Quasi-potentials in the Nonequilibrium Stationary States or a method to get explicit solutions of Hamilton-Jacobi equations

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Abstract

We assume that a system at a mesoscopic scale is described by a field $\phi(x,t)$ that evolves by a Langevin equation with a white noise whose intensity is controlled by a parameter $1/\sqrt{\Omega}$. The system stationary state distribution in the small noise limit ($\Omega \to \infty$) is of the form $P_{st}[\phi] \simeq \exp(-\Omega V_0[\phi])$ where $V_0[\phi]$ is called the *quasipotential*. V_0 is the unknown of a Hamilton-Jacobi equation. Therefore, V_0 can be written as an action computed along a path that is the solution from Hamilton's equation that typically cannot be solved explicitly. This paper presents a theoretical scheme that builds a suitable canonical transformation that permits us to do such integration by deforming the original path into a straight line. We show that this can be done when a set of conditions on the canonical transformation and the model's dynamics are fulfilled. In such cases, we can get the quasipotential algebraically. We apply the scheme to several one-dimensional nonequilibrium models as the diffusive and reaction-diffusion systems.

I. INTRODUCTION

Thermodynamics shows that many macroscopic properties of systems at equilibrium states are related to each other through a Thermodynamic Potential. Once we know, for instance, the Entropy for a one-component system as a function of energy and mass density, we can deduce many other observables: specific heat, compressibility, Pressure, or Temperature. However, Thermodynamics does not give us the explicit form of a system's Thermodynamic Potential. The Equilibrium Statistical Mechanics solves this problem by introducing the Gibbs invariant measure for the microscopic degrees of freedom. In our opinion, the elegant part of this connection between the microscopic and macroscopic descriptions is that the Gibbs measure depends on the object that defines the full microscopic system dynamics: the hamiltonian. Therefore, all the system's dynamical microscopic details are summed up and contained in the Thermodynamic Potentials. We see that systems at equilibrium have a complete set of theories that allow us to address much interesting macroscopic behavior, for instance, phase transitions.

However, systems at equilibrium are not the most common states in Nature. Typically the systems ate stationary states contain currents of any type, energy, mass,... that appear due to unbalanced boundary conditions and/or the effect of external agents that induce some driving. From the microscopic point of view, few things, but very relevant ones, have changed compared with the equilibrium case: we have typically a system of interacting particles whose dynamics is still Hamiltonian except that we now include on it the dynamical effects from the boundaries and/or the external agents. These apparently small changes break down the theories that apply to systems at equilibrium. First, we do not know how to build a complete macroscopic theory similar to Thermodynamics. Nevertheless, there have been many efforts to justify the existence of some intermediate or mesoscopic descriptions as the Boltzmann equation or macroscopic ones as the Navier-Stokes equations for fluids [1]. And second, the natural invariant measure defined on the phase space is of no practical use. For instance, we can use the SRB measure at the non-equilibrium attractor when the system is "very" chaotic [2]. It is expected that the volume of the nonequilibrium attractor is zero due to dissipation. Still, it could be assumed dense in phase space when the degrees of freedom tend to infinity. However, the attractor's topological structure typically depends strongly on the overall system dynamic trajectories, and therefore it is unknown [3]. We should compare this complex structure with, for example, the microcanonical measure at equilibrium where it is constant on the "a priori" well-known attractor that is the equal energy manifold $H(x,p)=E$.

