A Method to Construct Asymptotic Solutions Invariant under the Renormalization Group

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A renormalization group method with the Lie symmetry is presented for the singular perturbation problems. Asymptotic solutions are obtained as group-invariant solutions under approximate Lie groups admitted by perturbed diferential equations.

§1. Introduction

There have been many studies concerning application of the renormalization group theory of particle physics as a singular perturbation method to treat differential equations since the work of the Illinois group.[1)] In this method, integral constants appearing in the lowest-order perturbed solution are renormalized in order to remove secular or divergent terms appearing in the naive perturbation solution and give a well-behaved asymptotic solution. We call this renormalization group method in singular perturbation theory the "conventional RG method" in this paper. A geometrical aspect of the conventional RG method has been studied by means of the envelopes of solutions.[3)] The conventional RG method and its variants depend more or less on the naive perturbation analysis. In connection to this, it is interesting that the geometrical symmetry of a differential equation with respect to continuous transformations, i.e. the Lie symmetry, is known to be useful for deriving new solutions from known solutions.[4)] There have been some pioneering works aimed of constructing the renormalization group method in terms of the Lie symmetry.[5)6)] However, to this time, there is no case in which have been derived asymptotic solutions in singular perturbation problems by means of the Lie symmetry. The purpose of this paper is to derive such asymptotic solutions in the framework of the Lie group and symmetry.

§2. Method of the renormalization group with symmetry

For simplicity, let us consider the second-order ordinary differential equation

$$F_0(u,u',u'') + \varepsilon F_1(u,u') = 0, \tag{2·1}$$

where we have $u' = \frac{du}{dt}$ and $u'' = \frac{d^2u}{dt^2}$, F_0 and F_1 are some functions of their arguments, and ε is a small parameter. This system is assumed to be nearly solvable in the sense that the general solution is known for the unperturbed case, $\varepsilon = 0$. Our purpose is to construct an approximate solution of the system (2·1) for small ε by

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incorporating an approximate Lie symmetry. First, the equation (2·1) is rewritten as a system of first-order ordinary differential equations in the form

$$\begin{cases} u' = v, \\ F_0(u, v, v') + \varepsilon F_1(u, v) = 0, \end{cases} \tag{2·2}$$

or as a first-order ordinary differential equation with respect to a complex valuable $z := u + iv$ as

$$G_0(z, \bar{z}, z', \bar{z}') + \varepsilon G_1(z, \bar{z}) = 0. \tag{2·3}$$

Let (2·2) admits a Lie group transformation whose infinitesimal generator takes the form

$$X = \partial_\varepsilon + \zeta(t, u, v)\partial_t + \eta^u(t, u, v)\partial_u + \eta^v(t, u, v)\partial_v, \tag{2·4}$$

Then, under the transformation $(u, v) \mapsto (z, \bar{z})$, (2·4) can be rewritten as

$$X = \partial_\varepsilon + \xi(t, z, \bar{z})\partial_t + \eta^z(t, z, \bar{z})\partial_z + \eta^{\bar{z}}(t, z, \bar{z})\partial_{\bar{z}}, \tag{2·5}$$

where

$$\xi = \bar{\xi} = \zeta, \quad \eta^z = \eta^u + i\eta^v, \quad \eta^{\bar{z}} = \eta^u - i\eta^v = \overline{\eta^z}. \tag{2·6}$$

Next, let

$$\begin{aligned} X^{(1)}(t, z, \bar{z}, z', \bar{z}') = \partial_\varepsilon &+ \xi(t, z, \bar{z})\partial_t + \eta^z(t, z, \bar{z})\partial_z + \eta^{\bar{z}}(t, z, \bar{z})\partial_{\bar{z}} \\ &+ \eta^{z(1)}(t, z, \bar{z}, z', \bar{z}')\partial_{z'} + \eta^{\bar{z}(1)}(t, z, \bar{z}, z', \bar{z}')\partial_{\bar{z}'} \end{aligned} \tag{2·7}$$

be the prolongation of X. Then, the determining equation for (2·3), which determines each component of the vector field (2·5), is given by

$$X^{(1)}\{G_0(z, \bar{z}, z', \bar{z}') + \varepsilon G_1(z, \bar{z})\}\Big|_{G_0 + \varepsilon G_1 = 0} = 0. \tag{2·8}$$

