Recent Results on Driven Lattice Gases

F. de los Santos[1], M.A. Muñoz[2] and P.L. Garrido[2]

[1] *Centro de Física da Matéria Condensada da Universidade de Lisboa. Av. Prof. Gama Pinto 2. P-1649-003 Lisboa. Portugal.*
[2] *Instituto Carlos I de Física Teórica y Computacional and Departamento de Electromagnetismo y Física de la Materia. Universidad de Granada. E-18071 Granada. Spain.*

Abstract.
The standard field theoretic approach to the driven lattice gas model and a new recently proposed one are briefly reviewed. We comment on the singular nature of the infinite driving limit and on the role of the particle current term.

Driven Systems are subjected to external driving forces that prevent them from attaining thermal equilibrium. Instead, they may settle into *non-equilibrium steady-state*, *i.e.* states that, while out of equilibrium, are statistically time-independent. There are many examples of driven systems in nature, *v.g* granular systems, liquids under shear, etc, and many possible applications such as electrophoresis, traffic flow or fast ionic conductors (see [1] and references therein).

One of the simplest conceivable driven system is the *driven lattice gas* (DLG) [2], which was initially developed as a model for superionic conductors. It is defined on a regular d-dimensional lattice with periodic boundary conditions whose sites may be occupied by a particle o vacant, denoted by the occupation variables 1 and 0, respectively. Particles can hop to neighboring sites, the hopping rate being controlled by the Ising Hamiltonian, H, and an external uniform driving field, $\mathbf{E}$, pointing along one of the principal directions of the lattice. More specifically, the jumps occur with a rate per unit time given by $w((\Delta H - \boldsymbol{\ell}\cdot\mathbf{E})/T)$, where ΔH is the change in H after a particle-hole exchange, T is the temperature of a thermal bath coupled to the system, and $\boldsymbol{\ell}$ is a unit vector pointing from the particle to the hole. w is any function satisfying the detailed-balance condition, $w(-x) = e^{x}w(x)$, which ensures that when $E = |\mathbf{E}| = 0$ the familiar dynamic Ising model is recovered. For $E > 0$, the drive enhances jumps along its direction, suppresses jumps against it, and leaves unaffected those in the transverse directions. Note that detailed-balance holds, but only locally. That is, it is not satisfied for every pair of configurations, but just for those that differ by a single particle-hole exchange. Therefore, one can still speak of energy in the DLG. However, due to the periodic boundary conditions, the work done by a particle that moves between any two points depends on the path

CP574, *Modeling Complex Systems*, edited by P. L. Garrido and J. Marro

taken by the particle. Thus, the system is non-conservative and the rates are not derivable from a potential. In fact, there is a non-zero flux of energy through the system, which gains energy from the external field and dissipates it to the thermal bath. Finally, note that the dynamic rules conserve the total particle number. We shall restrict the discussion hereafter to half-filled lattices because is for that density value that critical phenomena is observed.

Figure 1 depicts the phase diagram of a bidimensional DLG. It displays a second-order phase transition at $T_c(E)$ from a disordered state to a highly anisotropic ordered one with a striped particle-rich region parallel to the drive. The transition temperature increases monotonically with E and saturates at $T_c(E=\infty) \approx 1.4T_c(E=0)$. Almost all Monte Carlo simulations deal with the $E=\infty$ case (no jumps against the field are allowed).

We now consider a mesoscopic description of the DLG. This approach was carried out for the first time by Leung and Cardy [3] and it can be summarized as follows: an order parameter is introduced as a coarse-grained version of the excess particle density (equivalently, a local magnetization $\phi(\mathbf{x},t)$ in the spin language). Since the magnetization is conserved, we write the dynamical equation as a continuity equation

$$\partial_t \phi + \nabla \cdot \mathbf{J} = 0, \tag{1}$$

where $\mathbf{J}$ is a spin or particle current. This is caused by chemical potential gradients and by the external field and, by virtue of the Fick's law for diffusion, can be written as $\mathbf{J} = \mathbf{J}_F + \mathbf{J}_E$, with $\mathbf{J}_F(\mathbf{x},t) = -\lambda \nabla \frac{\delta \mathcal{F}}{\delta \phi}$ and λ being a constant transport coefficient. One shall adopt the Landau-Ginzburg free energy as the mesoscopic counterpart of the Ising Hamiltonian,

$$\mathcal{F} = \int \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{\tau}{2}\phi^2 + \frac{u}{4!}\phi^4 \right\} d\mathbf{x}. \tag{2}$$