Some of these problems may be circumvented by studying systems with markovian dynamics. The attractors are compact sets that depend on the physical constraints of the variables. Therefore many of the complexities associated with the attractor topology go away compared with nonequilibrium particle systems. There have been many efforts to elucidate general properties of non-equilibrium lattice models: voter model, contact process, exclusion process,... [4]. For many years the stationary measure typically could only be obtained in few simple cases as the zero-dimensional stochastic models, systems with local detailed balance condition [5] or in the thermodynamic limit of the KMP model for heat conduction [6]. A breakthrough took place by the rigorous derivation of the stationary probability for the one-dimensional boundary driven Symmetric Simple Exclusion model (SSE) by Derrida et al. (2001) [7]. In SSE, only one particle may sit at one site. The dynamic is very simple: a randomly chosen particle may hop to an empty neighbor's site with some given probability. However, at the two boundaries, there are exit and incoming probability rates that may be different. Therefore, it may be created a net current of particles through the system. When the number of lattice sites, N, tend to infinity and the density at the boundaries is fixed and given by $\phi(0)=\phi_0$ and $\phi(1)=\phi_1$, $\phi_0>\phi_1$, they deduce that the probability to find a density profile $\eta = \{\eta(x), \forall x \in [0,1]\}$, where $x \in [0,1]$ and $\eta(x) \in [0,1]$, is given by a large deviation functional:

$$P[\eta] \simeq \exp\left[-N(V_0[\eta])\right] \quad (N \to \infty) \tag{1}$$

where $V_0[\eta]$ is called *quasi-potential* and it is given by:

$$V_0[\eta] = V_0[\phi^*] + G[\eta, \tilde{\eta}] \tag{2}$$

with

$$G[\eta, \tilde{\eta}] = \int_0^1 dx \left[\eta(x) \log\left[\frac{\eta(x)}{\tilde{\eta}(x)}\right] + (1-\eta(x))\log\left[\frac{1-\eta(x)}{1-\tilde{\eta}(x)}\right] + \log\left[\frac{\tilde{\eta}'(x)}{\phi_1 - \phi_0}\right]\right] \tag{3}$$

where $\tilde{\eta}(x)$ is an auxiliary function that is solution of the second order differential equation

$$\eta(x) = \tilde{\eta}(x) + \tilde{\eta}(x)(1-\tilde{\eta}(x))\frac{\tilde{\eta}''(x)}{\tilde{\eta}'(x)^2} \tag{4}$$

with boundary conditions: $\tilde{\eta}(0,1) = \phi_{0,1}$ and the stationary profile is given by $\phi^*(x) = \phi_0 + x(\phi_1 - \phi_0)$. They also mention that all the $\tilde{\eta}(x)$ functions such that

$$\frac{\delta G[\eta, \tilde{\eta}]}{\delta \tilde{\eta}(x)} = 0 \tag{5}$$

are the ones that solve the differential equation (4) which is a bite intriguing. The reader can find many interesting properties and insights of this explicit quasi-potential as, for instance, the existence, uniqueness, and monotonicity of the solutions $\tilde{\eta}(x)$ from the differential equation (4) at Ref.[7].

Let us remind that the quasi-potential for systems at equilibrium is directly related to the free energy functional that is a *Thermodynamic Potential.* That makes V_0 to be a fascinating object to analyze when looking for a non-equilibrium thermodynamic theory (if possible). For instance, let us focus just on the mathematical structure of Derrida's result. Please observe that the quasi-potential seems to be a local functional as it happens at equilibrium. However, the auxiliary field that is the solution of the second-order differential equation depends on the given $\eta(x)$ in a non-local way and on the boundary conditions. In our opinion, the elegant and inspiring part of this solution is how the non-local behavior that is typical in many non-equilibrium stationary states is mathematically codified.

Later, using similar techniques, Enaud and Derrida [8] obtained the quasi-potential for the boundary-driven asymmetric simple exclusion process (ASEP) for the driving field aligned with the density gradient. They obtained a quasi-potential's mathematical form similar to the SSEP case above showed: a local functional depending on an auxiliary field and a second-order differential equation for it. The strategy of studying lattice models with markovian dynamics was successful, and it gave us an important reference on the structure of the quasi-potentials. However, up to our knowledge, these results based on Derrida's matrix technique for one-dimensional models with exclusion process are the unique ones where the quasi-potential have been exactly derived.