Because we wish to find the approximate symmetries to leading order, we need only solve the following leading-order determining equation:

$$X^{(1)}\{G_0(z, \bar{z}, z', \bar{z}') + \varepsilon G_1(z, \bar{z})\}\Big|_{G_0 = 0} = O(\varepsilon). \tag{2·9}$$

By solving the approximate determining equation, (2·9), we obtain the infinitesimal generator X admitted by the system (2·3) in the leading-order approximation. As shown below in some examples, the approximate solutions of the determining equation (2·9) give various non-trivial approximate symmetries even if the system (2·3) admits only a few exact, trivial symmetries. It is by virtue of this fact that the present method is useful in obtaining approximate solutions. Using the approximate infinitesimal generator X, we construct a group-invariant solution of the system (2·3).

The group-invaritant solution, $z = z(\varepsilon, t)$, which satisfies

$$X\{z - z(\varepsilon, t)\}\Big|_{z - z(\varepsilon, t) = 0} = 0, \tag{2·10}$$

is obtained by solving the so-called Lie equation

$$\partial_\varepsilon z = -\xi(t, z, \bar{z})\partial_t z + \eta^z(t, z, \bar{z}). \tag{2·11}$$

Because our symmetry (2·5) always has a non-vanishing component in the ε direction, the Lie equation becomes a system of differential equations with respect to the perturbation parameter ε as described above. Noting this fact, we refer to the Lie equation (2·11) as the renormalization group equation. Solving the renormalization group equation by adopting solutions of the unperturbed system as boundary condition, i.e.

$$z(\varepsilon = 0, t) = z^{(0)}(t) \quad s.t. \quad G_0(t, z^{(0)}, \overline{z^{(0)}}, z^{(0)\prime}, \overline{z^{(0)}}') = 0, \tag{2·12}$$

we obtain an approximate solution of the system (2·3) for non-vanishing ε,

$$z = z(\varepsilon, t). \tag{2·13}$$

As the solution thus obtained is invariant with respect to the Lie group, we call it the invariant renormalized solution.

Here, some detailed discussion is necessary to regorously construct our remormalization group method with the Lie symmetry. In the above-described general procedure, we obtain the approximate symmetry only to leading order in ε, that is, ε^0. Nevertheless, the solution invariant with respect to the leading-order Lie symmetry yields an approximate solution up to $O(\varepsilon)$ of the system (2·3). Is this actually a solution of the given differential equations to order ε^1? The following proposition answers this question.

Proposition. Consider a system of ordinary differential equations of the form

$$G_0(z, \bar{z}, z', \bar{z}') + \varepsilon G_1(z, \bar{z}) = 0. \tag{2·14}$$

Assume that (2·14) approximately admits a Lie group transformation with an infinitesimal generator X, i.e.

$$X^{(1)}\{G_0(z, \bar{z}, z', \bar{z}') + \varepsilon G_1(z, \bar{z})\}\Big|_{G_0=0} = O(\varepsilon), \tag{2·15}$$

where $X^{(1)}$ is the first prolonged infinitesimal generator. Then, its invariant solution is a solution of (2·14) to order ε^1

Proof. Suppose that (2·14) admits

$$X = \partial_\varepsilon + \xi(t, z, \bar{z})\partial_t + \eta^z(t, z, \bar{z})\partial_z + \eta^{\bar{z}}(t, z, \bar{z})\partial_{\bar{z}}. \tag{2·16}$$

Then its invariant solution is a solution of

$$X\{z - z(\varepsilon, t)\}\Big|_{z=z(\varepsilon,t)} = 0 \tag{2·17}$$

$$\Longleftrightarrow \partial_\varepsilon z(\varepsilon, t) = -\xi(t, z, \bar{z})\partial_t z(\varepsilon, t) + \eta^z(t, z, \bar{z}), v(\varepsilon, t) = -\xi(t, u, v)\partial_t \tag{2·18}$$

Thus, the invariant solution can be expressed as

$$z(\varepsilon,t) = z^{(0)}(t) + \int_0^{\varepsilon} \{-\xi(t,z,\bar{z})\partial_t z(\varepsilon,t) + \eta^z(t,z,\bar{z})\}\, d\varepsilon \tag{2·19}$$

$$= z^{(0)} - \int_0^{\varepsilon} \left(\xi(t,z^{(0)},\bar{z}^{(0)})z^{(0)\prime} - \eta^z(t,z^{(0)},\bar{z}^{(0)})\right) d\varepsilon + O(\varepsilon^2) \tag{2·20}$$

$$= z^{(0)} - \varepsilon \left(\xi(t,z^{(0)},\bar{z}^{(0)})z^{(0)\prime} - \eta^z(t,z^{(0)},\bar{z}^{(0)})\right) + O(\varepsilon^2), \tag{2·21}$$

where $z^{(0)}(t)$ and $\bar{z}^{(0)}(t)$ are the solutions of the unperturbed system. By substituting (2·21) into (2·14) and taking account of (2·15), we can show that the solutions satisfy the given differential equations to order ε^1 □

