Necessary conditions for the existence of convective cores have been obtained by Naur and Osterbrock (1953) and Tayler (1967 a).

Schwarzschild's criterion for the onset of convection is, in the simplest case,

$$\frac{P}{T}\frac{dT}{dP} > \frac{\gamma - 1}{\gamma} \tag{3.20}$$

where the star is assumed to be a perfect gas with ratio of specific heats γ. Convective cores arise because of high temperature gradients in energy-generating regions. Stars with low surface temperatures have convective envelopes because the ratio of specific heats γ can become very close to unity in the ionization zones of abundant elements. As low-mass stars have low surface temperatures, they also have convective envelopes. Thus massive stars have large convective cores and radiative envelopes while low-mass stars have radiative cores and convective envelopes. Both at very high and very low mass there is a tendency towards the existence of fully convective stars. As mentioned earlier, a star with a convective core can burn more of its hydrogen while remaining near to the main sequence; thus the more massive a star the larger the fraction of its life it can be expected to spend near the main sequence.

The mass range for main-sequence stars is limited at both ends. At low mass the central temperature never becomes high enough for hydrogen burning and the temperature falls when the main part of the star becomes degenerate. With a normal stellar composition this critical mass is about $0{\cdot}1\ M_{\odot}$ (Hayashi and Nakano 1963, Kumar 1963 a, Ezer and Cameron 1967 b). There are corresponding masses below which helium burning cannot start in a pure helium star ($0{\cdot}35\ M_{\odot}$, Cox and Salpeter 1964) or carbon burning in a pure carbon star ($0{\cdot}8\ M_{\odot}$, Deinzer and Salpeter 1965, Takarada *et al.* 1966). At the high-mass end the stars become vibrationally unstable. In massive stars radiation pressure becomes very important and its effective ratio of specific heats of $\frac{4}{3}$ encourages instability; see §7 of this article. The stability limit for massive stars is approximately $17\ M_{\odot}/\mu^2$.

The detailed structure of main-sequence stars depends on their chemical composition and apart from that the results of different authors do not agree in fine detail because of the uncertainties in various physical parameters. Rather than tabulate the properties of main-sequence models, we shall summarize their basic character.

(i) For given chemical composition, main-sequence stars of different masses have a luminosity–effective-temperature relationship which is in qualitative agreement with the observed main sequence.

(ii) If the composition is changed the main sequence is moved in the following sense. If helium content is increased at the expense of hydrogen content the main sequence is moved to the left; if metal content is increased at the expense of hydrogen content the main sequence is moved to the right. Results of a recent calculation are shown in figure 5.

(iii) The theoretical mass–luminosity relation is in reasonable agreement with the observational relation obtained for nearby main-sequence stars with the most reliable masses (see Schwarzschild 1958). It does not, however, agree with mass–luminosity relations recently proposed by Eggen (1965).

(iv) Main-sequence stars of just over a solar mass probably have both convective cores and convective envelopes. Less massive stars have no convective core and more massive stars have radiative envelopes.

For a suitable choice of chemical composition, which is compatible with the observations, it seems that the qualitative agreement in (i) can be made quantitative. In any comparison between theory and observation it must be realized that stars remain in the neighbourhood of and above the main sequence for part of their post-main-sequence evolution (see § 5). Thus the lower part of the observed main sequence should be the true main sequence. (ii) shows that red giants cannot be

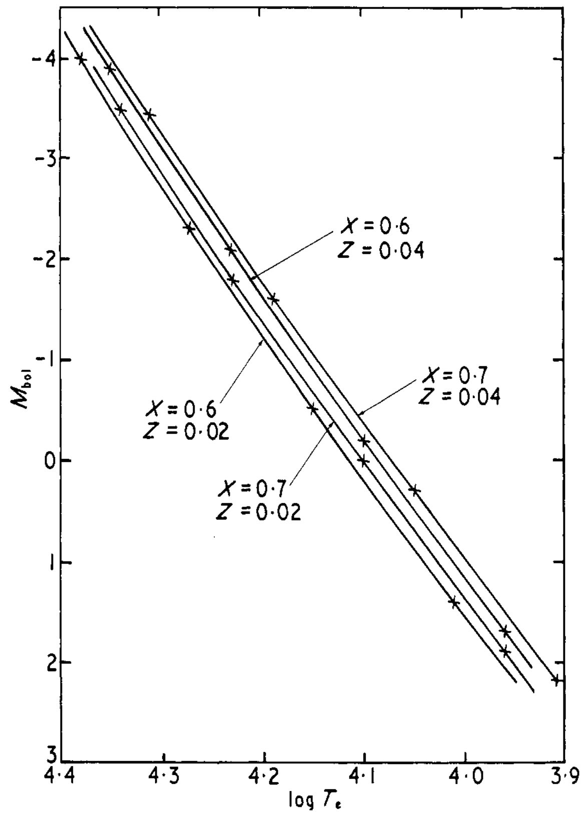

Figure 5. Theoretical main sequences for different chemical compositions. X is the hydrogen content by mass and Z is the mass fraction of elements heavier than helium. The results are taken from Strömgren (1965). (From Tayler 1967 b.)

formed from main-sequence stars if the stars remain well mixed as they evolve, for conversion of hydrogen into helium moves any individual star to the left in the HR diagram rather than to the giant region. It should also be noted that, if sub-dwarfs have a low metal content but normal helium, their position below the main sequence might be explained.

For much more detail of the structure of main-sequence stars see Iben (1967 a)†.

† Recent papers in which models are obtained for main-sequence stars include Auman and Bahng (1965), Bahng (1964), Bennick and Motz (1965), Bodenheimer *et al.* (1965), Boury (1960), Demarque (1960 a, b, 1961), Ezer (1961), Iben and Ehrman (1962), Pearce and Bahng (1965) and Varsavsky *et al.* (1962).

4. Pre-main-sequence evolution

This subject really divides into two parts: protostar formation and the pre-main-sequence evolution of protostars once they are formed. The former problem has recently been discussed at length by Layzer (1964) and Mestel (1965 a, b) while a recent review article on the evolution of protostars is by Hayashi (1966).

4.1. *Star formation*

This subject is still in a very early stage of its development and there is as yet no general agreement about the process of star formation. There are basically two different types of theory. The most common one assumes that both galaxies and stars are formed by instability and condensation in a more or less uniform medium of low density. However, other workers believe that stars originate from prestellar matter of extremely high density, possibly even nuclear density.

In the simplest cosmogonic picture it is supposed that galaxies are first formed by gravitational instability in the Universe as a whole, although attempts to account for galaxy formation from infinitesimal perturbations in the expanding Universe of general relativity or in the steady-state theory have not yet been entirely successful (see e.g. Lifshitz and Khalatnikov 1963, Hawking 1966, Novikov and Zeldovic 1967); this is, however, outside the subject matter of this article. Further gravitational instability in the galactic mass is supposed to lead to stars although Layzer has frequently expressed the view that galaxy and star formation need not be completely separate processes.

When gravitational instability within galaxies is considered there is a basic problem. It appears that the natural size of condensations is of the order of $10^3\ M_\odot$, the mass of a large galactic cluster, rather than $M_\odot$. If it is supposed that the original material of the galaxy was pure hydrogen so that no metals or dust were present, the figure is more like 10^5 or $10^6\ M_\odot$, which corresponds to the mass of a globular cluster. This suggests that star formation may be a hierarchical process in which the cluster masses first settle out and star formation follows afterwards. In any such theory the subcondensations must obtain randomized motion before contraction of the cluster as a whole leads to their coalescence. There is a mathematical theory of how the general star field is fed by slow evaporation of stars from clusters (Spitzer 1940, Chandrasekhar 1960) and there may thus be an advantage in a model which produces star clusters rather than individual stars.

One reason why star formation is not well understood is that it is certainly unrealistic to neglect the effect of rotation and magnetic fields. In fact, with the believed values of the interstellar magnetic field and vorticity, it has always been a problem to understand how the stars that we observe with low rotational velocities and magnetic fields could come into being. It is widely thought that, if a protostar is connected to its surroundings by a magnetic field, 'magnetic braking' can lead to sufficient transference of angular momentum for the formation of a star with a reasonable angular velocity. However, in that case some other process may be required to restrict the strength of the stellar magnetic field. The influence of rotation and magnetic fields on star formation is discussed in detail by Mestel (1965 a, b) and Schatzman (1962, 1966).

It should be noted that in addition to star clusters the expanding associations provide strong evidence for the simultaneous production of groups of stars. In fact it was the existence of stellar associations which led Ambartsumian to propose the theory of prestellar matter. He argued that a source of energy for the expansion of the association was required and he suggested that associations expanded from a prestellar state of high energy (for a brief discussion in English of Ambartsumian's views see Ambartsumian (1955)).

4.2. *Pre-main-sequence contraction*

In all early studies (Thomas 1930, Henyey *et al.* 1955) of pre-main-sequence stellar evolution it was believed that stars evolved with gradually increasing surface temperature and luminosity as shown in figure 6, with the main-sequence state

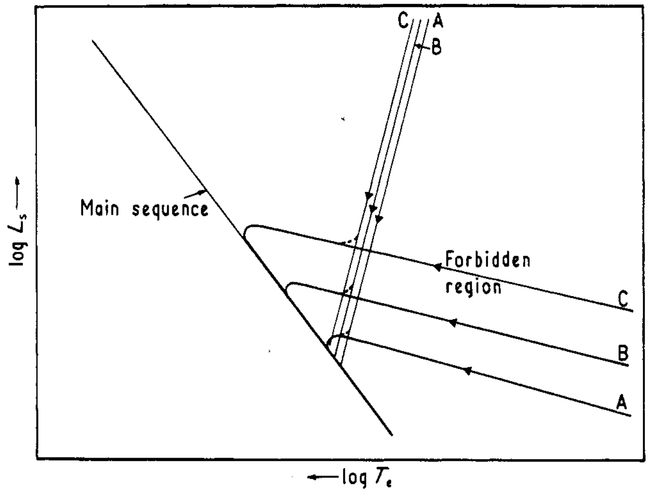

Figure 6. Approach to the main sequence. Fully radiative models follow the almost horizontal tracks while fully convective models follow the almost vertical tracks. Curves labelled A, B, C are for three different masses.

being essentially hotter and brighter than all of the preceding phases. There was perhaps a tendency to think that this must inevitably be true but that is certainly not correct. The maximum luminosity reached in the pre-main-sequence phase of gravitational contraction does not depend on the same factors as those which determine main-sequence luminosity; if a star never became opaque there would be no obvious limit to its luminosity and the maximum luminosity is therefore to some extent governed by the stellar opacity.

In the original calculations convection had been neglected in the outer layers of protostars and the simplified boundary condition $\rho = T = 0$ had been used. Hayashi (1961) showed that both of these assumptions were seriously wrong and that a star probably had a pre-main-sequence luminosity much higher than the main-sequence value. It had already been realized (Cox and Brownlee 1961) that convection would occur but it was the improved surface boundary condition which led to Hayashi's new result. The simplified boundary condition (3.8) is not a good approximation if it results in a model in which the level from which radiation escapes from the star is not close to that at which $T = T_e$. In the radiative protostar models the photospheric density was far too low and only strong convection could produce a model in which the level with $T = T_e$ was that from which radiation could just escape. In fact Hayashi found that protostars had very much deeper convective zones than was previously believed.

It was immediately clear that if Hayashi's theory were correct it would have several important consequences. Since the total release of gravitational energy in contraction to the main sequence is known accurately (since main-sequence structure is independent of previous history), the time to reach the main sequence would be significantly reduced, particularly for low-mass stars for which the Hayashi corrections are very important. It would also be possible for smaller stars than was previously believed to contract in the galactic lifetime. If stars had an overluminous pre-main-sequence phase this might account for the presence of stars above the lower main sequence in young clusters. If the deep convection zone survived almost to the main sequence, the surface concentrations of deuterium, lithium, beryllium and boron might be seriously reduced by thermonuclear reactions. Finally the possibility that the Sun might have had a high luminosity phase would have important implications for theories of the origin of the solar system. These consequences of Hayashi's theory will be discussed after his results and more recent ones have been described in more detail.

What Hayashi showed was that, if stars of a given mass were assumed to be in a quasi-static state, there was a region in the Hertzsprung–Russell diagram in which they could not lie. Moreover, the position of this forbidden region was not critically dependent on stellar mass and its boundary was almost vertical in the HR diagram (see figure 6). This means that, if a star approaches the main sequence through a succession of quasi-static states, it has a period in which it evolves with decreasing luminosity and almost constant surface temperature. The actual extent of this vertical track was not defined in the first paper but subsequently it was shown that it had an upper limit above which there were no quasi-static states. Clearly in its earliest stages a star is not quasi-static and it is not immediately clear that the contracting protostar reaches a quasi-static state at the highest possible position in the HR diagram rather than some way down the track. Thus Hayashi had essentially shown that a star might have an overluminous pre-main-sequence phase and not that it must have one.

This point has since been considered by several authors. Bodenheimer (1966 b) has considered the evolution of stars placed initially in the forbidden region and has shown that they do relax rapidly to the Hayashi track. von Sengbusch and Temesvary (1966) have suggested that whether or not the Hayashi theory is correct depends on whether further accretion of mass occurs after the initial formation of the protostar. If the star continues to grow in mass it may never become fully convective. Hayashi and Nakano (1965 a, b) and Hayashi (1966) have discussed the dynamic stage of evolution immediately after the star first becomes opaque. Their results are illustrated in figure 7, where it can be seen that there are two phases of almost vertical evolution in the HR diagram. The early stages of evolution are very rapid; for a star of one solar mass about 20 years are required to take the star to the top of the quasi-static Hayashi track and only about 100 days are required for the final flare-up.

Although Hayashi has now given a fairly complete description of the pre-main-sequence evolution of a star of given mass there are still many factors which must be investigated. What is the role of accretion and mass loss in this phase and how important are rotation and magnetic fields? The need for a good theory of convection has already been stressed and it is by no means clear that all sources of opacity at low temperatures have been investigated; two such sources are molecules (Gaustad 1963, Auman 1967) and carbon grains.