The choice for $\mathbf{J}_E$ rests on symmetry grounds: given that no flow can exist in regions locally full or empty, $\phi = \pm 1$, we postulate $\mathbf{J}_E = (1-\phi^2)\boldsymbol{\mathcal{E}}$. $\boldsymbol{\mathcal{E}}$ is the coarse-grained counterpart of $\mathbf{E}$. In addition, a Gaussian distributed, conserved noise $\boldsymbol{\xi}$ is included, modeling the coupling to the thermal bath. Due to anisotropy, additional parameters are introduced reflecting the fact that there is a preferential direction of hopping. Hence, all ∇ operators are split into components transverse ($\nabla_\perp$) and parallel ($\nabla_\parallel$) to $\mathbf{E}$. Putting all this terms together, and after irrelevant terms have been discarded in the renormalization group sense, we arrive at

$$\partial_t \phi = \lambda \left\{ (\tau_\perp - \nabla_\perp^2)\nabla_\perp^2 \phi + \tau_\parallel \nabla_\parallel^2 \phi + \frac{u}{3!}\nabla_\perp^2 \phi^3 \right\} + \mathcal{E}\nabla_\parallel \phi^2 + \nabla_\perp \cdot \boldsymbol{\xi}_\perp. \tag{3}$$

This is the Langevin equation postulated in [3] as a continuum description of the DLG. The parameters $\tau_\perp, \tau_\parallel, u$ and $\mathcal{E}$ are unspecified functions of the microscopic ones J (the coupling constant of the Ising Hamiltonian), T, E, but it is assumed that the detailed form of these functions is not needed for predicting macroscopic phenomena. A field theoretic analysis of equation (3) yields an order parameter critical exponent $\beta = 1/2$ for any value of the external field. Due to a Galilean symmetry, this result is exact to any order in an ε-expansion. This is at odds, however, with the estimates from Monte Carlo simulations with $E = \infty$ (the only case extensively studied) that predict non-classical critical behavior. It has been claimed that an anisotropic finite-size scaling analysis reconciles both predictions [4]. Nevertheless, as it was shown in [5], after invoking anisotropic simulation data analysis the discrepancy still holds and β is close to 1/3.

An effort in a different direction was proposed in [10], where it was suggested that the original field theory was deficient in the limit of infinite drive and a new one put forward. In this new approach one proceeds from a master equation [8]

$$\partial_t P_t(C) = \sum_{C'} \Big\{ W[C' \to C] P_t(C') - W[C \to C'] P_t(C) \Big\}, \tag{4}$$

where C stands for a configuration of coarse-grained variables $\phi_{\mathbf{r}}$ defined at each lattice as the average value of the occupation variables on a region of volume ϵ^{-1} around $\mathbf{r}$. Inspired by the original lattice dynamics, we postulate that the system evolves from a given configuration ϕ to another ϕ' by choosing at random a particle at point $\mathbf{r}$ and performing an exchange with its nearest neighbor in the $\boldsymbol{\ell}$ direction. Therefore, $\phi'_{\mathbf{x}} = \phi_{\mathbf{x}} + \epsilon(\delta_{\mathbf{x},\mathbf{r}} - \delta_{\mathbf{x},\mathbf{r}+\mathbf{a}})$ (see Fig. 2). The transition rates are the microscopic ones, $W[C \to C'] = w((\Delta H - \boldsymbol{\ell} \cdot \mathbf{E})/T)$, with H being expressed in terms of the new coarse-grained variables [9].

When ϵ is small enough, we can do the following identifications:

- $\phi_{\mathbf{x}} \to \phi(\mathbf{x})$
- $\phi'(\mathbf{x}) = \phi(\mathbf{x}) + \epsilon \nabla \delta(\mathbf{x} - \mathbf{r})$
- $H \to \mathcal{F} = \mathcal{S} + \mathcal{H} = \frac{1}{\epsilon} \int d\mathbf{x} \Big\{ \frac{\overline{\tau}}{2}\phi^2 + \frac{u}{4!}\phi^4 \Big\} + \frac{1}{\epsilon} \int d\mathbf{x} \Big\{ \frac{1}{2}(\nabla\phi)^2 + \frac{\tau}{2}\phi^2 \Big\}$
- $w = w\Big(\Delta \mathcal{S}\Big) w\Big(\Delta\mathcal{H} + \boldsymbol{\ell} \cdot \mathbf{E}(1 - \phi^2)\Big)$

A few comments are now in order: the variables $\phi_{\mathbf{x}}$ have been taken as continuous functions of $\mathbf{x}$ and the Ising Hamiltonian replaced by the usual Landau-Ginzburg free energy $\mathcal{F}$ which, being a mesoscopic free energy, comprises both entropic and energetic contributions [13]. Recall that from the microscopics of the DLG, the increment in energy from the drive only competes with the increment of energy coming from the Ising Hamiltonian. This is why the transition rates are factorized. In doing so, fundamental properties of the microscopic system are preserved: local detailed balance holds, invariance under translations in space and time and under the simultaneous change $E \to -E$ and $\phi \to -\phi$.