It is noted that this proposition is easily proved for a solution of arbitarary order in ε by expanding (2·19) to that order. Owing to this proposition, it is not necessary to know a naive perturbed solution in order to find the proper Lie symmetry. This contrasts with the situation in the RG method of Shirkov and Kovalev,[5)6)] in which the naive perturbed solution is needed.

§3. Examples

Let us consider two examples.

(1) *a Harmonic oscillator equation*

Here, we apply the renormalization group method with the Lie symmetry to the equation for a harmonic oscillator as an example of singular perturbaiton problems. We consider the equation

$$u'' + u = -\varepsilon u. \tag{3·1}$$

Then, defining $z := u + iu'$, we have

$$z' = -iz - \varepsilon\frac{1}{2}i\,(z+\bar{z}) \tag{3·2}$$

Next, let

$$X(t,z,\bar{z}) = \partial_\varepsilon + \xi(t,z,\bar{z})\partial_t + \eta^z(t,z,\bar{z})\partial_z + \eta^{\bar{z}}(t,z,\bar{z})\partial_{\bar{z}} \tag{3·3}$$

be the infinitesimal generator of an approximate Lie symmetry admitted by (3·2). Then, we write its first prolonged infinitesimal generator as

$$X^{(1)}(t,z,\bar{z},z',\bar{z}') = \partial_\varepsilon + \xi(t,z,\bar{z})\partial_t + \eta^z(t,z,\bar{z})\partial_z + \eta^{\bar{z}}(t,z,\bar{z})\partial_{\bar{z}} + \eta^{z(1)}(t,z,\bar{z},z',\bar{z}')\partial_{z'} + \eta^{\bar{z}(1)}(t,z,\bar{z},z',\bar{z}')\partial_{\bar{z}'}, \tag{3·4}$$

where

$$\eta^{z(1)}(t,z,\bar{z},z',\bar{z}') = \eta^z_t + z'(\eta^z_z - \xi_t) + \bar{z}'\eta^z_{\bar{z}} - z'^2\xi_z - z'\bar{z}'\xi_{\bar{z}}, \tag{3·5}$$

$$\eta^{\bar{z}(1)}(t,z,\bar{z},z',\bar{z}') = \eta^{\bar{z}}_t + \bar{z}'(\eta^{\bar{z}}_{\bar{z}} - \xi_t) + z'\eta^{\bar{z}}_z - \bar{z}'^2\xi_{\bar{z}} - z'\bar{z}'\xi_z. \tag{3·6}$$

Here, the subscript α ($\alpha = t, u, v$) of $\xi$and η represents the operation of ∂_α. Then, the approximate determining equation for (3·2) is

$$X^{(1)}\left\{z' + iz + \varepsilon\frac{1}{2}i\,(z+\bar{z})\right\}\Big|_{z'+iz=0} = O(\varepsilon). \tag{3·7}$$

This equation leads to

$$\eta^{z(1)}\Big|_{z'+iz=0} + i\eta^{z} + \frac{1}{2}i\,(z+\bar{z}) = 0. \tag{3·8}$$

The system (3·8) is linear in the unknown η^z and ξ with an inhomogeneous term due to the perturbation. Because we are interested in the effect of the perturbation, we seek a particular solution of the linear inhomogeneous system (3·8). Expanding ξ and η^u in powers of t, z and $\bar{z}$ as

$$\xi(t,z,\bar{z}) = \sum_{j,k,l\geq 0} \xi_{jkl} t^j z^k \bar{z}^l, \tag{3·9}$$

$$\eta^z(t,z,\bar{z}) = \sum_{j,k,l\geq 0} \eta_{jkl} t^j z^k \bar{z}^l. \tag{3·10}$$

and substituting these expressions into (3·8), we obtain a particular solution of lowest order,

$$\xi(t,z,\bar{z}) = 0, \tag{3·11}$$

$$\eta^z(t,z,\bar{z}) = -\frac{1}{2}itz + \frac{1}{4}z - \frac{1}{4}\bar{z}. \tag{3·12}$$