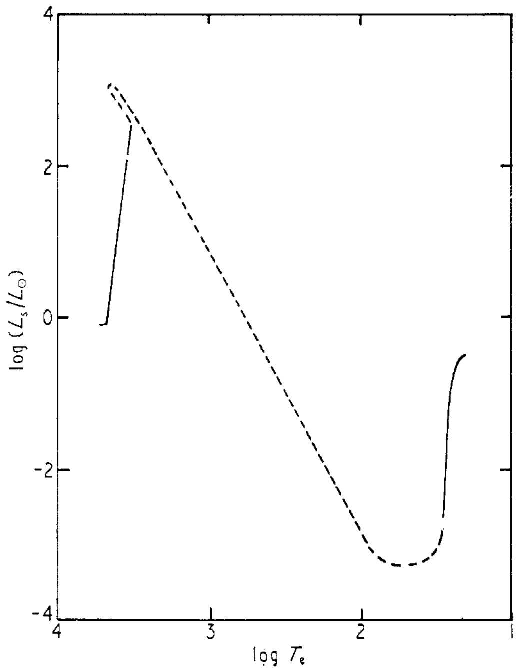

Figure 7. The pre-main-sequence contraction of a star of one solar mass. The broken line denotes a rapid dynamic phase. (From Hayashi 1966.)

4.3. *Detailed calculations of immediate pre-main-sequence evolution*

In the early calculations following Hayashi (1961) the contraction time to the main sequence of stars such as the Sun was thought to be much shorter than was previously believed. However, it is now believed that the original estimates of contraction time were far too short. We have previously stated that the first significant nuclear reactions are those converting hydrogen into helium. However, deuterium, lithium, beryllium and boron will react at lower temperatures than hydrogen and, as these reactions occur in the pre-main-sequence phase, they can hold up the contraction for a time which is significant on the contraction time scale although it is negligible compared with the main-sequence time scale. Iben (1965 a) further pointed out that in some stars another nuclear reaction would slow down the pre-main-sequence evolution. If it is assumed that carbon is more abundant than nitrogen the first reactions in the C–N cycle to occur are those which convert $^{12}_{6}C$ into $^{14}_{7}N$ and these reactions can occur before the main sequence is reached. It is now believed that for the Sun these nuclear considerations more than offset the reduction in contraction time predicted by the Hayashi effect and only for low-mass stars is it believed that there is a substantial reduction in contraction time.

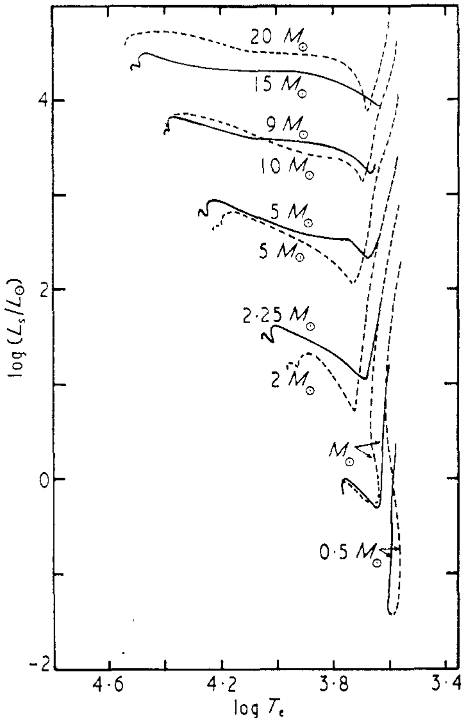

Figure 8. The approach to the main sequence for stars of different masses. The solid curves are results of Iben (1965 a) and the broken curves those of Ezer and Cameron (1967 a). (From Ezer and Cameron 1967 a.)

Sequences of evolutionary models have recently been calculated by Iben (1965 a) and Ezer and Cameron (1967 a) and the results of their calculations are shown in figure 8. The general agreement between their results is satisfactory and the general character of the evolutionary tracks is as follows:

(i) There is an essentially vertical decline in luminosity while the star remains fully convective.

(ii) For stars more massive than 0·7 $M_\odot$ there is a minimum in luminosity followed by evolution along a more nearly horizontal track in a largely radiative state, similar to the early results of Henyey *et al.* (1955). Since the boundary of the forbidden zone is only slightly dependent on mass while the main-sequence position is seriously dependent on mass, the more massive the star the longer the horizontal radiative section of the evolutionary track and the more nearly the contraction time scale is independent of the highly luminous phase.

(iii) For less massive stars there is no serious luminosity minimum before the main sequence is reached and there is still a very large convective region. Stars of 0·28 $M_\odot$ and less are still fully convective when they reach the main sequence according to Hayashi and Nakano (1963), and Ezer and Cameron (1967 b) find that slightly more massive stars become fully convective again once thermonuclear reactions start.

(iv) For stars of about 0·1 $M_\odot$ and less (Kumar 1963 a, Hayashi and Nakano 1963, Ezer and Cameron 1967 b) no serious main-sequence nuclear burning occurs. The star becomes degenerate and contracts towards the black-dwarf state. For such stars the reduction in contraction time due to the highly luminous phase is very important; before the work of Hayashi it was not thought that such low-mass stars could reach the main sequence in the galactic lifetime.

(v) In the final approach to the main sequence, particularly for the more massive stars, there are subsidiary 'hooks' in the evolutionary track. These 'hooks' occur when a nuclear fuel is ignited. When a nuclear fuel is ignited there is usually a reduction in density in the central region of the star and a consequent reduction in energy release and luminosity; this is one facet of the well-known stability of non-degenerate matter towards nuclear energy release.

4.4. *Comparison with observation*

4.4.1. *The case of FU Orionis.* As the time of dynamic contraction before the quasi-static sequence is reached is very short, it is very unlikely that a star will be observed in a state of dynamic collapse. However, Herbig (1964) pointed out that FU Orionis increased in luminosity by about 6 magnitudes in 120 days in 1936. Hayashi (1966) interprets this as the last flare-up before quasi-static contraction to the main sequence.

4.4.2. *The ages of young clusters.* At the Vatican Symposium on Stellar Populations in 1957 (O'Connell 1958) there was great difficulty in understanding the presence of stars above the lower main sequence in galactic clusters such as NGC 2264 and M8 (Walker 1956, 1957). At that time it did not seem that the faintest stars in the cluster should have evolved anywhere near to the main sequence in the supposed age of the cluster deduced from the contraction time of the brighter stars and the main-sequence turn-off point. This led to discussion that the faint stars must have been formed much earlier than the bright stars or that they had been more massive for most of their pre-main-sequence evolution and had subsequently fragmented. As soon as Hayashi had predicted an overluminous phase and more rapid contraction it seemed possible that the problem of the young clusters might be solved and this was the view taken by Hayashi, Hoshi and Sugimoto (1962) and Penston (1964).

Recent discussions of the ages of young clusters have been given by Iben and Talbot (1966) and Ezer and Cameron (1967 a). Iben and Talbot show that the hypothesis of coeval star formation will not work if the dispersion of stellar positions in the HR diagram is real and not due to observational errors. Although the problem is much less serious than it was before Hayashi's work, they think that it may still be necessary to suppose that stars in a single cluster have been formed gradually. This was previously discussed in detail by Herbig (1962 a, b). The observations of Herbig–Haro objects (Herbig 1951, Haro 1952, 1953) also suggest that star formation in clusters is a continuing process. The situation is not, however, completely clear because of various factors which have been omitted in the simple theory.

These factors include the influence of rotation and magnetic fields. They could either act to hold up the collapse of some stars or possibly could affect the efficiency of the convection. Another factor which is certainly very important is mass loss. A considerable number of the stars above the lower main sequence in young clusters such as NGC 2264 are variables. Many of them are irregular variables of the T Tauri type and they show evidence of strong surface activity. It has been shown by Herbig (1962 b) and Kuhi (1964, 1966) that there is considerable mass loss from T Tauri stars; according to Kuhi's observations this can be $10^{-7}\ M_{\odot}/\text{year}$ and such stars might lose half their mass before reaching the main sequence. It is clearly premature to talk of disagreement between theory and observation until the role of mass loss is better understood.

4.4.3. *The abundances of deuterium, lithium, beryllium and boron.* These elements are destroyed by nuclear reactions at temperatures of a few million degrees which means that we do not expect them to be present in the deep interiors of main-sequence stars. The possibility that stars have deep convection zones during their pre-main-sequence evolution means that the surface layers may also be depleted in these elements, for convection may carry the surface material down to a sufficiently high temperature for thermonuclear reactions. Their pre-main-sequence depletion has been considered by many authors including Bodenheimer (1965, 1966 a), Weymann and Sears (1965) and Ezer and Cameron (1965). If no other factors operated, observed abundances might give a good check on theories of pre-main-sequence evolution and even on theories of convection. However, there is also the possibility that the light elements might be produced by surface spallation nuclear reactions (Bashkin and Peaslee 1961, Fowler *et al.* 1962) in which proton bombardment breaks down atoms of elements such as carbon, nitrogen and oxygen. The presence of intense stellar activity in the T Tauri phase suggests that such nuclear reactions then or earlier are possible. The observations are complicated by the recent report (Boesgaard 1967, unpublished) that lithium and beryllium abundances are anticorrelated. Because these elements are so readily destroyed by thermonuclear reactions it is difficult to suppose that one generation of stars inherits them from a previous generation. It seems likely that they are produced in the earliest stages of star formation or, in the case of deuterium and ${}^{7}_{3}\mathrm{Li}$, cosmologically (Wagoner *et al.* 1967), and then largely destroyed before the main sequence is reached. However, the question is far from settled.

4.5. *Early solar evolution and the origin of the solar system*

The past life history of the Sun is of particular interest to us in this context because of its influence on the formation of the solar system. There is also particular interest in explaining why the terrestrial abundances of deuterium, lithium, beryllium and boron are greater than the solar abundances. For this reason many of the papers following Hayashi's (Ezer and Cameron 1963, Weymann and Moore 1963, Faulkner *et al.* 1963) concentrated on the pre-main-sequence evolution of the Sun. In recent years the preferred theories of the origin of the solar system have been of the solar nebula type (Hoyle 1960, Fowler *et al.* 1962, Burnett *et al.* 1965) in which the mass of the planetary system is shed by the Sun in a late stage of its contraction to the main sequence. In Hoyle's theory it was assumed that the Sun evolved according to the theory of Henyey *et al.* (1955) so that the solar luminosity was smaller when the solar system was formed than it is now. According to the Hayashi theory the solar luminosity has in the past been 500 times its present value. Faulkner *et al.* (1963) found it difficult to reconcile a high-luminosity Sun with the presence of water in meteorites (*inter alia*) and they suggested that some factor inhibited the Hayashi phase in the Sun if not in all stars. They first suggested an unknown source of opacity at low temperatures would do this but were unable to suggest a strong enough one and they also suggested that the solar magnetic field might reduce convective efficiency. Although no convincing reason for disbelieving the Hayashi effect has yet been found it is clear that its existence will continue to be questioned.

5. Post-main-sequence evolution

5.1. *Historical introduction*

It was realized very soon that homogeneous, quasi-static stars, of whatever chemical composition, could not have the observed properties of red giants. However, various authors† were able to show that stars with a single discontinuity of composition and with material of higher molecular weight inside could lie in the red-giant region. The problem was how to produce such stars as, following the earlier work of Vogt (1925) and Eddington (1925, 1929), it was believed that essentially all stars would be kept well mixed by meridional circulation.

Hoyle and Lyttleton (1942) proposed that the required discontinuity of composition arose through the accretion of interstellar matter, mainly hydrogen, by the star; the red-giant phase would then be a transient phase and in order to have a large number of giants substantial accretion would have to occur in a time short compared with the mixing time. At this stage the radio astronomers had not yet discovered and mapped the neutral hydrogen of the Galaxy. The accretion theory subsequently met difficulties on two counts: it could produce giants but it was not clear why giants in clusters should lie on a well-defined curve in the HR diagram, and the interstellar hydrogen clouds were found to be neither dense enough nor slowly moving enough for significant accretion to occur frequently. Fortunately at about this time Sweet (1950), Öpik (1951) and Mestel (1953) showed that Eddington had seriously overestimated the efficiency of mixing currents in stars and they showed that only the most rapidly rotating stars could be well mixed. This opened the way to the study of unmixed stars. The early evolution off the main sequence was studied by Tayler (1954 a, 1956) and Kushwaha (1957) and they were able to show how something like the lower part of a cluster HR diagram could arise by natural evolution of an original main sequence (figure 9).

Earlier Schönberg and Chandrasekhar (1942) had studied the structure of stars in which the central region was exhausted of hydrogen and in which the nuclear energy was being released in a shell around the centre. In the absence of an energy source they supposed that the central regions would be isothermal but they found that the star could not remain in a quasi-static state if the mass of the

† Hoyle and Lyttleton (1942, 1949), Harrison (1944, 1946, 1947), Li Hen and Schwarzschild (1949), Oke and Schwarzschild (1952), Bondi and Bondi (1950, 1951), Öpik (1938) and Gamow and Keller (1945).

isothermal core was more than about 10% of the mass of the star. As the exhausted core became larger it entered a stage of gravitational collapse which could only be stopped if it became degenerate. Sandage and Schwarzschild (1952) were the first authors to study the post-Schönberg–Chandrasekhar evolution and later Hoyle and Schwarzschild (1955) made a very detailed study of stars of a little over a solar mass and obtained a qualitative understanding of the globular-cluster HR diagram up to the top of the giant branch. Kushwaha (1957) was the first person to follow evolution from the main sequence right up to the appearance of an isothermal core.

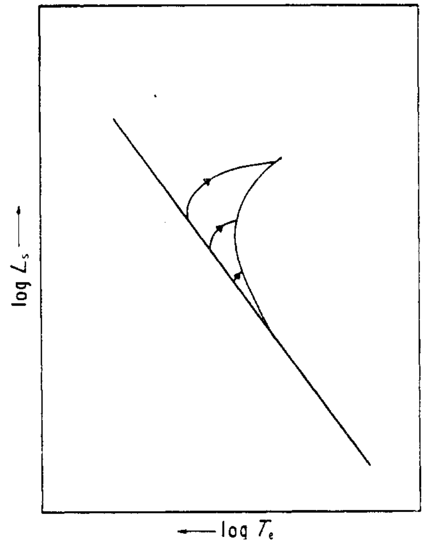

Figure 9. Evolution away from the main sequence. Because massive stars evolve most rapidly the cluster diagram near the turn-off point is produced.