A simplified formal path towards the quasi-potentials study was already on the stake by using Fokker-Planck descriptions of non-equilibrium situations [5]. In the context of non-equilibrium many-body systems, it is assumed that there are a set of macroscopic fields that evolve following a known deterministic dynamics and a weak stochastic term (typically white noise) reminiscence of the microscopic fluctuations. Let us mention the pioneering work of Graham et al. [9] where it is studied the general properties of the quasi-potential for systems with finite degrees of freedom. For instance, they introduced the possibility of a Lagrangian transition that would imply the non-differentiability of the quasi-potential in some regions of configurational space. In fact, this property seems to be natural for many systems at non-equilibrium stationary states. Let us also point out that they also developed a gradient expansion of the quasi-potential for the supercritical complex Ginzburg-Landau equation [10]. These works have been applied with great success in several models and fields. Let us remark here just its use in the study of biological systems where the quasi-potentials give a complete description of, for instance, the most probable path that a complex network of chemical reactions follows it to go from a local minimum to another one [11, 12].

Bertini and coworkers introduced further improvement by formulating the Macroscopic Fluctuation Theory (MFT) [13]. MFT was formulated based on many previous rigorous results connecting microscopic stochastic lattice models with their corresponding macroscopic dynamical equations. Large deviation formulas were also obtained, and thus, the mesoscopic description of such systems. They compiled all this information to define a theory that is a generalization of the already known fluctuating hydrodynamics [14]. That is, systems described by hydrodynamic continuum fields that evolve following a Langevin-like equation. First, they computed the quasi-potential for the zeroth range process, and they found that it is local. Moreover, they applied MFT to the continuum mesoscopic version of the SSE model. They found that the quasipotential obtained by Derrida et al. (Eqs. (2) and 3) was also the solution of the Hamilton-Jacobi equation that defined the quasi-potential in the MFT. Bertini et al. [15] obtained the quasi-potential for the mesoscopic version of the Kipnis, Marchioro, Presutti model for heat conduction (KMP)[6]. In fact, they proposed a functional $G[\eta, \tilde{\eta}]$ inspired by eq. (3), and they showed that it was the solution of the corresponding Hamilton-Jacobi equation from MFT. They also found that the quasipotential for the boundary driven ASEP from Enaud and Derrida [8] was also the solution for the MFT [16] and expanded such result when the field is strong enough, and it points against the gradient [17]. In this case, they explicitly found a Lagrangian Transition, that is, the quasipotential has non-differential behavior. All these results showed that MFT had solid theoretical grounds to describe non-equilibrium systems at the mesoscopic level correctly. Let us mention that the quasi-potentials study is just a part of the set of properties of non-equilibrium systems that MFT describes self-consistently. A fascinating review of many aspects that MFT sheds some light on can be found in ref.[18].

We have seen that the exact results from Derrida et al. using their matrix method and the inspired works from Bertini et al. by defining MFT have open the way for a deep understanding of the quasi-potential's mathematical structure. However, to go beyond this point, it is needed new insights that permit us to study more systematically other systems or/and dimensions. This paper focuses on looking for a general algebraic method to obtain the quasi-potential from MFT.

As we will see, in the MFT context, the quasi-potential is solution of a Hamilton-Jacobi equation of the form:

$$H[\eta, \frac{\delta V_0[\eta]}{\delta \eta}] \equiv \int_\Lambda dx\, h\left[\eta(x), \frac{\delta V_0[\eta]}{\delta \eta(x)}\right] = 0 \tag{6}$$