Then, the infinitesimal generator admitted approximately by (3·2) is

$$X(t,u,v) = \partial_\varepsilon + \left(-\frac{1}{2}itz + \frac{1}{4}z - \frac{1}{4}\bar{z}\right)\partial_z + \left(\frac{1}{2}it\bar{z} - \frac{1}{4}z + \frac{1}{4}\bar{z}\right)\partial_{\bar{z}}. \tag{3·13}$$

The invariant solution is derived from the corresponding Lie equation or renormalization group equation, and we have

$$\partial_\varepsilon z = -\frac{1}{2}itz + \frac{1}{4}z - \frac{1}{4}\bar{z}. \tag{3·14}$$

This renormalization group equation is consistent with the naive expanded solution of (3·2),

$$z(\varepsilon,t) = Ae^{-i(t+\theta)} + \varepsilon\left(-\frac{1}{2}iAte^{-i(t+\theta)} + \frac{1}{4}Ae^{-i(t+\theta)} - \frac{1}{4}\bar{A}e^{i(t+\theta)}\right) + O(\varepsilon^2). \tag{3·15}$$

Then, defining $z_0 := Ae^{-i(t+\theta)}$, Eq.(3·15) reads

$$z(\varepsilon,t) = z_0 + \varepsilon\left(-\frac{1}{2}itz_0 + \frac{1}{4}z_0 + \frac{1}{4}\bar{z}_0\right), \tag{3·16}$$

which is the Euler scheme difference equation corresponding to the renormalization equation (3·14). Using the unperturbed solution as the boundary condition, i.e.

$$z(\varepsilon=0,t)=Ae^{-i(t+\theta)}, \tag{3·17}$$

we obtain the invariant renormalized solution

$$u(\varepsilon,t)=2\left(t^2-\frac{1}{4}\right)^{-\frac{1}{2}}e^{\frac{\varepsilon}{4}}R\sin\left(\frac{1}{2}\left(t^2-\frac{1}{4}\right)^{\frac{1}{2}}\varepsilon\right)(2t\cos(t+\theta)-\sin(t+\theta))$$
$$+e^{\frac{\varepsilon}{4}}R\cos\left(\frac{1}{2}\left(t^2-\frac{1}{4}\right)^{\frac{1}{2}}\varepsilon\right)\sin(t+\theta), \tag{3·18}$$

where $R=|A|$. Although the expression (3·18) is somewhat complicated, it is easy to show that this invariant solution asymptotically approaches the renormalized solution derived using the conventional RG method. Since we are interested in the long-time behavior of solution [i.e. $t\sim(1/\varepsilon)$ in this case] we set $(t^2-\frac{1}{4})\approx t^2$ and $e^{\frac{\varepsilon}{4}}\approx 1$ and ignore $\sin(t+\theta)$ in the first term. Then, the solution (3·18) reduces to the conventional renormalized solution,

$$u(\varepsilon,t)=R\sin\left(\left(1+\varepsilon\frac{1}{2}\right)t+\theta\right). \tag{3·19}$$

Adding the general solution of the determining equation (3·8) to the particular solution given in Eqs.(3·11) and (3·12), we can construct a simpler invariant renormalized solution. For example, choose a particular solution of (3·8) as

$$\xi(t,u,v)=-\frac{1}{2}t, \tag{3·20}$$

$$\eta^z(t,u,v)=\frac{1}{4}(z-\bar{z}). \tag{3·21}$$

Then, the Lie equation becomes

$$\partial_\varepsilon z=\frac{1}{2}t\partial_t z+\frac{1}{4}(z-\bar{z}). \tag{3·22}$$

Thus, we have the simpler invariant renormalized solution

$$u(\varepsilon,t)=R\sin(e^{\frac{\varepsilon}{2}}t+\theta), \tag{3·23}$$

which also is identical with the conventional renormalized solution to order ε^1. These two invariant renormalized solutions are depicted in Fig. 1 along with the exact solution, the naive perturbation solution and the conventional renormalized solution. We observe good agreement among these solutions even for $\varepsilon=0.2$, except for the naive perturbation solution.