5.2. *Factors influencing post-main-sequence evolution*

The detailed behaviour of a star as it evolves away from the main sequence depends quite critically on its mass; there is also a dependence on chemical composition but this is generally not so important. Factors which influence the way a star evolves include the following.

5.2.1. *Existence of a convective core.* Low-mass stars have no convective core on the main sequence whereas massive stars have large convective cores. Stars are found to stay in the region of the main sequence until hydrogen is exhausted in the central regions. This happens much more rapidly, measured in terms of the star's total consumption of fuel, for a fully radiative star than for a star with a large convective core in which fuel is continually carried into the interior. However, once a massive star does leave the neighbourhood of the main sequence its subsequent progress is likely to be more rapid as its hydrogen-exhausted core already exceeds the Schönberg–Chandrasekhar limit when it is formed. As long as the initial mass function† of a cluster is smooth and slowly varying, the density

† The initial mass function $F(M)$ is defined so that $F(M)dM$ is the number of stars with masses in the range dM around M.

of stars in any region of the HR diagram should be proportional to the time an individual star spends in the region. The rapid evolution away from the main sequence of massive stars immediately gives a qualitative explanation of the existence of the Hertzsprung gap in the HR diagrams of the young galactic clusters.

5.2.2. *Occurrence of electron degeneracy.* As mentioned earlier the virial theorem predicts that stars must continue to heat up as they lose energy as long as an increase in pressure and density is accompanied by an increase in temperature. However, at high enough densities the electrons in the stellar material become degenerate and this may enable a star to become cooler as it becomes denser. Low-mass stars generally have higher densities at a given stage in their evolution than high-mass stars and they are thus more prone to degeneracy. In very-low-mass stars, as we have already seen, degeneracy occurs before any significant nuclear reactions and the objects never become genuine stars. Degeneracy at a later stage may cut off the nuclear evolution of rather more massive stars and they may evolve through further contraction into a white-dwarf state.

Even if degeneracy is not capable of such a strong influence it can have a significant effect on the course of stellar evolution. As one example, the early models of Sandage and Schwarzschild evolved too far to the right in the HR diagram, partly because electron degeneracy had been neglected. It has also been known since the work of Mestel (1952) that ignition of a nuclear fuel in degenerate material is potentially explosive. An initial rise in temperature is not accompanied by a rise in pressure and the temperature continues to rise until the material becomes non-degenerate and a pressure rise does occur. By this time the adjustment may be catastrophic. This has led to great difficulties in carrying the evolution of stars of about one solar mass beyond the phase studied by Hoyle and Schwarzschild.

5.2.3. *Neutrino-emitting reactions.* It has recently become clear that when central temperatures are much greater than 10^8 °K reactions emitting neutrinos are likely to have an important effect on subsequent stellar evolution. At present many evolutionary calculations have not included the neutrino reactions in detail although attempts have been made to estimate their effect on the results obtained. The effect of neutrino reactions on stellar evolution depends on whether or not the centre of the star is degenerate. As neutrinos escape freely from normal stars, this tends to lower the temperature and hence the pressure of a non-degenerate core. In turn this leads to an accelerated collapse of the core and to a rise in temperature. Once the central temperature passes 10^9 °K, neutrino-emitting reactions carry away orders of magnitude more energy than photons and considerably reduce the further evolutionary lifetime of a massive non-degenerate star. If the star's centre is degenerate the cooling of the core is not accompanied by any significant change in pressure and it is thus possible that neutrino reactions might keep the central temperature down sufficiently to prevent the next source of nuclear energy from being 'tapped'.

5.2.4. *Convective envelopes.* If at any stage in a star's evolution its surface temperature becomes lower than about 5000 °K it acquires an important outer convection zone. If the star continues in a quasi-static state, Hayashi's considerations about the forbidden zone in the HR diagram apply to it as well as to contracting protostars. Thus at any time in a star's evolution there is a limit to how far it can evolve to the right in the HR diagram while remaining in a quasi-static state.

5.3. *Evolution of relatively massive stars*

By relatively massive stars will be understood those stars in which the helium burning (3.15) starts before the centre of the star becomes degenerate; according to Iben (1967 a) this means stars more massive than about 2·25 $M_\odot$, where of course this figure depends on chemical composition†. Iben (1967 a) has recently

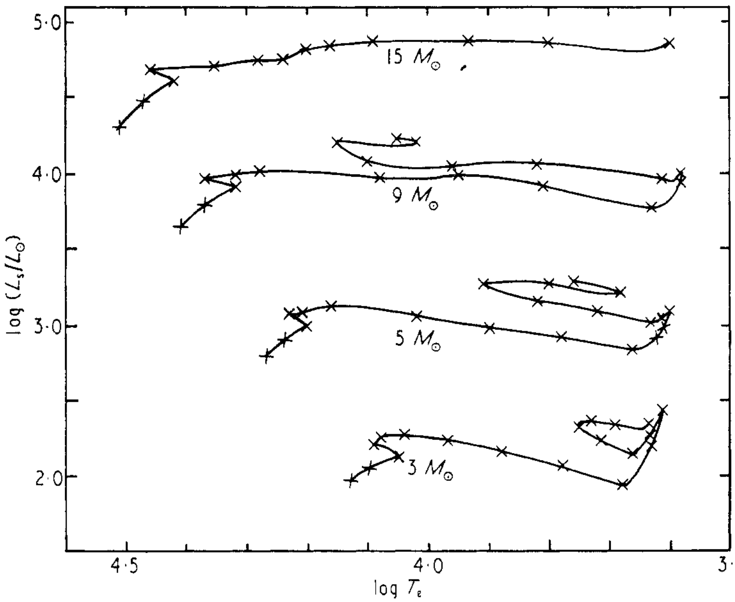

Figure 10. Post-main-sequence evolution of relatively massive stars.

reviewed all work on post-main-sequence evolution and his article should be consulted for a much more detailed discussion than can be given here. The most striking feature of these recent calculations is the complexity of the calculated evolutionary tracks with their multiple passages across the HR diagram. The results obtained by Iben are shown in figure 10; the results of other workers are generally similar. The predicted multiple passages across the HR diagram suggest that the comparison of theory and observation may not be as simple as seemed likely when only the early evolution off the main sequence had been calculated.

In these calculations one difficulty arises which has not previously been mentioned. All stars in this mass range have a convective core on the main

† Recent papers include those by Hayashi, Jugaku and Nishida (1959), Hayashi, Nishida and Sugimoto (1961, 1962), Hayashi and Cameron (1962, 1964), Hofmeister (1967 a, b), Hofmeister *et al.* (1964 a, b), Iben (1965 b, 1966 a, b, c), Kippenhahn *et al.* (1965, 1966), Kotok (1966), Polak (1962), Sakashita *et al.* (1959), Sakashita and Hayashi (1959, 1961), Schwarzschild and Härm (1958), Stothers (1963, 1964, 1966 a, b), Tanaka (1966) and Weigert (1966).

sequence. For the lighter stars the convective core decreases in size as evolution proceeds and a radiative zone of variable chemical composition comes into being outside a well-mixed convective core. For the more massive stars ($\gtrsim 10\, M_\odot$) the core grows in size and this could be expected to lead to a discontinuity of chemical composition between the convective core and radiative envelope; however, it is found that no models can exist with such a discontinuity in composition. It appears that some sort of instability must occur and it was suggested by Tayler (1954 b) and Schwarzschild and Härm (1958) that there would be a semi-convection zone of variable composition outside a smaller convective core and that inside this zone the condition (3.20) for convection would be marginally satisfied but that no energy would be carried by convection. Such a zone has been included in most later models of evolving massive stars although this has never been fully justified.

It has been mentioned in the introduction that the contraction of the interior of a star may be accompanied by expansion of the outside. In fact it is found rather generally in these calculations that the material on the two sides of a zone in which nuclear energy is being released moves in opposite directions. Thus in their calculations of a star of $5\, M_\odot$, Kippenhahn *et al.* (1965) find that when the nuclear energy is released by hydrogen burning in a shell outside an exhausted core the centre contracts and the surface expands. At a later stage, when both hydrogen- and helium-burning shell sources exist, the centre is again contracting, the region between the two energy sources is expanding and the surface is contracting. Finally there is a stage in which the hydrogen-shell source ceases to exist and then the star as a whole expands again. Thus the multiple passes in the HR diagram seem to be related to the nuclear processes ruling in the interior.

These calculations of the evolution of relatively massive stars can be compared with observations of galactic clusters and qualitatively there is agreement between theory and observations. The time scales for passage from the main sequence to the region of the giants are all small compared with the lifetime in either the main sequence or the giant regions. This gives qualitative agreement with the occcurrence of the Hertzsprung gap in the HR diagrams of galactic clusters, for, if we assume that a cluster has a smooth mass function, there should be very few stars in the region of the gap. Stars that are observed in the Hertzsprung gap include Cepheid variables and the theoretical models of stars crossing the Hertzsprung gap have been tested for pulsational instability and instability has been found; this will be discussed further in § 7.

In detail the agreement is not so good. The density of stars at any point in an evolutionary track should be inversely proportional to the lifetime in that region and thus theory should account for the relative number of red giants and stars which have just left the main sequence in galactic clusters. In the case of h and χ Persei (Wildey 1963) there seem to be too many giants by a factor of about 10. Before it was realized that neutrino-emitting reactions would be very important around the onset of core carbon burning, agreement was reasonable. However, it now appears that there should be a serious energy loss from neutrino emission from reaction (3.18). Hayashi and Cameron (1962, 1964) have in fact suggested that the number of red giants in h and χ Persei is evidence against the CVC theory of weak interactions. However, as other astrophysical evidence tends to favour universality in weak interactions various other suggestions have been made to account for the number of red giants in h and χ Persei. Iben (1967 a) thinks that the period of star formation in h and χ Persei may have been prolonged. W. A. Fowler (1967, private communication) suggests that a small alteration in nuclear reaction cross sections might cause carbon burning to start at a rather lower temperature and this might account for the discrepancy as the neutrino-emitting reactions are highly temperature dependent.

Iben discusses the agreement between theory and observation for an older cluster NGC 1866 studied by Arp and Thackeray (1967) and analysed by Arp (1967). For most parts of the HR diagram he finds a good agreement between theory and observation but in the region of hydrogen shell burning and core contraction before helium burning he finds about four times as many stars as theory predicts.

One property of highly evolved models which has been watched with great interest is the depth of the outer convection zone. When an evolved star develops a deep outer convection zone, there is the possibility that hydrogen from the outer layers can be mixed down to regions where it has previously been exhausted. In the stars in which the only energy source is a helium-burning shell source there exists the possibility that this transport of hydrogen to the helium-rich region could lead to an explosive ignition of hydrogen and to mass loss or some other instability. In this context it is important to study overshooting from convective regions (Saslaw and Schwarzschild 1965). Moving elements reach the edge of a region where criterion (3.20) is satisfied with a finite velocity and convection may therefore extend some distance into the supposedly stable region. In their study of the 5 $M_\odot$ star Kippenhahn *et al.* (1965) did find that an outer convection zone extended so that there was little mass between it and the helium shell source; however, the distance was still large and it did not seem likely that convective overshooting would lead to explosive hydrogen ignition.

The most advanced evolutionary stage at present reached by direct calculation from the main sequence is the 5 $M_\odot$ model of Kippenhahn *et al.* (1966). At the end point of their calculations there is a considerable carbon-burning convective core; there is no longer a helium-burning shell source but there is once again an outer hydrogen shell source. Further calculations to more advanced evolutionary stages, particularly for more massive stars, may be expected.

5.4. *Evolution of low-mass stars*

The earliest detailed evolutionary calculations for low-mass stars were those of Hoyle and Schwarzschild for population II globular cluster stars of about $M_\odot$. As mentioned earlier, they followed the evolution up to the onset of helium burning at the top of the giant branch and they obtained qualitative agreement with this part of the HR diagram of globular clusters. In addition it appeared plausible from their calculations that the different position of the top of the giant branch in globular clusters and old galactic clusters like M67 was due to the higher metal content in the galactic clusters. After the onset of helium burning they supposed that the stars jumped to the horizontal branch but they found it difficult to obtain satisfactory horizontal branch models. Even today, thirteen years after this early work, there have scarcely been any models which have been computed through the onset of helium burning, although there has been no lack of papers on the subject†. The factor that has caused all the difficulty is the ignition of helium in degenerate material.

This ignition is potentially explosive. If the material is sufficiently degenerate it may cause explosion of the whole star and it has been suggested that this might lead to supernovae of type 1 which are of low mass. If the explosion is less violent it may lead to mass loss or at least to serious mixing of material from the inner regions with the outer layers. This means that an evolutionary calculation may

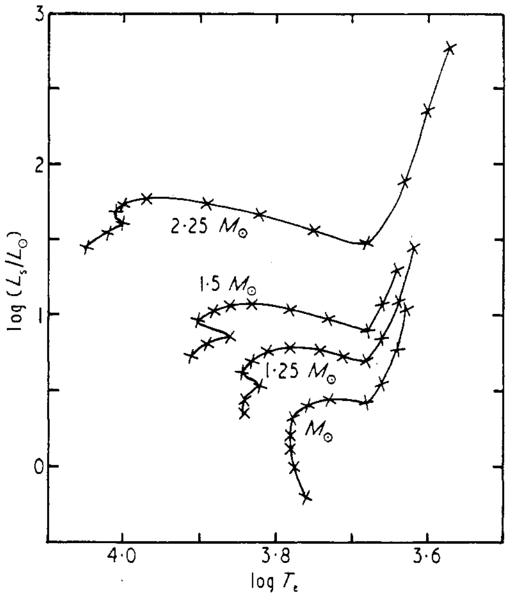

Figure 11. Post-main-sequence evolution of stars between $M_\odot$ and $2{\cdot}25\ M_\odot$. The results are those of Iben (1967 b, c).

be far from easy. In addition there are mathematical difficulties; in the early calculations of Schwarzschild and Härm (1962), at the peak of the helium flash the time interval between successive stellar models was only 2 s and the peak luminosity of the core was $10^{12}\ L_\odot$, although most of this was absorbed in the star's interior. Since then it has been realized that two factors omitted in the early calculations may be of importance. Firstly the core will not be truly isothermal before the helium flash and since gravitational energy is continually being deposited at the surface of the core it is not immediately clear that the centre of the star will be its hottest point. Since the reaction rate of the triple α reaction (3.15) is much more strongly dependent on temperature than on density, this suggests the possibility that the helium flash might start some way out from the centre and that it might be less violent than was previously supposed. Secondly the energy loss from plasma and photoneutrinos should be taken into account; this could also have the effect of reducing the violence of the explosion. In addition Schwarzschild

† Eggleton (1966 a, b), Härm and Schwarzschild (1964, 1966), Hayashi, Hoshi and Sugimoto (1962), Osaki (1963), Schwarzschild and Härm (1962), Schwarzschild and Selberg (1962) and Sugimoto (1964). Recent calculations of evolutionary tracks up to the onset of helium burning are shown in figure 11.

and Härm (1965) have pointed out that the helium flash in the core is not the only thermal instability associated with helium burning. When helium is burning in a very thin non-degenerate shell outside a helium-exhausted core a small rise in temperature will lead to the expansion of the shell, but they find that although its density drops the temperature may continue to rise and thermal runaway may result.