Formally, this equation is solved by using the method of characteristics [19]. That is, $H[\eta, \pi]$ is a hamiltonian that defines a dynamical system. We will see that the quasi-potential $V_0[\eta]$ can be found as essentially an integral of $\pi(t)\partial_t \eta(t)$ along the trajectory (solution of the Hamilton equations with H) that connects the stationary state and η. Except for trivial cases, it is unknown how to solve the hamilton equations to get such a trajectory and then compute the integral. At this point, the existence of the auxiliary field $\tilde{\eta}$ in the exact solution for the SSE and the need to solve the Hamilton equations induced us to ask the following question: Is it possible to define a canonical transformation $(\phi, \pi) \to (\tilde{\phi}, \tilde{\pi})$ such that, in the new variables, we can make the time-integral to get V_0? We explore this idea by using a type 1 canonical transformation defined by the generator of the transformation $L[\phi, \tilde{\phi}]$. It is impossible to write down from scratch a L generator such that in the new variables, we could do the time integrals explicitly. Therefore, we first assume the existence of a map $\phi(x) = \phi[\tilde{\phi}; x]$ between the hamiltonian paths at each t. We show that in this case, the quasi-potential is obtained by a parametric integral that connects, by a straight line, the stationary state with $\tilde{\eta}$ weighted by a functional derivative of the generator and other from the map: A and K respectively. Therefore, once we know A and K, we can get V_0 explicitly by doing a "trivial" time integral. The next question to answer was: which A and K local functionals are compatible with the existence of a canonical transformation defined by L for a given dynamics defined by H? We'll see that A should fulfill the well-known compatibility associated with the existence of a functional when its derivatives are given. On the other hand, the dynamic connection between ϕ and $\tilde{\phi}$ through the map implies an equation that relates A and K for a given hamiltonian H. Therefore, we found a set of necessary conditions that A and K should follow. We cannot prove that such conditions uniquely determine the functionals A and K. However we have designed an algebraic way to build them by assume explicit analytical forms for such functionals, for instance, $A[\phi, \tilde{\phi}; x] = a(\phi(x), \tilde{\phi}(x))$ and $\phi(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), ..., \tilde{\phi}^{(n)}(x))$. The goal is to look algebraically for the functional forms a and f being consistent with the required conditions. As we will see, the interesting part of this method is that we do not need to solve any differential equation. In some way, we manage to map a functional partial differential equation (the original Hamilton-Jacobi equation) to an algebraic polynomial-like system of equations. The attempted solutions should be just an identity for them. This scheme is not perfect because we should also need to restrict the original dynamical model to some concrete forms for each trial functionals forms. Moreover, not always the pair, model and elected functionals, do have a solution, and therefore this is a digging-like method to find some gold nuggets. Nevertheless, we reproduce all the known quasi-potentials with this method, and we discover some new ones.

We present all these results in the following manner. In section II, we define the Langevin dynamics of the system and fast review how to get the quasi-potential and some concepts that we will use in the paper. In section III, we do the canonical transformation and see how it affects the quasi-potential's formal solution. We introduce the necessity for the map between ϕ and $\tilde{\phi}$. We derive the quasi-potential as a function of A and K, and we find the conditions for them. Finally, we assume some general functional forms for them, and we express the conditions in an operational form. Section IV is devoted to showing the quasi-potentials we have found corresponding to the one-dimensional diffusive-system that includes the SSEP and ASEP. In section V we show some quasi-potentials for one-dimensional reaction-diffusion models. We explain the algebraic details on how we derive our results in Appendix III.

II. LANGEVIN DESCRIPTION OF MESOSCOPIC SYSTEMS

We assume that our systems at a mesoscopic level of description are characterized by a unique scalar field $\phi(x,t) \in \mathbb{R}$ where $x \in \Lambda \subset \mathbb{R}^d$, d is the spatial dimension and t is the time. We have initially restricted ourselves to this case in this paper for the sake of simplicity. Still, one can straightforward generalize all the results below to systems described by vector fields. The system dynamics is given by a mesoscopic Langevin equation with a white noise. For instance, in the case of a reaction dynamics (RD) it is:

$$\partial_t \phi(x,t) = F[\phi; x, t] + h[\phi; x, t]\xi(x,t) \tag{7}$$

where F and h are given local functional of $\phi(x,t)$, $\nabla\phi(x,t)$, ... $\xi(x,t)$ is an uncorrelated gaussian random field:

$$\begin{aligned} \langle \xi(x,t) \rangle &= 0 \\ \langle \xi(x,t)\xi(x',t') \rangle &= \frac{1}{\Omega}\delta(x-x')\delta(t-t') \end{aligned} \tag{8}$$