Next, a Kramers-Moyal [8] expansion of the master equation leads to a Fokker-Planck equation whose stochastically equivalent Langevin equation reads

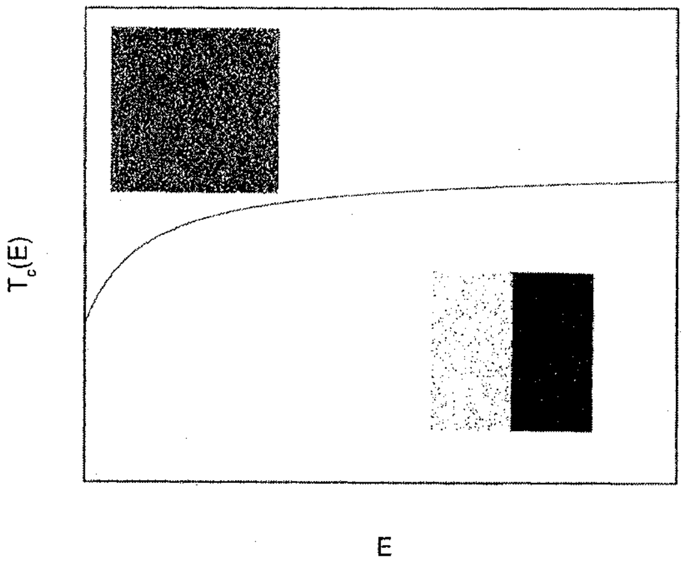

FIGURE 1. Phase diagram for a bi-dimensional DLG. The driving field points downwards.

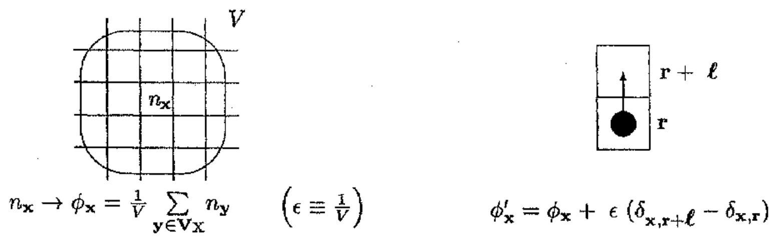

FIGURE 2. Schematic representation of the continuous limit described in the text.

$$\partial_t \phi = \sum_{a=1}^{d} \nabla_a \Big[w(\lambda_a^S) w(\lambda_a^H + \lambda_a^E) - w(-\lambda_a^S) w(-\lambda_a^H - \lambda_a^E) \qquad (5)$$

$$+ \sqrt{w(\lambda_a^S) w(\lambda_a^H + \lambda_a^E) + w(-\lambda_a^S) w(-\lambda_a^H - \lambda_a^E)}\, \xi_a \Big].$$

$\xi_a(\mathbf{x}, t)$ is a Gaussian white noise and $\lambda_a^X = -\nabla_a \frac{\delta X}{\delta \phi}$ In contrast to previous proposals, the dependence of (5) on the microscopic rates is apparent. We now focus on the critical region and discard irrelevant terms in the renormalization group sense by naive power counting. After performing the following scale transformations $t \to \mu^{-z} t, r_\perp \to \mu^{-1} r_\perp, r_\| \to \mu^{-\sigma} r_\|$ and $\phi \to \mu^\delta \phi$, we expand the Langevin in power series while keeping only the leading terms. The time scale, the transverse spatial interaction and the transverse noise are impose to be invariant under the scale transformations, implying $z = 4$ and $\delta = (\sigma + d - 3)/2$. We shall set $\sigma = 2$ because $\nabla_\perp^4 \phi$ and $\nabla_\|^2 \phi$ are the leading gradient terms of the theory. Thus, after dropping irrelevant terms we are lead to

$$\partial_t \phi(\mathbf{r}) = \frac{1}{2}\Big[-\Delta_\perp^2 \phi + (\tau + \overline{\tau}) \Delta_\perp \phi + \frac{g}{6} \Delta_\perp \phi^3 \Big] - E\, w'(E)\, \nabla_\| \phi^2$$

$$+ \Big[\overline{\tau}\, \frac{w(E) + w(-E)}{4} - \tau\, w'(E) \Big]\, \nabla_\|^2 \phi + \sum_\perp \nabla_\perp \xi_\perp(\mathbf{r}, t). \qquad (6)$$