Härm and Schwarzschild (1964) studied the possibility that convection following the helium flash in their earlier (Schwarzschild and Härm 1962) models might lead to the mixing of hydrogen from the outer layers into the helium core. They found a very extended convective core which just appeared to miss the hydrogen-rich layers and the subsequent work of Saslaw and Schwarzschild (1965) suggested that the overshooting of convection would not alter this result. Sugimoto (1964) and Härm and Schwarzschild (1966) refined the helium-flash calculations and found a less violent flash than did Schwarzschild and Härm (1962); even so Sugimoto was not convinced that an explosion could not be caused by the helium flash and such an explosion might lead to a type 1 supernova. Schwarzschild and Härm (1967) have since studied thermal instability in stars with thin helium-burning shells and have found that successive thermal pulses can occur. In none of these calculations have neutrino reactions been fully taken into account and in many of them the finite thermal conductivity of the core has been neglected. Eggleton (1966 b) has included both of these factors and has found that in some circumstances the onset of helium burning occurs in a shell around the centre of the star. The occurrence of shell helium burning depends on the chemical composition of the star.

It is clear that it is still not possible to give an unambiguous description of evolution of a low-mass star from the main sequence to the globular-cluster horizontal branch, in particular because of uncertainties associated with mass loss and mixing at the onset of helium burning. The structure of horizontal-branch stars has been discussed by many authors, but with various free parameters which can in principle be chosen by comparison with observations. This will be discussed in § 6.

5.5. *The ages of star clusters and the Sun*

Perhaps the most important single consequence of the realization that stars would develop chemical inhomogeneities as they evolved was the possibility of estimating the ages of star clusters. Previously the ages of stellar associations could be estimated from their expansion velocities and an overestimate of the age of main sequence stars could be obtained from a comparison of luminosity and nuclear energy sources. This was sufficient to show that the most massive main sequence stars must be much younger than the Sun. However, as mentioned in § 2, an observation of the position of the turn-off point in cluster HR diagrams enabled a direct estimate to be made of the age of the cluster. Once age estimates were available the relationship of chemical composition and position in the Galaxy to age could be investigated so that information could be obtained on the chemical and dynamical evolution of the Galaxy. Finally the ages of the oldest star clusters could be compared with the age of the Universe according to various cosmological theories, with the possibility that contradiction might lead to a cosmological theory being discarded.

For clusters of a similar chemical composition the position of the main-sequence turn-off point is a monotonic function of cluster age but the absolute values of the ages deduced from comparison between theory and observation are possibly uncertain by as much as 50%. Sandage (1962) estimated an age as high as $2{\cdot}5 \times 10^{10}$ years for globular clusters and $1{\cdot}5 \times 10^{10}$ years and 10^{10} years for the old galactic clusters NGC 188 and M67. However, Woolf (1962) pointed out that the globular cluster ages were almost certainly far too high. All of these ages have been revised downward as a result of comparison with more recent calculations of stellar evolution. Thus for the old galactic clusters Demarque and Larsen (1964 a) estimate an age of 10^{10} years for NGC 188 and Iben (1967 a) obtains $5{\cdot}5 \times 10^{9}$ years for M67 and $1{\cdot}1 \times 10^{10}$ years for NGC 188. For the globular clusters Demarque and Larsen (1964 b) give about 2×10^{10} years while for M92 Faulkner and Iben (1966) give $1{\cdot}5 \times 10^{10}$ years.

As far as the Sun is concerned geological evidence and ideas concerning the origin of the solar system suggest an age of 5×10^{9} years. In this case a comparison can be made between the predicted main-sequence position of the Sun and its present slightly evolved position to see whether the amount of evolution is compatible with an age of 5×10^{9} years. We have, of course, far more information about the Sun than we have about other stars and we can make a more detailed comparison of theory and observation. This problem has been considered by Demarque and Percy (1964) who have been able to fit the observations with a solar age of about 5×10^{9} years.

The ages predicted for the oldest star clusters are still rather high compared with the Hubble time deduced from the expansion of the Universe (1 to $1{\cdot}3 \times 10^{10}$ years), whereas for most relativistic cosmological theories the age of the Universe and hence of all its component parts should be less than the Hubble time. This has stimulated calculations of stellar evolution in a Brans–Dicke (1961) cosmology with varying gravitational coupling constant. Thus Pochoda and Schwarzschild (1964), Roeder and Demarque (1966) and Ezer and Cameron (1966) have studied solar evolution with variable G and Roeder (1967) has considered the more general problem of stellar evolution and the age of star clusters. The great amount of knowledge we have about the Sun and solar system enables limits to be placed on the rapidity with which G can have varied in the past. Thus Pochoda and Schwarzschild found that G should not be decreasing more rapidly than $(t_0/t)^{1/5}$, where t_0 is the present time while Roeder and Demarque suggested a slower rate $(t_0/t)^{1/11}$. Using this value Roeder finds that the age of middle-aged galactic clusters might be reduced by only about 10%. With the uncertainties both in the Hubble age and the cluster ages calculated with the assumption of constant G, there is at present not a clear enough contradiction for variable G to be necessary.

A general discussion of age determinations is given by Sears and Brownlee (1965).

6. Advanced evolutionary stages

In § 5 we have considered evolution away from the main sequence. We have been unable to follow any star's evolution to the end of its life and there are several types of star whose structure and evolutionary significance have not been discussed.

In some cases we believe we can usefully discuss their structure and evolution although we do not have full knowledge of their previous life history. Such groups of stars include horizontal-branch globular-cluster stars, planetary nebulae and white dwarfs and supernovae.

6.1. *Supernovae*

In a supernova explosion an amount of energy is released which is comparable with the gravitational binding energy of the star and so it is reasonable to suppose that the supernova explosion shatters the whole star. For a long time it was generally believed that a star significantly more massive than the Chandrasekhar limiting mass must become a supernova as only by a great mass loss could the star reach a state in which it could die quietly. Now it is regarded as possible that a star could enter a state of gravitational collapse. However, supernovae do exist and their structure and evolution may be discussed.

Type 2 supernovae are believed to be a very advanced stage of evolution in rather massive stars (10–30 $M_{\odot}$). The only known sources of energy which can supply the supernova outburst are gravitational collapse and nuclear reactions and it is believed that such a supernova arises as follows. There is a decreasing yield of energy from nuclear reactions in the centre of a star and eventually, when the centre is largely composed of iron and neighbouring elements in the periodic table, no further nuclear energy is available and the centre must collapse. This is certainly true if the centre is still non-degenerate as it will be in a massive enough star. The speed of collapse will depend on how rapidly the star is losing energy from its surface at this stage, but for two reasons the collapse is more rapid than would be expected from consideration of the radiative energy loss alone.

At the temperatures ruling near the end stage of nuclear evolution, the energy loss from neutrino reactions is many orders of magnitude greater than the photon energy loss, particularly if the Feynman–Gell-Mann CVC theory of weak interactions is correct. This speeds up the rate at which energy is required and it has been suggested by Chiu (1961) that this could cause catastrophic collapse to occur before the exothermic nuclear reactions are completed. The second factor is that at these high temperatures and the corresponding densities the nuclear reactions are very rapid and some approximation to statistical nuclear equilibrium is set up; how close an approximation is in dispute. As the temperature increases statistical equilibrium favours a phase change from iron to helium and neutrons, and this is a highly endothermic reaction for which energy can only be supplied by catastrophic gravitational collapse. Fowler and Hoyle (1964) believe that this phase change triggers the final collapse.

Fowler and Hoyle suggest that this collapse causes very rapid heating in regions of the star further out from the centre and an explosive ignition of the nuclear fuel remaining there. It is this explosion which causes the outer layers to be blown off and produces the visible supernova. It is the supernova explosion which is usually associated with nucleogenesis in stars in the sense that it is here that highly processed material can be returned to space to be ready to be incorporated into future generations of stars; this material may have been processed either by exothermic reactions in the star's earlier history or by endothermic reactions, such as fast-neutron capture, in the explosion itself.

Several authors (Colgate and White 1966, Arnett 1966, 1967, Ohyama 1963, Ono *et al.* 1960 a, b, 1961) have considered the dynamics of a supernova outburst and Colgate and White have suggested one important modification to the theory of Fowler and Hoyle. Although they agree that explosive nuclear reactions play a role in the supernova outburst they believe that there is an even more important source of explosive energy. We have already seen that one of the causes of collapse is the vast amount of energy carried away by neutrinos with an effectively infinite mean free path. Colgate and White find that, in fact, when the core implosion is well under way it is possible for the neutrino mean free path to become less than the stellar radius so that the energy released in the centre can be deposited further out in the star and can be the main source of the supernova outburst.

There are of course some uncertainties in the whole theory because there are as yet no calculations of stellar evolution between the onset of carbon burning and the supernova event. It is also clear that the detailed behaviour of the highly imploded core must be affected by its possession of angular momentum.

6.2. *Horizontal branch stars*

As direct integration through the helium flash has proved difficult and because of the resulting uncertainties in the amount of mass loss and mixing that might occur at the time of the flash, there has been much investigation of double-energy-source models with a helium-burning core and a hydrogen shell source. A wide range of values for core and envelope size and for chemical composition has been taken in the hope that semi-empirical information could be found about what happens during the flash. In early papers by Nishida (1960), Nishida and Sugimoto (1962) and Osaki (1963) it became clear that the double-energy-source stars did lie in the region of the horizontal branch but that the exact position of the stars in the branch and even the direction of evolution across the branch depended on fine details of ratio of envelope mass to core mass and chemical composition. More recent papers on the horizontal branch include Suda and Virgopia (1966), Virgopia and Suda (1966), Faulkner (1966) and Faulkner and Iben (1966), and a brief account of the conclusions of the two latter authors follows.

Faulkner's investigation started from the observation that the shape of the horizontal branch in globular clusters is correlated with their metal content in the sense that clusters with the lowest metal content have a predominantly blue horizontal branch with relatively few stars at the red end while the reverse is true for clusters with higher metal content†. Faulkner produced double-energy-source models for a range of metal contents. He found a good qualitative agreement with the observations and further concluded that the best agreement of theory and observation was obtained if it was assumed that there was no mixing in the transition from the giant branch to the horizontal branch and that the horizontal branch stars had a helium content ($\sim 35\%$ by mass) which is much larger than is usually assumed for population II stars.

Faulkner and Iben (1966) then discussed the whole structure of the globular cluster HR diagram. They found that they could account for the entire shape of

† It should be noted that even the higher metal content is very low compared with the solar metal content. R. Cannon (1967, private communication) informs the author that the correlation between metal content and horizontal-branch structure no longer seems secure.

the HR diagram of the globular cluster M92 if they assumed that both red giants and horizontal-branch stars had essentially the same mass ($0{\cdot}7\ M_{\odot}$) and a high helium content (35%). They thus concluded that neither mass loss nor mixing had seriously affected evolution through the helium flash. The mass suggested for the globular cluster stars is considerably less than that originally suggested by Hoyle and Schwarzschild (1955) and it might therefore be thought that this low mass would lead to an unacceptably large age for the cluster. However, the reduction in hydrogen content relative to the earlier models acts as a corrective and they found an age of about $1{\cdot}5 \times 10^{10}$ years. It should be noted that the mass and chemical composition is in general agreement with that found by Christy (1966 a) for the RR Lyrae stars which are also horizontal branch stars.

The suggestion that the globular-cluster stars, which are believed to be amongst the oldest in the Galaxy, have a high helium content is very important in considering the chemical evolution of the Galaxy. If these authors are correct it seems likely that the helium in the Galaxy was produced by a cosmological process rather than by nuclear reactions in the earliest generations of stars. Their result is in contradiction with observations of some stars, including horizontal-branch stars (Greenstein 1966, Greenstein and Münch 1966, Sargent and Searle 1966, Searle and Rodgers 1966) which apparently have an exceedingly low helium abundance. A general discussion of problems concerning the cosmic helium abundance has been given by Tayler (1967 b).

The further phase of evolution of horizontal-branch stars after the main phase of core helium burning has been studied by Hayashi, Hoshi and Sugimoto (1965) and Sugimoto and Yamamoto (1966). They have found that as the helium-exhausted carbon core grows the main source of nuclear energy becomes not a helium shell source but a hydrogen shell source. However, eventually the degenerate helium outside the carbon core becomes hot enough for a second helium flash to occur and in fact they also find a third helium flash. Sugimoto and Yamamoto suggest that the second helium flash produces an extensive convection zone which may become large enough for carbon-rich material from the helium-burning shell to be mixed into the surface convection zone. These stars are situated near the top of the giant branch in globular clusters and it is therefore suggested that some stars at the top of globular-cluster giant branches should be carbon stars, that is show abnormally large carbon abundances in their atmospheres.