and we follow the Ito's scheme. The dynamics becomes deterministic when $\Omega \to \infty$:

$$\partial_t \phi_D(x,t) = F[\phi_D; x, t] \tag{9}$$

We assume along this paper that the deterministic dynamics has a unique stationary state and that it is locally stable:

$$F[\phi^*; x] = 0 \quad , \quad \phi^*(x) = \lim_{t\to\infty} \phi_D(x,t) \tag{10}$$

for almost any initial state $\phi_D(x,0) = \phi_0(x) \in \Lambda$. Our system may have periodic boundary conditions, fix boundary conditions ($\phi_D(x,t) = f(x)$, $\forall x \in \partial\Lambda$) or a mixture of both.

When the noise intensity is very small, the stationary probability distribution is of the form:

$$P_{st}[\eta] \simeq \exp\left[-\Omega V_0[\eta]\right] \quad (\Omega \to \infty) \tag{11}$$

where $V_0[\eta]$ is the so-called *quasi-potential*. It can be derived from the expression:

$$V_0[\eta] = V_0[\phi^*] + \int_{-\infty}^{0} dt \int_\Lambda dx\, \pi(x,t)\partial_t\phi(x,t) \tag{12}$$

The fields $(\pi(x,t), \phi(x,t))$ are the solution of the Hamilton's equations:

$$\begin{aligned}
\partial_t\phi(x,t) &= \frac{\delta H[\phi(t),\pi(t)]}{\delta\pi(x,t)} \\
\partial_t\pi(x,t) &= -\frac{\delta H[\phi(t),\pi(t)]}{\delta\phi(x,t)}
\end{aligned} \tag{13}$$

where the hamiltonian H is given by:

$$H[\phi,\pi] = \int_\Lambda dx\, \pi(x)\left[F[\phi;x] + \frac{1}{2}\pi(x)h[\phi;x]^2\right] \tag{14}$$

and the Hamilton's equations should be solved with the time boundaries: $(\phi(x,-\infty), \pi(x,-\infty)) = (\phi^*(x), 0)$ and $(\phi(x,0), \pi(x,0)) = (\eta(x), \pi(x))$ $\forall x \in \Lambda$.

Let us point out some properties that we will use below:

- $\phi^*(x)$ is the absolute minimum of the quasi potential:

$$\left.\frac{\delta V_0[\phi]}{\delta\phi(x)}\right|_{\phi(x)=\phi^*(x)} = 0 \quad \forall x \in \Lambda \tag{15}$$

  That is so because in the strict limit $\Omega \to \infty$ we should get the stationary deterministic solution (10). In other words:

$$P_{st}[\eta] = \prod_{x\in\Lambda} \delta(\eta(x) - \phi^*(x)) \tag{16}$$

- $H[\phi^*, 0] = 0$ by construction and therefore, $H[\phi(t), \pi(t)] = 0$.

- $\pi(x,t) = \delta V_0[\eta]/\delta\eta(x)|_{\eta=\phi(t)}$ from eq.(12) .

- Notice that for fix boundary conditions $(\phi(x,t), \pi(x,t)) = (\phi(x), 0)$ $\forall x \in \partial\Lambda$ and t. We have included $\pi(x,t)|_{x\in\partial\Lambda} = 0$. This condition reflects that the boundary is thought as an equilibrium thermal bath having the property $\partial V_B(\phi)/\partial\phi(x) = 0$, with V_B an equilibrium potential. That is, $\pi_B(x)|_{x\in\partial\Lambda} = 0$ and, by continuity $\pi_B(x) = \pi(x)$ $\forall x \in \partial\Lambda$. This choice, of course, affects the nature of the fluctuations about the system's stationary state, but it has the advantage that it permits us to build systems at equilibrium. Then, just by changing the boundaries, we can create nonequilibrium stationary states.

- There can be more than one path solution of the hamilton equations that go from $(\phi^*, 0)$ up to (η, π_n) where n could change for each of the paths. Then it is implicitly understood that one should take in eq. (12) the path that minimizes the value of V_0.