(2) *Rayleigh equation*

We next consider the Rayleigh equation,

$$u''+u=\varepsilon\left(u'-\frac{1}{3}u'^3\right), \tag{3·24}$$

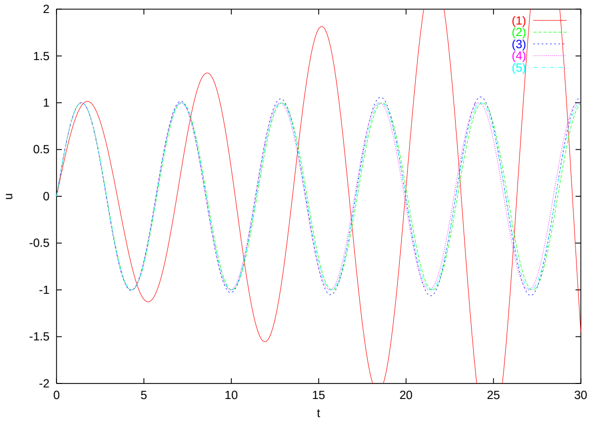

Fig. 1. Comparison of perturbative solutions of Eq.(3·1). (1) represents the naive perturbation solution of order ε, (2) the exact solution, (3) and (4) the invariant renormalizad solutions (3·18) and (3·23), respectively, and (5) conventional renormalized solution, where $\theta = 0, R = 1$,and $\varepsilon = 0.2$

which can be rewitten

$$z' = -iz + \varepsilon\left(\frac{1}{2}z - \frac{1}{2}\bar{z} + \frac{1}{24}z^3 - \frac{1}{8}|z|^2 z + \frac{1}{8}|z|^2\bar{z} - \frac{1}{24}\bar{z}^3\right). \tag{3·25}$$

In this case, the determining equation for the components $(\xi, \eta^z, \eta^{\bar{z}})$ of the infinitesimal generator is

$$\eta^{z(1)}\Big|_{z'+iz=0} + i\eta^z + \frac{1}{2}z - \frac{1}{2}\bar{z} + \frac{1}{24}z^3 - \frac{1}{8}|z|^2 z + \frac{1}{8}|z|^2\bar{z} - \frac{1}{24}\bar{z}^3 = 0. \tag{3·26}$$

Following the same procedure as in the previous example, we obtain the following particular solution of (3·26):

$$\xi(t,u,v) = 0, \tag{3·27}$$

$$\eta^z(t,z,\bar{z}) = \frac{1}{4}i\bar{z} + \frac{1}{2}tz + \frac{1}{48}iz^3 - \frac{1}{96}i\bar{z}^3 - \frac{1}{16}i|z|^2\bar{z} - \frac{1}{8}t|z|^2 z. \tag{3·28}$$

From this, we find the infinitesimal generator approximately admitted by Eq.(3·24) to be

$$\begin{aligned} X(t,u,v) = \partial_\varepsilon &+ \left(\frac{1}{4}i\bar{z} + \frac{1}{2}tz + \frac{1}{48}iz^3 - \frac{1}{96}i\bar{z}^3 - \frac{1}{16}i|z|^2\bar{z} - \frac{1}{8}t|z|^2 z\right)\partial_z \\ &+ \left(-\frac{1}{4}iz + \frac{1}{2}t\bar{z} + \frac{1}{96}iz^3 - \frac{1}{48}i\bar{z}^3 + \frac{1}{16}i|z|^2 z - \frac{1}{8}t|z|^2\bar{z}\right)\partial_{\bar{z}}. \end{aligned} \tag{3·29}$$

The corresponding Lie equation or renormailization group equation reads

$$\partial_\varepsilon z = \frac{1}{4}i\bar{z} + \frac{1}{2}tz + \frac{1}{48}iz^3 - \frac{1}{96}i\bar{z}^3 - \frac{1}{16}i|z|^2\bar{z} - \frac{1}{8}t|z|^2 z. \tag{3·30}$$

This renormalization group equation is also consistent with the naive expanded solution of (3·24),[2)]

$$z = Ae^{i(t+\theta)} + \varepsilon\left\{\frac{A}{2}\left(1-\frac{A^2}{4}\right)ite^{-i(t+\theta)} + \frac{A}{4}\left(1-\frac{A^2}{4}\right)ie^{i(t+\theta)} + \frac{A^3}{48}ie^{-3i(t+\theta)} + \frac{A^3}{96}ie^{3i(t+\theta)}\right\}. \tag{3·31}$$