6.3. *Planetary nebulae and white dwarfs*

The structure of white dwarfs has been reasonably well understood since the work of Chandrasekhar (1939) and this will not be discussed here. The recent developments have been the study of the approach to the white-dwarf state and suggestions that planetary nebulae and white dwarfs form a single evolutionary sequence.

The evolution to the white-dwarf state of a star with too low a mass to ignite the carbon in the helium-exhausted core was studied by Hayashi, Hoshi and Sugimoto (1962, 1965) and the evolutionary track of such a star in the HR diagram is shown in figure 12. This evolutionary track passes through the region occupied by the planetary nebulae and suggests the possibility that planetary nebulae occur at a stage of stellar evolution between the horizontal branch and the white dwarf. Cox and Salpeter (1961) had previously suggested that they might arise from horizontal-branch stars with only a very small hydrogen envelope outside a helium core. O'Dell (1963) and Harman and Seaton (1964) from observations of planetary nebulae concluded that the central stars were shrinking and the nebulae were expanding on a very short time-scale, which would be consistent with their following an evolutionary track similar to that of Hayashi, Hoshi and Sugimoto. Thus O'Dell found the central stars contracting from one solar radius to one-hundredth

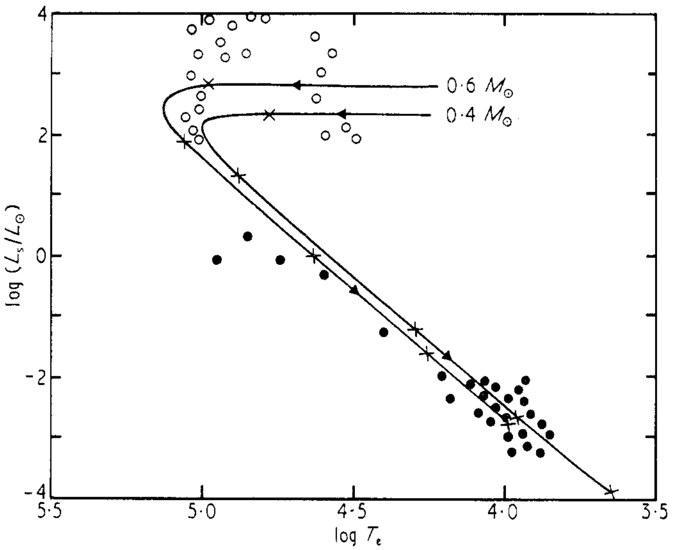

Figure 12. Late stages of evolution of low-mass stars. ○ Central stars of the planetary nebulae; ● white dwarfs.

of a solar radius in about 25 000 years and suggested that it was possible that the majority of white dwarfs had passed through a planetary nebula stage. Since then several authors (Rose 1966 a, b, Vila 1966, 1967, L'Ecuyer 1966) have studied the structure and evolution of stars mainly composed of helium which might be the nuclei of planetary nebulae and there seems general agreement with the evolutionary scheme described above.

Chin *et al.* (1966) have in particular concentrated on the role of neutrino interactions in this evolution. They point out that there is a gap of about a factor of 100 in luminosity between the faintest nuclei of planetary nebulae and the brightest white dwarfs. The absence of stars in this region indicates that it is a region of very rapid evolution, and they find it impossible to account for this rapid evolution unless there is a neutrino energy loss greater than the photon energy loss. Provided the CVC theory of weak interactions is valid, the energy loss from plasma neutrinos, reaction (3.19), appears to be large enough to account for the absence of stars in the gap. Thus Chin *et al.* as well as Fowler and Hoyle (1964) claim astronomical support for the CVC theory while Hayashi and Cameron (1962) find it difficult to interpret the red-giant numbers in h and χ Persei if the theory is correct.

7. The intrinsic variable stars

7.1. *Introduction*

There are several classes of star which show periodic variations in luminosity and other characteristics and for which it is established that the variations are intrinsic to a single star rather than, for example, due to the eclipse by a companion in a binary system. It is naturally of interest to try to understand why these stars are variable and what is their evolutionary significance. It is the latter point with which we are concerned in this article and only a minimum discussion will be given of the structure of variable stars and the related problem of stellar stability. A recent discussion of stellar stability is given by Ledoux (1965) and the theory of variable stars by Zhevakhin (1963) and Christy (1966 d).

The intrinsic variable stars have been studied seriously since Eddington's (1917) first paper on the pulsation theory of Cepheid variables. It is supposed that the light variations are due to a radial pulsation of the star and in the earliest papers adiabatic free oscillations were considered. The stellar models used were only approximate but it was found that the periods of free oscillations were comparable with the observed periods of Cepheid variables, although the simple linear adiabatic theory did not, for example, give the correct phase relationship between variations in luminosity and radial velocity. In addition only a small minority of variables have the simple symmetrical light curves predicted by the linear theory. In the earliest work the excitation of the oscillations was not explained although it was realized that in the absence of continuous excitation the oscillations should be damped more rapidly than was observed; δ Cephei has a period of $5\frac{1}{3}$ days and its period has been found to change by no more than about 6 s a century. This in turn suggests that the occurrence of variability is not accidental and that it, as well as the character of the instability, depends on the star's internal structure. The classical theory of the variable stars is well described in the book by Rosseland (1949).

With the developments in observations of HR diagrams of star clusters it became clear that variable stars were concentrated in particular regions in the HR diagrams and this suggested that the problem of the variables was concerned with stellar evolution. Whatever exciting mechanism acts it should be one which comes naturally into operation at a particular stage of a star's evolution. The germ of the ideas on this exciting mechanism was contained in another paper by Eddington (1941), where he suggested that the existence of an ionization zone of hydrogen in late-type stars probably played an important role. Zhevakhin (1953) subsequently suggested that the helium ionization zone might be more important.

Theoretical calculations of adiabatic stellar oscillations showed that for simple models the oscillations were replaced by instability if the ratio of specific heats of stellar material was below $\frac{4}{3}$; it is in fact easy to see that this follows from the virial theorem for self-gravitating systems in quasi-static equilibrium. In real stars the value of γ is variable and this means that the true criterion for dynamical instability depends on whether or not an appropriately weighted mean of γ exceeds $\frac{4}{3}$. When non-adiabatic effects are included it is found that a star can be vibrationally unstable even when it is dynamically stable and a growth is superposed on an oscillation with a frequency basically similar to that already found. There are stages in stellar evolution when dynamical instability with $\gamma < \frac{4}{3}$ might be important, molecular dissociation in a protostar and nuclear dissociation in a highly evolved star as discussed in the last section, but the variable stars are excited by a much less violent instability.

Christy (1966 d) discusses the excitation mechanism in great detail. In the simplest possible treatment it is found that the rate of increase in pulsation energy W is given by

$$\frac{dW}{dt} = \frac{1}{2}\int_0^{M_s} (\gamma - 1)\frac{\delta\rho}{\rho}\left\{\delta\epsilon - \delta\left(\frac{\partial L}{\partial M}\right)\right\} dM \tag{7.1}$$

which is an expression of the pulsational energy gain from nuclear reactions and radiation flow. It is now believed that the nuclear contribution is unimportant and (7.1) predicts that heat flow will increase the pulsation energy if heat is absorbed when the density increases. The importance of the ionization zones of the abundant elements, hydrogen and helium, is now apparent. If a small compression of a star carries neutral hydrogen, for example, to a region where its temperature encourages ionization, then it may absorb a large amount of energy from the outward flow of radiation. The system will also be destabilized if there is an increase of opacity on compression and this can also be important in ionization zones. Christy stresses that although the basic sources of excitation are now understood the detailed models of variable stars are so complicated that it is not easy to state what is responsible for the excitation in any individual star.

The identification of the importance of the ionization zones has two consequences. In the first place the pulsation of the star is found to be a much more superficial phenomenon than was originally believed, as the amplitude of the oscillations decreases very rapidly with distance towards the centre of the star. In turn this means that the detailed composition and structure of the central regions do not need to be known very accurately to obtain an approximate prediction of instability; only the stability of the outer layers, assuming that the core is immovable and supplies a constant outward flux of energy, need be studied. This further means that to a first approximation all stars in certain regions of the HR diagram can be expected to be variable and detailed evolutionary calculations with full interiors serve to say whether a star of particular mass and chemical composition will at any time have outer layers which place it in the critical region. Secondly, we have some idea of the evolutionary significance of variable stars. For relatively high-mass stars the surface temperature in the neighbourhood of the main sequence is so high that the ionization zones are not very important and we do not expect to find intrinsic variables of the type discussed here. As such a star evolves and moves to the right in the HR diagram its surface temperature drops and this leads to the possibility that instability may arise. This will be discussed further below.

The main recent development in the theory of variable stars has been non-linear calculations†. The linear theory only predicts instability whereas the non-linear calculations can study the limitation of the instability at finite amplitude. A satisfactory non-linear theory can be expected to do far more than merely explain the occurrence of the pulsation and calculate the period; it should also predict

† See for example the series of papers by Christy (1964, 1966 a, b, c, d, 1968).

the amplitude of the pulsation and explain the detailed structure of the light and velocity variations and their phase relationships. It has been known for a long time that changes in the shape of light curves are correlated with period and this correlation is predicted in the latest non-linear calculations by Christy.

There is still one defect in most theoretical studies of variable stars which is that convection has not been properly treated in the calculations. In general, convection has been included in the equilibrium models whose stability has been studied but in the stability calculations the fraction of the energy carried by convection has been held fixed. There is, however, the possibility of an important interaction between the two instabilities with the possibility either that convection can feed energy into the pulsation or vice versa. Unfortunately this is a very difficult problem as a time-dependent theory of convection is required. However, some attempts to study it have been made and the first results suggest that the interaction of convection and pulsation might account for some of the present discrepancies between theory and observation.

7.2. *RR Lyrae stars*

The evolutionary place of the RR Lyrae variables is quite clear; they are horizontal-branch stars such as those discussed in § 6.2. Like those there is a break in their detailed life history at the time of the helium flash but an understanding of the RR Lyrae phenomenon might give some information about what must have happened at the onset of helium burning.

Christy (1966 a) has recently made an extensive non-linear investigation of RR Lyrae models. As mentioned earlier only the outer layers of the star need be considered in detail and all possible atmospheres can be characterized by values of four parameters, luminosity, effective temperature, chemical composition and surface gravity, or equivalently mass of the star. Christy investigated one hundred possible models for RR Lyrae variables. Instability was found to occur for models in the region of the observed RR Lyrae strip in the HR diagram although the boundary of the strip was significantly dependent on the helium content of the atmosphere; the high-temperature boundary of the strip increased by about 500 °K for each 15% increase in the mass fraction of helium. It was found that the best agreement between observed and calculated light and velocity curves was obtained if the stars had about 30% helium by mass. If the luminosities of the stars agree with observations the masses must be about $0{\cdot}5\ M_{\odot}$.

The two conclusions about the masses and chemical compositions of the RR Lyrae stars are very important and they should be compared with the discussion we have already given about horizontal branch stars. The masses found by Christy are rather low even compared with the latest horizontal branch estimates by Faulkner and Iben.

7.3. *Cepheid variables*

As mentioned in § 5.3, evolutionary tracks of relatively massive stars have been found to cross the Cepheid region up to five times and this means that there is a possibility of placing Cepheids in an evolutionary sequence. Shortly after these evolutionary tracks were obtained Baker and Kippenhahn (1965) investigated the stability of the models against small perturbations and found that instability was predicted for all five crossings of the Cepheid region. They also pointed out that the linear oscillation period of the models varied as the strip was crossed, increasing as the star moved to the right in the HR diagram and decreasing for crossings in the opposite direction. This suggested the possibility of identifying the direction of evolution of a star for which small-period changes could be observed. It should be noted that this is a rare opportunity of observing non-catastrophic stellar evolution. The position of the theoretical instability strip was close to the observed strip with the theoretical strip extending some hundred degrees too low in effective temperature.

These results have been followed by a more thorough investigation by Hofmeister (1967 b). In a previous paper (Hofmeister 1967 a) she has obtained some revised models for stars of 5 $M_\odot$ and 9 $M_\odot$. In these calculations she varied the chemical composition of the stars taking in one case 60% hydrogen and 4·4% heavy elements and in the other case 74% hydrogen and 2·1% heavy elements. She found that the shape of the evolutionary tracks in the HR diagram depended strongly on the chemical composition and that for the low metal content there were no further crossings of the Cepheid strip after the first evolution to the red-giant region. On all counts she found that the models with a high metal content gave the best agreement with observations. The position of the instability strip still differed from observations in the way found by Baker and Kippenhahn, and both she and they suggested that this discrepancy would be resolved if a variable convective flux were allowed. Some investigations by Gough (1966) suggest that this is probably true. However, Kamijo (1967) has suggested a destabilization due to convection in Cepheids.

Hofmeister compared the number of Cepheids in proportion to main-sequence stars of similar mass predicted by her models to the number observed. For the high-metal-content models the numbers agreed to within a factor of 2 and no better agreement than that could be expected when the uncertainties of both theoretical and observational numbers are considered. For the lower metal content there were too few Cepheids by a factor of 10 and that seemed significant. For the low-metal stars only increasing periods could be obtained while the higher metal content allowed both increasing and decreasing periods in agreement with observations. She thus concluded that Cepheids could be placed firmly in an evolutionary sequence provided that they could be assumed to have a high metal and high helium content. It should of course be borne in mind that these statistical arguments do not rule out the possibility of there being some Cepheids with a lower metal content.

As mentioned in the discussion of RR Lyrae variables the linear theory can only give some of the properties of the stars and Christy (1968) has recently extended his study of non-linear models to Cepheids. His calculations have shown a change of shape of light curve with period of pulsation which is in general agreement with observation. However, if he wishes to obtain the best possible agreement between theory and observation, the masses of the stars have to be lower by a factor of 2 than would be predicted by the evolutionary models of Iben and Hofmeister. It is therefore suggested that considerable mass loss must have occurred in the red-giant phase. This would not necessarily lead to a significant change in luminosity in post-red-giant evolution in the same way as the horizontal branch luminosity is only weakly dependent on stellar mass. However, before this mass-loss hypothesis is accepted there will need to be a thorough investigation of other possibilities or else direct observational evidence of mass loss at the required high rate.