- For Diffusive Dynamics (DD) everything is equal except for the Hamiltonian (14) that is in this case:

$$H[\phi,\pi] = \int_\Lambda dx\, \nabla\pi(x)\cdot\left[G[\phi;x] + \frac{1}{2}\chi[\phi;x]\nabla\pi(x)\right] \tag{17}$$

  where G is the determinist part of the current and χ is related with the noise intensity.

The above definitions and properties are well known in the literature, and that's why we pass through them fast. We refer the readers to ref.[20] for the details about how the above expressions are derived for systems with RD and DD and several comments about the properties of the stationary state.

That is, the problem of finding V_0 is formally solved. However, it is almost impossible at the practical level to obtain the solutions from Hamilton's equations (13). This paper is devoted to building a strategy to be able to make explicitly the time-integral in eq. (12).

III. THE CANONICAL TRANFORMATION

Let us build a general *Type 1* canonical transformation on a generic field hamiltonian $H(\phi,\pi)$ through the generator $L[\phi,\tilde{\phi}]$:

$$(\phi,\pi)\to(\tilde{\phi},\tilde{\pi}) \quad : \quad \pi(x) = \frac{\delta L[\phi,\tilde{\phi}]}{\delta\phi(x)} \equiv A[\phi,\tilde{\phi};x] \quad , \quad \tilde{\pi}(x) = -\frac{\delta L[\phi,\tilde{\phi}]}{\delta\tilde{\phi}(x)} \equiv B[\phi,\tilde{\phi};x] \tag{18}$$

These equations define a one-to-one relationship between the two sets of variables during the system's evolution under the hamiltonian H. The quasipotential (12) is written in the new variables:

$$V_0[\eta] = V_0[\phi^*] + \int_{-\infty}^0 d\tau \int_\Lambda dx\, \frac{\delta L[\phi,\tilde{\phi}]}{\delta\phi(x)}\bigg|_{\substack{\phi=\phi(\tau)\\ \tilde{\phi}=\tilde{\phi}(\tau)}} \partial_\tau\phi(x,\tau) \tag{19}$$

L does not depend on t explicitly, and therefore we can use the relation:

$$\partial_t L[\phi(t),\tilde{\phi}(t)] = \int_\Lambda dx\, \left[\frac{\delta L[\phi,\tilde{\phi}]}{\delta\phi(x)}\bigg|_{\substack{\phi=\phi(t)\\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t\phi(x,t) + \frac{\delta L[\phi,\tilde{\phi}]}{\delta\tilde{\phi}(x)}\bigg|_{\substack{\phi=\phi(t)\\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t\tilde{\phi}(x,t)\right] \tag{20}$$

to get

$$V_0[\eta] = V_0[\phi^*] + L[\eta, \tilde{\eta}] - L[\phi^*, \tilde{\phi}^*] + \int_{-\infty}^{0} d\tau \int_{\Lambda} dx\, \tilde{\pi}(x,\tau)\partial_\tau \tilde{\phi}(x,\tau) \tag{21}$$

where the fields $\tilde{\eta}$ and $\tilde{\phi}^*$ are the canonical transformed η and ϕ^* respectively.

At this point, it could look like that we have not gained too much because we still have to do time integral to get the quasi-potential. However, we have the possibility to design a convenient form for $(\tilde{\phi}, \tilde{\pi})$ so that the integral in eq. (12) can be done. Therefore, our strategy is to look for the necessary conditions to find such optimal transformation. We may divide this search into three well-defined elements: (1) We find the theoretical way to convert such time integral that depends on the paths defined by Hamilton's equations into a new integral along a simple path. We'll see that this new integral depends on a pair of local functionals that are a kind of weight for the new path. (2) Assuming that we have given such functionals, we look for the necessary conditions they should follow to have a well-posed canonical transformation. And finally, (3) we define a way to derive those functionals and to do, afterward, explicitly the new parametric integral and so to get the quasi-potential.