Three regimes of critical behavior are found depending on the value of E:

- $E = 0$. Were there no external field, it is easy to verify by direct substitution that equation (6) reduces to a Model B [11], the continuum counterpart of the Ising model with conserved order parameter.
- $0 < E < \infty$. This is the equation postulated by Leung and Cardy (3). We identify $Ew'(E)$ as the mesoscopic counterpart of the microscopic field E, and $\tau + \overline{\tau}$ and $\overline{\tau}(w(E) + w(-E))/4 - \tau w'(E)$ as the two effective temperatures associated with the transverse and longitudinal directions, respectively. The associated order parameter exponent is $\beta = 1/2$ [6].
- $E = \infty$. In this case equation (6) simplifies to

$$\partial_t \phi = -\Delta_\perp^2 \phi + (\tau + \overline{\tau}) \Delta_\perp \phi + \frac{\overline{\tau}}{2} \Delta_\| \phi + \frac{u}{6} \Delta_\perp \phi^3 + 2 \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi}_\perp, \qquad (7)$$

  because the rates are dominated by terms of the form e^{-E}. Interestingly enough, the order parameter critical exponent associated with (7) is $\beta = 1/3$ in good agreement with the estimations coming from Monte Carlo simulations [12].

We have already seen how our continuous approach respects the symmetries present in the microscopic model. The advised reader, however, will have realized that equation (7) has an extra symmetry lacking in the DLG: the $\phi \to -\phi$ symmetry. It is as if there were no particle current present. In fact, equation (7) was first proposed as a mesoscopic description of a DLG in which the drive fluctuates accordingly to an even distribution. It is clear that such a random force cannot induce a current in the system. A number of arguments supporting the fact that the current is not a relevant feature for the critical properties in the DLG under infinite external drive were given in [10]. From the present point of view, we argue that on taking the $E = \infty$ limit the current term coefficient vanishes due to the saturation of the transition rates in the Master equation. Finally, as a matter of consistency, we shall say that equation (7) is also the resulting equation that our formalism predicts for a random DLG irrespective the value of E considered [13].

Summing up, we have presented an alternate field theoretic approach to the driven lattice gas model. For finite driving force we recover the equation of Leung and Cardy. However, the limit of infinite large driving force corresponds to a different universality class, that of the randomly driven lattice gas. In this case the current becomes irrelevant and the prediction for order parameter critical exponent matches the estimates from Monte Carlo simulations.

REFERENCES

1. Marro J. and Dickman R.: *Nonequilibrium Phase Transitions in Lattice Models*, Cambridge University Press (Cambridge 1999); Schmittmann B. and Zia R.K.P.: *Statistical Mechanics of Driven Diffusive Systems*, in Phase Transitions and Critical Phenomena, eds. C. Domb and J. Lebowitz, Academic, London (1995).
2. Katz S., Lebowitz J.L. and Spohn H., *Phys. Rev.* B, **28**, 1655 (1983).
3. Leung K.-t. and Cardy J., *J. Stat. Phys.* **44**, 567 (1986).
4. Leung K.-t., Phys. Rev. Lett. **66**, 453 (1991). Wang J.S., *J. Stat. Phys.* **82**, 1409 (1996). Leung K.-t. and Wang J.S., *Int. J. Mod. Phys. C* **10**, 853 (1999).
5. Achahbar A. and Marro J., *J. Stat. Phys.* **78**, 1493 (1995); Marro J., Achahbar A., Garrido P.L. and Alonso J.J., *Phys. Rev.* E **53**, 6038 (1996).
6. Janssen H.K. and Schmittmann B., *Z. Phys. B* **64**. 503 (1986).
7. Garrido P.L., de los Santos F. and Muñoz M.A., *Phys. Rev. E* **57**, 752 (1998); de los Santos F. and Garrido P.L. , *J. Stat. Phys.* **96**, 303 (1999).
8. van Kampen N.G., *Stochastic Processes in Physics and Chemistry*, North Holland (1992).
9. Garrido P.L., unpublished.
10. Garrido P.L., Muñoz M.A. and de los Santos F., *Phys, Rev. E* **61**, R4683 (2000).
11. Hohenberg P.C. and Halperin B.I., *Rev. Mod. Phys.***49**, 435 (1977).
12. Schmittmann B. and Zia R.K.P., *Phys. Rev. Lett.* **66**, 357 (1991); Schmittmann B., *Europhys. Lett.* **24**, 109 (1993).
13. de los Santos F. , Muñoz M.A. and Garrido P.L., preprint.