8. Non-spherical stars

8.1. *General discussion*

In all that has been said above we have concentrated attention on the single, spherically symmetrical star largely because this is the simplest and the best understood problem. However, there are rapidly rotating stars, stars with strong magnetic fields and close binaries. Moreover, in theories of star formation an explanation is required why most stars do not rotate rapidly or have strong magnetic fields.

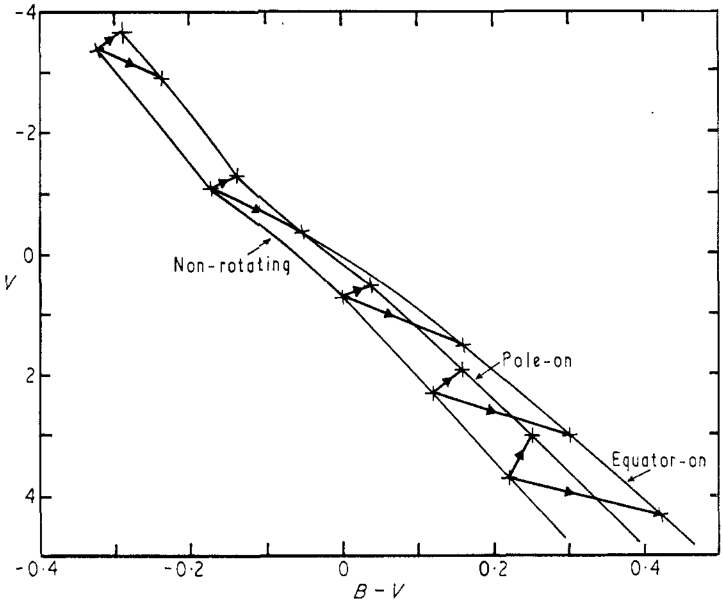

Figure 13. The rotational spread of the main sequence. The displacement in the HR diagram is shown for stars rotating pole-on and equator-on. (From Roxburgh and Strittmatter 1965.)

There are basically two types of effect associated with rotation and magnetic fields. The one that will be called structural is that stars with high rotational velocities are seriously non-spherical. Thus rotating stars are flattened at the poles. In the simplest cases we can calculate that this lack of sphericity will lead to a different luminosity for the star (usually a reduction) but in addition there will be differences in its appearance according as it is seen pole-on or equator-on; in any case the star will appear redder (see figure 13). Thus on the observational side we can expect main sequences to be blurred by the occurrence of rapidly rotating stars and that slightly evolved non-rotating stars will coincide in position with rapidly rotating main-sequence stars. Recent papers on the rotational spread of the main sequence include Roxburgh and Strittmatter (1965, 1966 b), Rubin (1966) and Faulkner *et al.* (1968).

If the rotational velocity becomes very large the centrifugal force may become comparable with the force of gravity at the star's equator; in this case there may be mass loss from the neutral curve in the combined gravitational and centrifugal potential (figure 14). Clearly as the star expands and contracts at different stages

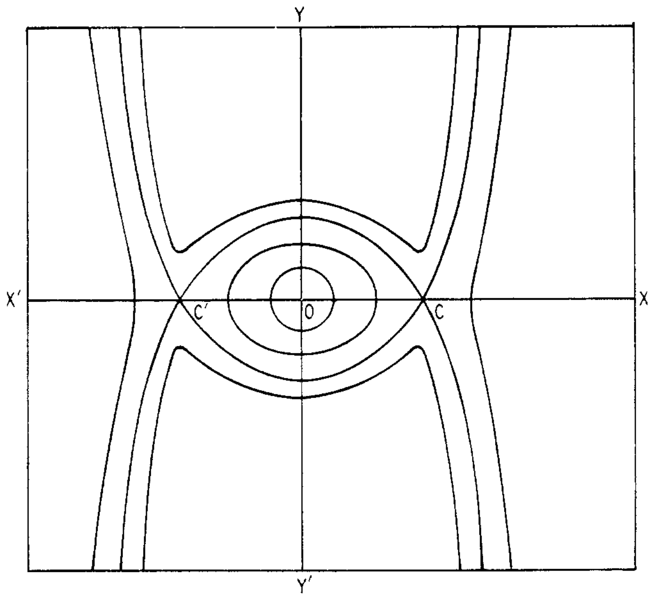

Figure 14. Equipotentials for a rotating star. Mass loss from the equator can be expected if the star fills the equipotential containing C and C′.

of its evolution there could in principle be phases of mass shedding succeeded by phases in which the star is well within the critical equipotential. This purely dynamical effect can be even more important in the evolution of close binary systems and this will be discussed later.

In addition to these rather obvious effects of departures from spherical symmetry there are some rather more subtle consequences of perturbing forces. If we suppose that a star in which all the energy transport is by radiation is rotating with an arbitrary rotational velocity or contains a magnetic field, it is possible to show that in general these assumptions are inconsistent. This 'paradox' was discovered by von Zeipel (1924) and Eddington (1925), and Vogt (1925) showed that the rotation or magnetic fields would drive a meridional circulation in the star (figure 15). It was these currents which were expected to keep even a slowly rotating star well mixed until Sweet (1950) showed that their speed had been overestimated. However, although they are generally much slower than thermal convection currents they do convect matter and vorticity, and because the stellar material is highly ionized they convect magnetic fields and can lead to the existence of magnetic fields where there were none before.

As soon as the existence of meridional circulation is admitted it becomes very difficult to obtain consistent steady-state models for stars which are rotating or contain magnetic fields. The reason for this is that it is thought that ordinary dissipative mechanisms such as viscosity are unimportant except in convection zones and that a fluid element in a radiative region would be expected to retain its angular momentum during circulation, unless a poloidal magnetic field effects angular momentum transfer. In the absence of such a field two possibilities arise: either angular momentum is constant along circulation streamlines or a state of zero circulation is eventually reached. In the first case there is inevitably a singularity

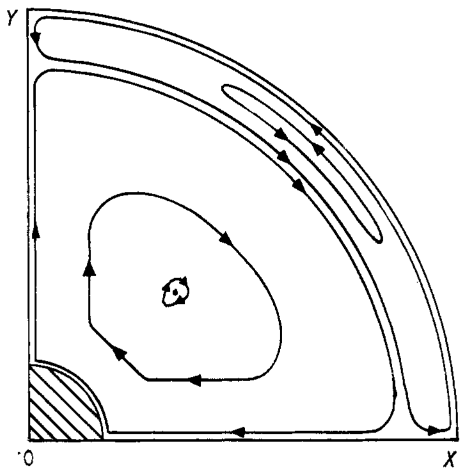

Figure 15. Meridional circulation in a uniformly rotating star with a convective core. The convective core is shaded. The arrows show the direction of the circulation. The break-up into two independent circulations has been discussed by Mestel (1966).

in the angular velocity on the axis of rotation if the surface of the star is observed to rotate at all, so that dissipation must become important locally. Some distributions of angular momentum and magnetic field which do not drive circulations have been found. Such solutions without a magnetic field have been found by Roxburgh (1964 a, b) and with a magnetic field by Roxburgh (1966 a) and Roxburgh and Strittmatter (1966 a, b). However, doubts can be cast on the significance of either of these solutions. In the models of Roxburgh (1964 a, b) the angular velocity decreases outwards and Fricke (1967) has shown that the equilibrium is unstable against small perturbations. In fact Goldreich and Schubert (1967) have shown that the equilibrium is always unstable if the angular momentum per unit mass decreases outwards. In addition Howard *et al.* (1967) have suggested that there may be processes more efficient than viscosity in equalizing angular velocities of stars so that it is possible that stars rotate as solid bodies despite the tendency of meridional circulation to transport angular momentum. Dicke (1967) and Fricke and Kippenhahn (1967, private communication) have, however, suggested that the mechanism proposed by Howard *et al.* (1967) (Ekman boundary layer) might not be effective in stars. If there is a poloidal magnetic field, states of steady circulation may exist in which the lines of circulation essentially coincide with those of the magnetic field, but if the magnetic field is basically dipolar there must be regions where the circulation crosses the field. There are a large number of unsolved problems and it is at present difficult to estimate the influence of rotation and magnetic fields on stellar evolution.

Another possible effect of rotation and magnetic fields arises from their interaction with convection. Theory and experiment on convection in liquids show that the onset of convection can be delayed by magnetic fields and rotation. There is thus the possibility that a strong magnetic field or rapid rotation might affect the existence or efficiency of convection in stars. This would be of particular interest in the Hayashi phase of pre-main-sequence contraction because at this stage we expect strong magnetic fields and high rotational velocities. Conversely, if the convection is not seriously affected by the magnetic field and the rotation, it may be strong enough to break down the length scale of the field and velocities so that dissipation due to resistivity and viscosity is considerably enhanced. In this case it might be the convection in the Hayashi phase which helps to prevent the angular velocities and magnetic fields from being unmanageable in the main-sequence phase. This is a difficult non-linear problem and it is not easy to progress beyond order-of-magnitude estimates.

8.2. *Evolution of close binaries*

One of the main uncertainties in theories of stellar evolution is the extent to which mass loss plays an important role. In most cases it is difficult to estimate the rate at which mass loss will occur but in one problem it is possible to make progress with a minimal number of assumptions. This problem is the evolution of close binary systems.

Before discussing their evolution we must first consider their formation. In the case of binaries where the separation of the two components is not vastly different from the diameters of the individual stars it is clear that a simple picture in which the two components undergo their pre-main-sequence evolution with only slight interaction cannot be correct, for if the stars were a close binary soon after formation they would become a very wide binary as a result of pre-main-sequence contraction. Two types of theory have been proposed to try to account for the existence of close binaries. In the first it is supposed that there is some coupling between spin and orbital angular momentum of the two stars which enables their separation to be reduced as they evolve; in recent discussions by Mestel (1968) and Huang (1966) it is suggested that magnetic braking by material in a stellar wind which is tied to magnetic field lines might be the solution to the problem. The other obvious possibility is that the star was originally single and only divided because of rotational instability at a fairly late stage in its pre-main-sequence evolution. This possibility has recently been discussed by Roxburgh (1966 b).

Let us consider now the post-main-sequence evolution of close binaries. In a plane containing the axis of rotation and the centres of the stars the equipotentials of the combined gravitational and centrifugal fields of the two stars are schematically shown in figure 16. As the stars evolve the more massive star will first become a red giant and if the separation of the components is small enough it will in time fill its lobe of the critical equipotential† which first connects the two stars. Any

† The Roche lobe.

further expansion will lead to material escaping from the more massive to the less massive star and there is thus mass exchange between the two components.

Calculations on the evolution by mass exchange of close binaries have recently been performed by several groups of workers (Kippenhahn and Weigert 1967, Kippenhahn *et al.* 1967, Giannone and Weigert 1967, Paczynski 1966, 1967 a, b, Paczynski and Ziolkowski 1967, Plavec 1967), and the work of the German group will now be described briefly. The calculations are simplified in several ways. The

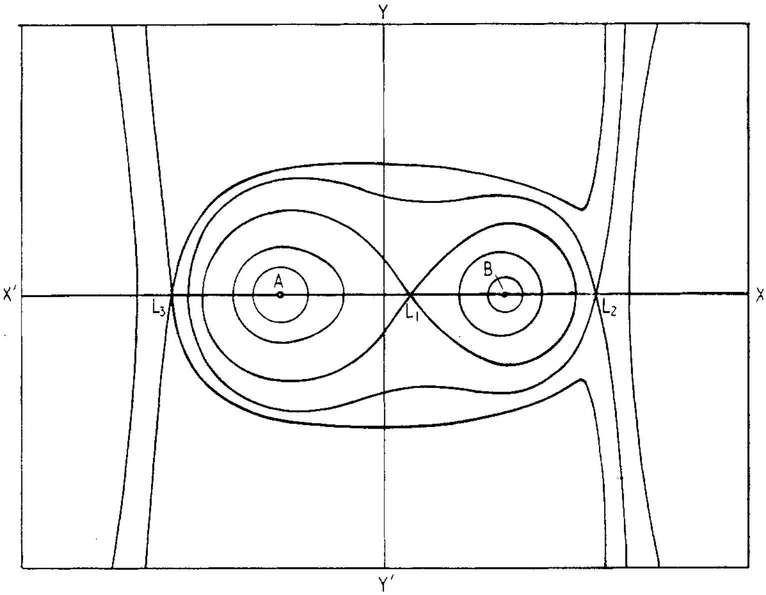

Figure 16. Equipotentials for a synchronously rotating binary system. Mass exchange occurs if either star fills its lobe of the critical equipotential passing through L_1.

lack of spherical symmetry of the individual stars is not taken into account and it is assumed that the stars rotate synchronously. It is possible that in an actual system processes of angular momentum transfer accompanying mass transfer would alter this.

In the first paper Kippenhahn and Weigert considered the evolution of a double star with components of masses 9 $M_\odot$ and 5 $M_\odot$. First they supposed that the stars were so close that mass loss occurred in the stage of hydrogen burning when the massive star was still near to the main sequence. In this case after mass loss the original secondary was now a main-sequence star with a mass of 11 $M_\odot$ and the original primary was a red giant of mass 3 $M_\odot$. Most of this mass loss occurred in the very short period of 6×10^4 years so that no significant evolution of the remaining main-sequence star could occur during that time. At the close of this phase of evolution the new secondary still filled its Roche lobe. In the second case they supposed that the separation was wider and that mass exchange did not begin until after the central hydrogen was exhausted. In this case the mass loss was even greater and the original primary became an almost pure helium star of 2 $M_\odot$ and settled down near the helium-burning main sequence.

Kippenhahn *et al.* studied the evolution of a binary with components $2\,M_\odot$ and $M_\odot$ and an initial separation of $6{\cdot}6\,R_\odot$. The mass loss again starts after hydrogen has been exhausted in the core and at the end of the mass loss the original primary has a mass of only $0{\cdot}26\,M_\odot$. This is too low for the ignition of helium and the star becomes a white dwarf, while the original secondary is a main-sequence star of mass $2{\cdot}74\,M_\odot$. This shows how ill-matched binaries can arise: a low-mass white dwarf as a companion for a more massive main-sequence star.