(1) V_0'S CONVENIENT FORM

In the transformed variables $(\tilde{\phi}, \tilde{\pi})$ we can define its quasi-potential by eq. (12), $\tilde{V}_0[\tilde{\phi}]$. Therefore

$$\tilde{\pi}(x)|_{\tilde{T}} = \left.\frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)}\right|_{\tilde{T}} \tag{22}$$

where $\tilde{T}$ represent any pair $(\tilde{\phi}(x,t), \tilde{\pi}(x,t))$ that are solution of the Hamilton's equations (13) with the canonical transformed hamiltonian: $\tilde{H}(\tilde{\phi}, \tilde{\pi}) = H(\phi, \pi)$ and the corresponding boundary conditions. This implies two relevant properties:

- V_0 in eq. (12) can be written:

$$V_0[\eta] = V_0[\phi^*] + L[\eta, \tilde{\eta}] - L[\phi^*, \tilde{\phi}^*] + \tilde{V}_0[\tilde{\eta}] - \tilde{V}_0[\tilde{\phi}^*] \tag{23}$$

- There exists a functional relation between the paths $\phi(t)$ and $\tilde{\phi}(t)$ solutions of the respective hamiltonians that minimize their potentials:

$$\phi(x) = \phi[\tilde{\phi}; x] \tag{24}$$

This can be seen by restricting the canonical transformation (22) to the trajectories:

$$\left.\frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)}\right|_{\tilde{T}} = -\left.\frac{\delta L[\phi, \tilde{\phi}]}{\delta \tilde{\phi}(x)}\right|_{T,\tilde{T}} \tag{25}$$

and therefore (24) is assumed to exist.

Observe that the quasipotential is linearly related to the generator of the canonical transformation. Derrida et al. in Ref.[7] got from the exact microscopic measure the form of the quasipotential. They observed that the functional $G[\phi, \tilde{\phi}]$ that is just the quasipotential assuming that ϕ and $\tilde{\phi}$ are independent variables, had the property that its extreme gave the equation that relates ϕ and $\tilde{\phi}$: $\phi(x) = \phi[\tilde{\phi}; x]$. Similarly, Bertini and co-workers obtained a similar property when studying the quasi-potential associated with a model of heat flow [15]. However, in both cases, they couldn't explain the reason for such property. In our derivation for the quasipotential, such property is just related to the fact that $\tilde{\phi}$ is not just an auxiliary field but the transformed ϕ variable under the canonical transformation: from eq.(23) we define $G[\phi, \tilde{\phi}] = V_0[\phi^*] + L[\phi, \tilde{\phi}] + \tilde{V}_0[\tilde{\phi}] - \tilde{V}_0[\tilde{\phi}]$. Then, the condition $\delta G[\phi, \tilde{\phi}]/\delta \tilde{\phi}(x) = 0$ on $T, \tilde{T}$, is just the equation that relates the variables ϕ with the transformed ones $\tilde{\phi}$ by eq. (25).

Let us now define the *restricted* transformation $\bar{L}$ by substituting ϕ by its relation with $\tilde{\phi}$ along T in (24):

$$\tilde{L}[\tilde{\phi}] = L[\phi[\tilde{\phi}], \tilde{\phi}] \tag{26}$$

then

$$\frac{\delta \tilde{L}[\tilde{\phi}]}{\delta \tilde{\phi}(x)} = -\frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} + \int_\Lambda dy\, A[\phi[\tilde{\phi}], \tilde{\phi}; y] K[\tilde{\phi}; y, x] \tag{27}$$

where

$$K[\tilde{\phi}; x, y] = \frac{\delta \phi[\tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \tag{28}$$

and A is defined in eq. (18).