Defining $z_0 := Ae^{-i(t+\theta)}$, Eq.(3·31) reads

$$z = z_0 + \varepsilon\left\{\frac{1}{4}i\bar{z}_0 + \frac{1}{2}tz_0 + \frac{1}{48}iz_0^3 + \frac{1}{96}i\bar{z}_0^3 - \frac{1}{16}i|z_0|^2\bar{z}_0 - \frac{1}{8}t|z_0|^2 z_0\right\}, \tag{3·32}$$

which is the Euler scheme difference equation corresponding to the renormalization equation (3·30). The renormalization group equation (3·30) is approximated for large t by

$$\partial_\varepsilon z = \frac{1}{2}tz\left(1-\frac{1}{4}|z|^2\right). \tag{3·33}$$

Under the transformation of coordinates $(z,\bar{z}) \mapsto (A,\alpha)$, with

$$A := |z|, \quad \alpha := \frac{i}{2}\mathrm{Log}\left(\frac{\bar{z}}{z}\right), \tag{3·34}$$

Eq.(3·33) becomes

$$\left\{\begin{array}{lcl} \partial_\varepsilon A & = & \frac{A}{2}\left(1-\frac{A^2}{4}\right)t, \\ \partial_\varepsilon \alpha & = & 0. \end{array}\right. \tag{3·35}$$

or

$$\left\{\begin{array}{lcl} \partial_\tau A & = & \frac{A}{2}\left(1-\frac{A^2}{4}\right), \\ \partial_\tau \alpha & = & 0, \end{array}\right. \tag{3·36}$$

where $\tau := \varepsilon t$. This renormalization group equation is equivalent to that obtained with the conventional renormalization group method.[1)]

Because the determining equation of the Lie symmetry is linear in each component of the infinitesimal generator, it is easy to calculate higher-order corrections to the leading-order Lie symmetry. The determining equation to order ε^1 is

$$\eta^{z(1)}|_{z'+iz-\varepsilon\left(\frac{1}{2}z-\frac{1}{2}\bar{z}+\frac{1}{24}z^3-\frac{1}{8}|z|^2z+\frac{1}{8}|z|^2\bar{z}-\frac{1}{24}\bar{z}^3\right)} + i\eta^z - \varepsilon\left(\frac{1}{2}z-\frac{1}{2}\bar{z}+\frac{1}{24}z^3-\frac{1}{8}|z|^2z+\frac{1}{8}|z|^2\bar{z}-\frac{1}{24}\bar{z}^3\right) = O(\varepsilon^2) \tag{3·37}$$

The solution of (3·37) gives the following approximate symmetry to order ε^1 for large t:

$$\eta^z = \frac{1}{2}tz\left(1-\frac{1}{4}|z|^2\right) + \varepsilon\frac{1}{4}itz\left(1+\frac{1}{4}|z|^2-\frac{1}{16}|z|^4\right). \tag{3·38}$$

Then, we obtain the renormalization group equation to order ε^2,

$$\frac{dz}{d\varepsilon} = \frac{1}{2}z\left(1 - \frac{1}{4}|z|^2\right)t + \varepsilon\frac{1}{4}iz\left(1 + \frac{1}{4}|z|^2 - \frac{1}{16}|z|^4\right)t, \tag{3·39}$$

or

$$\frac{dz}{d\tau} = \frac{1}{2}z\left(1 - \frac{1}{4}|z|^2\right) + \varepsilon\frac{1}{4}iz\left(1 + \frac{1}{4}|z|^2 - \frac{1}{16}|z|^4\right). \tag{3·40}$$

§4. Concluding remarks

In the method presented here, the renormalization equation appears as the main ingredient in the Lie group theory. More specifically, we find that the Lie equation for the Lie point group in the expanded space that includes the perturbation parameter is the renormalization group equation. In the conventional RG method, it is necessary to calculate naive perturbation solutions of a given nonlinear system, which is often tedious, due to the nonlinearity of the system. To avoid this step, the proto-RG approach was proposed in the perturbation analysis of a perturbed system.[7] The present method does not require any pertubational analyses to determine approximate solutions of a perturbed system but, instead, approximate symmetries. Because the determinig equation is always a linear system for the symmetry, it is easier to calculate an approximate symmetry than to obtain an approximate solution of the nonlinear system. Thus, the present method always remains within the framework of Lie group theory and completely frees us from the need to calculate naive perturbation solutions.

It should be noted that the present method can be applied not only to continuous systems but also to discrete systems, owing to the general framework of the Lie group.

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