There is a possibility of further evolution in this system if the new primary can in turn fill its Roche lobe and transfer hydrogen-rich material to the white dwarf. This has been studied for a white dwarf of initial mass $0{\cdot}5\,M_\odot$ by Giannone and Weigert. They have shown that ignition of hydrogen in a shell in this star can lead to thermal instability of the type discussed by Schwarzschild and Härm (1965). A succession of thermal pulses can occur and they found that the star brightened by 5 magnitudes and this might be related to the outburst of a dwarf nova (see e.g. Kraft 1963).

It is clear that many interesting results may follow from the consideration of the evolution of close binary systems. There is now a belief (Mumford 1967) that most, if not all, novae are binary systems and phenomena of the type discussed by Giannone and Weigert but on a larger scale might provide a theory of novae. McCrea (1964) suggested that the main-sequence blue stragglers in galactic clusters might be partners in binary systems and it now seems quite possible that this is so. They would be the original secondaries which have increased in mass and evolved much less than their mass would suggest. Such a process might also account for the presence of white dwarfs in a young galactic cluster such as Hyades (Auer and Woolf 1965).

9. Concluding remarks

It should be clear from what has been said above that we feel that we have a fairly good theoretical understanding of the structure and evolution of single, spherically symmetrical stars. There are uncertainties of detail because of the inadequacy of our knowledge of the cross sections of many of the atomic and nuclear processes occurring in stars. In addition there are mathematical complications when evolutionary phases are very rapid or when there is a large number of competing physical processes. In future theoretical calculations are likely to be taken to later evolutionary stages than have yet been reached directly. There is at least a good qualitative agreement between theory and observation. This agreement is likely to be tested more precisely with improvements in observational data such as measurements of a wider range of stellar radii and observations of stellar spectra over a wider wavelength range from rockets and satellites. However, the foundations of stellar evolution theory seem secure.

The same is not true of situations where large-scale instabilities occur or when perturbing forces cause stars to be non-spherical. Two large-scale instabilities, mass loss and fully developed convection, have been mentioned frequently in this article and it is clear that an increasing effort must be devoted to their study. Only a minority of stars may be rapidly rotating or possess strong magnetic fields when they are in their main-sequence phase, so that the subject of strong perturbing forces might be considered less important. Nevertheless, such strong perturbing forces may affect almost all stars when they are forming and it is clearly of interest to understand why some stars possess large angular momentum and magnetic fields while others do not.

References

ALLER, L. H., and MCLAUGHLIN, D. B. (Eds), 1965, *Stars and Stellar Systems*, Vol. VIII, *Stellar Structure* (Chicago: Chicago University Press).
AMBARTSUMIAN, V. A., 1955, *Observatory*, **75**, 72.
ARNETT, W. D., 1966, *Can. J. Phys.*, **44**, 2553.
—— 1967, *Can. J. Phys.*, **45**, 1621.
ARP, H., 1967, *Astrophys. J.*, **149**, 91.
ARP, H., and THACKERAY, A. D., 1967, *Astrophys. J.*, **149**, 73.
AUER, L. H., and WOOLF, N. J., 1965, *Astrophys. J.*, **142**, 182.
AUMAN, J. R., 1967, *Astrophys. J. (Suppl.)*, **14**, 171.
AUMAN, J. R., and BAHNG, J., 1965, *Astrophys. J.*, **142**, 170.
BAADE, W., 1944 a, *Astrophys. J.*, **100**, 137.
—— 1944 b, *Astrophys. J.*, **100**, 147.
—— 1963, *Evolution of Stars and Galaxies* (Cambridge, Mass.: Harvard University Press).
BAHNG, J., 1964, *Astrophys. J.*, **140**, 1041.
BAKER, N., and KIPPENHAHN, R., 1965, *Astrophys. J.*, **142**, 868.
BASHKIN, S., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 1.
BASHKIN, S., and PEASLEE, D. C., 1961, *Astrophys. J.*, **134**, 981.
BENNICK, H. H., and MOTZ, L., 1965, *Astrophys. J.*, **141**, 195.
BIERMANN, L., 1948, *Z. Astrophys.*, **25**, 135.
BODENHEIMER, P., 1965, *Astrophys. J.*, **142**, 451.
—— 1966 a, *Astrophys. J.*, **144**, 103.
—— 1966 b, *Astrophys. J.*, **144**, 709.
BODENHEIMER, P., FORBES, J. E., GOULD, N. L., and HENYEY, L. G., 1965, *Astrophys. J.*, **141**, 1019.
BÖHM-VITENSE, E., 1958, *Z. Astrophys.*, **46**, 108.
BONDI, C. M., and BONDI, H., 1950, *Mon. Not. R. Astr. Soc.*, **110**, 287.
—— 1951, *Mon. Not. R. Astr. Soc.*, **111**, 397.
BOURY, A., 1960, *Bull. Soc. R. Sci. Liège*, **29**, 306.
BRANS, C., and DICKE, R. H., 1961, *Phys. Rev.*, **124**, 925.
BURBIDGE, E. M., BURBIDGE, G. R., FOWLER, W. A., and HOYLE, F., 1957, *Rev. Mod. Phys.*, **29**, 547.
BURBIDGE, G. R., and BURBIDGE, E. M., 1958, *Handb. Phys.*, **LI**, 134–295, Ed. S. Flügge (Berlin: Springer-Verlag).
BURNETT, D. S., FOWLER, W. A., and HOYLE, F., 1965, *Geochim. Cosmochim. Acta*, **29**, 1209.
CHANDRASEKHAR, S., 1939, *An Introduction to the Study of Stellar Structure* (Chicago: Chicago University Press; New York: Dover Publications (1957)).
—— 1960, *Principles of Stellar Dynamics*, Enlarged edn (New York: Dover Publications).
CHIN, C.-W., CHIU, H.-Y., and STOTHERS, R., 1966, *Ann. Phys., N.Y.*, **39**, 280.
CHIU, H.-Y., 1961, *Ann. Phys., N.Y.*, **16**, 321.
—— 1965, *Neutrino Astrophysics* (New York: Gordon and Breach).
—— 1966, *Ann. Rev. Nucl. Sci.*, **16**, 591.
CHRISTY, R. F., 1964, *Rev. Mod. Phys.*, **36**, 555.
—— 1966 a, *Astrophys. J.*, **144**, 108.
—— 1966 b, *Astrophys. J.*, **145**, 337.
—— 1966 c, *Astrophys. J.*, **145**, 340.
—— 1966 d, *Ann. Rev. Astr. Astrophys.*, **4**, 353.

CHRISTY, R. F., 1968, *Quart. J. R. Astr. Soc.*, **9**, 13.
COLGATE, S. A., and WHITE, R. H., 1966, *Astrophys. J.*, **143**, 626.
COWLING, T. G., 1966, *Quart. J. R. Astr. Soc.*, **7**, 121.
COX, A. N., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 3.
COX, A. N., and BROWNLEE, R. R., 1961, *Sky Telesc.*, **21**, 252.
COX, J. P., and SALPETER, E. E., 1961, *Astrophys. J.*, **133**, 764.
—— 1964, *Astrophys. J.*, **140**, 485.
DEINZER, W., and SALPETER, E. E., 1965, *Astrophys. J.*, **142**, 813.
DEMARQUE, P. R., 1960 a, *Astr. J.*, **65**, 396.
—— 1960 b, *Astrophys. J.*, **132**, 366.
—— 1961, *Astrophys. J.*, **134**, 9.
DEMARQUE, P. R., and LARSEN, R. B., 1964 a, *Astrophys. J.*, **140**, 544.
—— 1964 b, *Astr. J.*, **69**, 136.
DEMARQUE, P. R., and PERCY, J. R., 1964, *Astrophys. J.*, **140**, 541.
DICKE, R. H., 1967, *Astrophys. J.*, **149**, L121.
EDDINGTON, A. S., 1917, *Observatory*, **40**, 290.
—— 1925, *Observatory*, **48**, 73.
—— 1926, *The Internal Constitution of the Stars* (London: Cambridge University Press; New York: Dover Publications (1959)).
—— 1929, *Mon. Not. R. Astr. Soc.*, **90**, 54.
—— 1941, *Mon. Not. R. Astr. Soc.*, **101**, 182.
EGGEN, O. J., 1965, *Ann. Rev. Astr. Astrophys.*, **3**, 235.
EGGLETON, P. P., 1966 a, *Mon. Not. R. Astr. Soc.*, **132**, 479.
—— 1966 b, *Doctoral Diss.*, University of Cambridge.
EZER, D., 1961, *Astrophys. J.*, **133**, 159.
EZER, D., and CAMERON, A. G. W., 1963, *Icarus*, **1**, 422.
—— 1965, *Can. J. Phys.*, **43**, 1497.
—— 1966, *Can. J. Phys.*, **44**, 593.
—— 1967 a, *Can. J. Phys.*, **45**, 3429.
—— 1967 b, *Can. J. Phys.*, **45**, 3461.
FAULKNER, J., 1966, *Astrophys. J.*, **144**, 978.
FAULKNER, J., GRIFFITHS, K., and HOYLE, F., 1963, *Mon. Not. R. Astr. Soc.*, **126**, 1.
—— 1965, *Mon. Not. R. Astr. Soc.*, **129**, 363.
FAULKNER, J., and IBEN, L., 1966, *Astrophys. J.*, **144**, 995.
FAULKNER, J., ROXBURGH, I. W., and STRITTMATTER, P. A., 1968, *Astrophys. J.*, **151**, 203
FEYNMAN, R. P., and GELL-MANN, M., 1958, *Phys. Rev.*, **109**, 193.
FOWLER, W. A., CAUGHLAN, G. R., and ZIMMERMAN, B. A., 1967, *Ann. Rev. Astr. Astrophys.*, **5**, 525.
FOWLER, W. A., GREENSTEIN, J. L., and HOYLE, F., 1962, *Geophys. J. R. Astr. Soc.*, **6**, 148.
FOWLER, W. A., and HOYLE, F., 1964, *Astrophys. J.* (*Suppl.*), **9**, 201.
FRANK-KAMENETSKII, D. A., 1962, *Physical Processes in Stellar Interiors*, English edn (Jerusalem: Israel Program for Scientific Translations).
FRICKE, K., 1967, *Doctoral Diss.*, University of Göttingen.
GAMOW, G., and KELLER, G., 1945, *Rev. Mod. Phys.*, **17**, 125.
GAUSTAD, J. E., 1963, *Astrophys. J.*, **138**, 1058.
GIANNONE, P., and WEIGERT, A., 1967, *Z. Astrophys.*, **67**, 41.
GOLDREICH, P., and SCHUBERT, G., 1967, *Science*, **156**, 1101.
GOUGH, D. O., 1966, *Doctoral Diss.*, University of Cambridge.
GRATTON, L. (Ed.), 1963, *Star Evolution* (New York: Academic Press).
GREENSTEIN, J. L., 1966, *Astrophys. J.*, **144**, 496.
GREENSTEIN, J. L., and MÜNCH, G., 1966, *Astrophys. J.*, **146**, 618.
HAMADA, T., and SALPETER, E. E., 1961, *Astrophys. J.*, **134**, 683.
HANBURY BROWN, R., DAVIS, J., and ALLEN, L. R., 1967 a, *Mon. Not. R. Astr. Soc.*, **137**, 375.
HANBURY BROWN, R., DAVIS, J., ALLEN, L. R., and ROME, J. M., 1967 b, *Mon. Not. R. Astr. Soc.*, **137**, 393.

Härm, R., and Schwarzschild, M., 1964, *Astrophys. J.*, **139**, 594.
—— 1966, *Astrophys. J.*, **145**, 496.
Harman, R. J., and Seaton, M. J., 1964, *Astrophys. J.*, **140**, 824.
Haro, G., 1952, *Astrophys. J.*, **115**, 572.
—— 1953, *Astrophys. J.*, **117**, 73.
Harrison, B. K., Thorne, K. S., Wakano, M., and Wheeler, J. A., 1965, *Gravitational Theory and Gravitational Collapse* (Chicago: Chicago University Press).
Harrison, M. H., 1944, *Astrophys. J.*, **100**, 343.
—— 1946, *Astrophys. J.*, **103**, 192.
—— 1947, *Astrophys. J.*, **105**, 322.
Hawking, S. W., 1966, *Astrophys. J.*, **145**, 544.
Hayashi, C., 1961, *Publs Astr. Soc. Japan*, **13**, 450.
—— 1966, *Ann. Rev. Astr. Astrophys.*, **4**, 171.
Hayashi, C., and Cameron, R. C., 1962, *Astrophys. J.*, **136**, 166.
—— 1964, *Astr. J.*, **69**, 140.
Hayashi, C., Hoshi, R., and Sugimoto, D., 1962, *Prog. Theor. Phys., Japan* (*Suppl.*), **22**, 1.
—— 1965, *Prog. Theor. Phys., Japan*, **34**, 885.
Hayashi, C., Jugaku, J., and Nishida, M., 1959, *Prog. Theor. Phys., Japan*, **22**, 531.
Hayashi, C., and Nakano, T., 1963, *Prog. Theor. Phys., Japan*, **30**, 460.
—— 1965 a, *Prog. Theor. Phys., Japan*, **33**, 554.
—— 1965 b, *Prog. Theor. Phys., Japan*, **34**, 754.
Hayashi, C., Nishida, M., and Sugimoto, D., 1961, *Prog. Theor. Phys., Japan*, **25**, 1053.
—— 1962, *Prog. Theor. Phys., Japan*, **27**, 1233.
Henyey, L. G., Le Levier, R., and Levee, R. D., 1955, *Publs Astr. Soc. Pacif.*, **67**, 154.
—— 1959, *Astrophys. J.*, **129**, 2.
Herbig, G. H., 1951, *Astrophys. J.*, **113**, 697.
—— 1962 a, *Astrophys. J.*, **135**, 736.
—— 1962 b, *Adv. Astr. Astrophys.*, **1**, 147.
—— 1964, *Contributions from the Lick Observatory*, No. 178.
Hofmeister, E., 1967 a, *Z. Astrophys.*, **65**, 164.
—— 1967 b, *Z. Astrophys.*, **65**, 194.
Hofmeister, E., Kippenhahn, R., and Weigert, A., 1964 a, *Z. Astrophys.*, **59**, 242.
—— 1964 b, *Z. Astrophys.*, **60**, 57.
Howard, L. N., Moore, D. W., and Spiegel, E. A., 1967, *Nature, Lond.*, **214**, 1297.
Hoyle, F., 1960, *Quart. J. R. Astr. Soc.*, **1**, 28.
Hoyle, F., and Lyttleton, R. A., 1942, *Mon. Not. R. Astr. Soc.*, **102**, 218.
—— 1949, *Mon. Not. R. Astr. Soc.*, **109**, 614.
Hoyle, F., and Schwarzschild, M., 1955, *Astrophys. J.* (*Suppl.*), **2**, 1.
Huang, S. S., *Annls Astrophys.*, **29**, 331.
Iben, I., 1965 a, *Astrophys. J.*, **141**, 993.
—— 1965 b, *Astrophys. J.*, **142**, 1447.
—— 1966 a, *Astrophys. J.*, **143**, 483.
—— 1966 b, *Astrophys. J.*, **143**, 505.
—— 1966 c, *Astrophys. J.*, **143**, 516.
—— 1967 a, *Ann. Rev. Astr. Astrophys.*, **5**, 571.
—— 1967 b, *Astrophys. J.*, **147**, 624.
—— 1967 c, *Astrophys. J.*, **147**, 650.
Iben, I., and Ehrman, J. R., 1962, *Astrophys. J.*, **135**, 770.
Iben, I., and Talbot, R. J., 1966, *Astrophys. J.*, **144**, 968.
Kamijo, F., 1967, *Z. Astrophys.*, **67**, 227.
Kippenhahn, R., Kohl, K., and Weigert, A., 1967, *Z. Astrophys.*, **66**, 58.
Kippenhahn, R., Thomas, H.-C., and Weigert, A., 1965, *Z. Astrophys.*, **61**, 241.
—— 1966, *Z. Astrophys.*, **64**, 373.
Kippenhahn, R., and Weigert, A., 1967, *Z. Astrophys.*, **65**, 251.
Kotok, E. V., 1966, *Soviet Astr.*, **9**, 948.
Kraft, R. P., 1963, *Adv. Astr. Astrophys.*, **2**, 43.

KUHI, L. V., 1964, *Astrophys. J.*, **140**, 1409.
—— 1966, *Astrophys. J.*, **143**, 991.
KUMAR, S. S., 1963 a, *Astrophys. J.*, **137**, 1121.
—— 1963 b, *Astrophys. J.*, **137**, 1126.
KUSHWAHA, R. S., 1957, *Astrophys. J.*, **125**, 242.
LANGER, E., HERTZ, M., and COX, J. P., 1966, *Joint Institute for Laboratory of Astrophysics, University of Colorado, Rep. No.* 88.
LAYZER, D., 1964, *Ann. Rev. Astr. Astrophys.*, **2**, 341.
L'ECUYER, J., 1966, *Astrophys. J.*, **146**, 845.
LEDOUX, P., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 10.
LI HEN, and SCHWARZSCHILD, M., 1949, *Mon. Not. R. Astr. Soc.*, **109**, 631.
LIFSHITZ, E. M., and KHALATNIKOV, I. M., 1963, *Adv. Phys.*, **12**, 185.
MASEVICH, A. G., KOTOK, E. V., DLURZHNEVSKAYA, O. B., and MAZANI, A.,1965, *Sov. Astr.*, **9**, 263.
MCCREA, W. H., 1964, *Mon. Not. R. Astr. Soc.*, **128**, 147.
MENZEL, D. H., BHATNAGAR, P. L., and SEN, H. K., 1963, *Stellar Interiors* (London: Chapman and Hall).
MESTEL, L., 1952, *Mon. Not. R. Astr. Soc.*, **112**, 598.
—— 1953, *Mon. Not. R. Astr. Soc.*, **113**, 716.
—— 1965 a, *Quart. J. R. Astr. Soc.*, **6**, 161.
—— 1965 b, *Quart. J. R. Astr. Soc.*, **6**, 265.
—— 1965 c, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 5.
—— 1966, *Z. Astrophys.*, **63**, 196.
—— 1968, *Mon. Not. R. Astr. Soc.*, **138**, 359.
MUMFORD, G. S., 1967, *Publs Astr. Soc. Pacif.*, **79**, 283.
NAUR, P., and OSTERBROCK, D. E., 1953, *Astrophys. J.*, **117**, 306.
NISHIDA, M., 1960, *Prog. Theor. Phys., Japan*, **23**, 896.
NISHIDA, M., and SUGIMOTO, D., 1962, *Prog. Theor. Phys., Japan*, **27**, 145.
NOVIKOV, I. D., and ZELDOVIC, YA. B., 1967, *Ann. Rev. Astr. Astrophys.*, **5**, 627.
O'CONNELL, D. J. K. (Ed.), 1958, *Stellar Populations* (Amsterdam: North-Holland).
O'DELL, C. R., 1963, *Astrophys. J.*, **138**, 67.
OHYAMA, N., 1963, *Prog. Theor. Phys., Japan*, **30**, 170.
OKE, J. B., and SCHWARZSCHILD, M., 1952, *Astrophys. J.*, **116**, 3.
ONO, Y., SAKASHITA, S., and OHYAMA, N., 1961, *Prog. Theor. Phys., Japan* (*Suppl.*), **20**, 85.
ONO, Y., SAKASHITA, S., and YAWAZAKI, H., 1960 a, *Prog. Theor. Phys., Japan*, **23**, 155.
—— 1960 b, *Prog. Theor. Phys., Japan*, **23**, 294.
ÖPIK, E., 1938, *Publs Obs. Univ. Tartu*, **30**, 37.
—— 1951, *Mon. Not. R. Astr. Soc.*, **111**, 78.
OSAKI, Y., 1963, *Publs Astr. Soc. Japan*, **15**, 428.
PACZYNSKI, B., 1966, *Acta Astron.*, **16**, 231.
—— 1967 a, *Acta Astron.*, **17**, 1.
—— 1967 b, *Acta Astron.*, **17**, 193.
PACZYNSKI, B., and ZIOLKOWSKI, J., 1967, *Acta Astron.*, **17**, 7.
PEARCE, W. P., and BAHNG, J., 1965, *Astrophys. J.*, **142**, 164.
PENSTON, M. V., 1964, *Observatory*, **84**, 141.
PLAVEC, M., 1967, *Bull. Astr. Insts Csl.*, **18**, 253.
POCHODA, P., and SCHWARZSCHILD, M., 1964, *Astrophys. J.*, **139**, 587.
POLAK, E. J., 1962, *Astrophys. J.*, **136**, 465.
REEVES, H., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 2.
ROEDER, R. C., 1967, *Astrophys. J.*, **149**, 131.
ROEDER, R. C., and DEMARQUE, P. R., 1966, *Astrophys. J.*, **144**, 1016.
ROSE, W. K., 1966 a, *Astrophys. J.*, **144**, 1001.
—— 1966 b, *Astrophys. J.*, **146**, 838.

ROSSELAND, S., 1949, *The Pulsation Theory of Variable Stars* (London: Oxford University Press).
ROXBURGH, I. W., 1964 a, *Mon. Not. R. Astr. Soc.*, **128**, 157.
—— 1964 b, *Mon. Not. R. Astr. Soc.*, **128**, 237.
—— 1966 a, *Mon. Not. R. Astr. Soc.*, **132**, 201.
—— 1966 b, *Astrophys. J.*, **143**, 111.
ROXBURGH, I. W., and STRITTMATTER, P. A., 1965, *Z. Astrophys.*, **63**, 15.
—— 1966 a, *Mon. Not. R. Astr. Soc.*, **133**, 1.
—— 1966 b, *Mon. Not. R. Astr. Soc.*, **133**, 345.
RUBIN, R., 1966, *Mon. Not. R. Astr. Soc.*, **133**, 339.
RUDERMAN, M. A., 1965, *Rep. Prog. Phys.*, **28**, 411 (London: Institute of Physics and Physical Society).
SAKASHITA, S., and HAYASHI, C., 1959, *Prog. Theor. Phys., Japan*, **22**, 830.
—— 1961, *Prog. Theor. Phys., Japan*, **26**, 942.
SAKASHITA, S., ONO, Y., and HAYASHI, C., 1959, *Prog. Theor. Phys., Japan*, **21**, 315.
SANDAGE, A. R., 1962, *Astrophys. J.*, **135**, 349.
SANDAGE, A. R., and SCHWARZSCHILD, M., 1952, *Astrophys. J.*, **116**, 463.
SARGENT, W. L. W., and SEARLE, L., 1966, *Astrophys. J.*, **145**, 652.
SASLAW, W. C., and SCHWARZSCHILD, M., 1965, *Astrophys. J.*, **142**, 1468.
SCHATZMAN, E., 1962, *Annls Astrophys.*, **25**, 18.
—— 1966, *Stellar Evolution*, Eds R. F. Stein and A. G. W. Cameron (New York: Plenum Press), p. 215.
SCHÖNBERG, M., and CHANDRASEKHAR, S., 1942, *Astrophys. J.*, **96**, 161.
SCHWARZSCHILD, M., 1958, *The Structure and Evolution of Stars* (Princeton, N.J.: Princeton University Press).
SCHWARZSCHILD, M., and HÄRM, R., 1958, *Astrophys. J.*, **128**, 348.
—— 1962, *Astrophys. J.*, **136**, 158.
—— 1965, *Astrophys. J.*, **142**, 855.
—— 1967, *Astrophys. J.*, **150**, 961.
SCHWARZSCHILD, M., and SELBERG, H., 1962, *Astrophys. J.*, **136**, 150.
SEARLE, L., and RODGERS, A. W., 1966, *Astrophys. J.*, **143**, 809.
SEARS, R. L., and BROWNLEE, R. R., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 11.
VON SENGBUSCH, K., and TEMESVARY, S., 1966, *Stellar Evolution*, Eds R. F. Stein and A. G. W. Cameron (New York: Plenum Press), p. 209.
SPITZER, L., 1940, *Mon. Not. R. Astr. Soc.*, **100**, 396.
STEIN, R. F., and CAMERON, A. G. W. (Eds), 1966, *Stellar Evolution* (New York: Plenum Press).
STOTHERS, R., 1963, *Astrophys. J.*, **138**, 1074.
—— 1964, *Astrophys. J.*, **140**, 510.
—— 1966 a, *Astrophys. J.*, **143**, 91.
—— 1966 b, *Astrophys. J.*, **144**, 959.
STRÖMGREN, B., 1965, *Stars and Stellar Systems*, Vol. VIII, Eds L. H. Aller and D. B. McLaughlin, chap. 4 (Chicago: Chicago University Press).
STRUVE, O., 1950, *Stellar Evolution* (Princeton, N.J.: Princeton University Press).
SUDA, K., and VIRGOPIA, N., 1966, *Astrophys. J.*, **143**, 75.
SUDARSHAN, E. G., and MARSHAK, R. E., 1958, *Phys. Rev.*, **109**, 1860.
SUGIMOTO, D., 1964, *Prog. Theor. Phys., Japan*, **32**, 703.
SUGIMOTO, D., and YAMAMOTO, Y., 1966, *Prog. Theor. Phys., Japan*, **36**, 17.
SWEET, P. A., 1950, *Mon. Not. R. Astr. Soc.*, **110**, 548.
TAKARADA, K., SATO, H., and HAYASHI, C., 1966, *Prog. Theor. Phys., Japan*, **36**, 504.
TANAKA, Y., 1966, *Publs Astr. Soc. Japan*, **18**, 47.
TAYLER, R. J., 1954 a, *Astrophys. J.*, **120**, 332.
—— 1954 b, *Doctoral Diss.*, University of Cambridge.
—— 1956, *Mon. Not. R. Astr. Soc.*, **116**, 25.
—— 1966, *Rep. Prog. Phys.*, **29**, 489 (London: Institute of Physics and Physical Society).
—— 1967 a, *Mon. Not. R. Astr. Soc.*, **135**, 225.

Tayler, R. J., 1967 b, *Quart. J. R. Astr. Soc.*, **8**, 13.
Thomas, L. H., 1930, *Mon. Not. R. Astr. Soc.*, **91**, 122.
Varsavsky, C. M., Gratton, F., and Pöppel, W. G. L., 1962, *Annls Astrophys.*, **25**, 261.
Vila, S. C., 1966, *Astrophys. J.*, **146**, 437.
—— 1967, *Astrophys. J.*, **149**, 613.
Virgopia, N., and Suda, K., 1966, *Astrophys. J.*, **143**, 87.
Vogt, H., 1925, *Astr. Nachr.*, **223**, 229.
Wagoner, R. V., Fowler, W. A., and Hoyle, F., 1967, *Astrophys. J.*, **148**, 3.
Walker, M. F., 1956, *Astrophys. J.* (*Suppl.*), **2**, 365.
—— 1957, *Astrophys. J.*, **125**, 636.
Weigert, A., 1966, *Z. Astrophys.*, **64**, 395.
Weymann, R., and Moore, E., 1963, *Astrophys. J.*, **137**, 552.
Weymann, R., and Sears, R. L., 1965, *Astrophys. J.*, **142**, 174.
Wildey, R. L., 1963, *Astrophys. J.* (*Suppl.*), **8**, 439.
Woolf, N. J., 1962, *Astrophys. J.*, **135**, 644.
von Zeipel, H., 1924, *Mon. Not. R. Astr. Soc.*, **84**, 665.
Zhevakhin, S. A., 1953, *Astr. Zh.*, **30**, 161.
—— 1963, *Ann. Rev. Astr. Astrophys.*, **1**, 367.
Zwicky, F., 1965, *Stars and Stellar Systems,* Vol. VIII, Eds L. H. Aller and D. B. McLaughlin (Chicago: Chicago University Press), chap. 7.

Note added in proof. For some time now an experiment has been under way to measure the neutrino flux from the Sun from the β decay of $^{8}_{5}B$ which occurs in a subsidiary reaction to (3.13). This should give an accurate estimate of the central temperature of the Sun. We are informed by A. G. W. Cameron and W. A. Fowler (private communications) that the flux detected by R. Davis is ten times lower than expected. This discrepancy is greater than the estimated errors of theory and observation and it seems that some factor may have been omitted in present solar models (and in other stellar models).