Assuming that $\tilde{L}[\tilde{\phi}]$ exists (we will address this issue below), we know that (see Appendix 1):

$$\begin{aligned} \tilde{L}[\tilde{\eta}] &= \tilde{L}[\tilde{\phi}^*] - \tilde{V}_0[\tilde{\eta}] + \tilde{V}_0[\tilde{\phi}^*] \\ &\quad + \int_0^1 d\lambda \int_\Lambda dx\, (\tilde{\eta}(x) - \tilde{\phi}^*(x)) \int_\Lambda dy\, A[\phi[\tilde{\phi}(\lambda)], \tilde{\phi}(\lambda); y] K[\tilde{\phi}(\lambda), y, x] \end{aligned} \tag{29}$$

where

$$\tilde{\phi}(x, \lambda) = \tilde{\phi}^*(x) + \lambda(\tilde{\eta}(x) - \tilde{\phi}^*(x)) \tag{30}$$

and the quasi-potential (23) can be written as:

$$
\begin{aligned}
V_0[\eta] &= V_0[\phi^*] \\
&+ \int_0^1 d\lambda \int_\Lambda dx\,(\tilde{\eta}(x) - \tilde{\phi}^*(x)) \int_\Lambda dy\, A[\phi[\tilde{\phi}(\lambda)], \tilde{\phi}(\lambda); y] K[\tilde{\phi}(\lambda); y, x]
\end{aligned}
\tag{31}
$$

We see that the time integral in (12) has been "deformed" by a straight path connecting the stationary state $(\tilde{\phi}^*, 0)$ and the target state $(\tilde{\eta}, \tilde{\pi})$. Moreover, in order to get V_0 we do not need to know the full canonical transformation but the functionals A and K. In expression (31) is hidden a practical problem: for a given canonical transformation L, we get easily the functional A (it is just a functional derivative of L), but we cannot obtain the functional relation between ϕ and $\tilde{\phi}$ (eq. (24)) because we should solve the Hamilton equations explicitly and get the paths to build the map. In conclusion, it seems that we are stuck with the same problem that we initially had. However, the relation (31) can be useful if we change our point of view. That is, given a dynamics and assuming some functional forms for A and $\phi[\tilde{\phi}; x]$ What are the conditions on them to guarantee that they have an associated canonical transformation? With this strategy in mind, we know that two general relations determine the structure of the unknown functionals: (1) The conditions for the existence of L and therefore for $\tilde{L}$ and (2) the equations of motion for ϕ and π that creates the path.

(2) CONDITIONS FOR THE CANONICAL TRANSFORMATION'S EXISTENCE

We know that giving the two functionals $A[\phi, \tilde{\phi}; x]$ and $B[\phi, \tilde{\phi}; x]$ in eq.(18), they may considered as the first derivatives of $L[\phi, \tilde{\phi}]$ if and only if their doubled crossed derivatives are independent on the applied order. That is:

$$
\begin{aligned}
&\text{(C1):}\ \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} = \frac{\delta A[\phi, \tilde{\phi}; y]}{\delta \phi(x)} \\
&\text{(C2):}\ \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = \frac{\delta B[\phi, \tilde{\phi}; y]}{\delta \phi(x)} \\
&\text{(C3):}\ \frac{\delta B[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = \frac{\delta B[\phi, \tilde{\phi}; y]}{\delta \tilde{\phi}(x)}
\end{aligned}
\tag{32}
$$

By other hand, equation (25) implies that $B[\phi, \tilde{\phi}; x]$ should be at least of the form:

$$
B[\phi, \tilde{\phi}; x] = \frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} + \int_\Lambda dy\,(\phi(y) - \phi[\tilde{\phi}; y]) B_\perp[\phi, \tilde{\phi}; x, y] \tag{33}
$$

Using this expression for B we can write the conditions *(C2)* and *(C3)* in the following form:

$$\text{(C2'):}\ \frac{\delta A[\phi,\tilde{\phi};x]}{\delta\tilde{\phi}(y)} = B_\perp[\phi,\tilde{\phi};y,x] + \int_\Lambda dz(\phi(z)-\phi[\tilde{\phi};z])\frac{\delta B_\perp[\phi,\tilde{\phi};y,z]}{\delta\phi(x)} \tag{34}$$

$$\begin{aligned}\text{(C3'):}\quad & \int_\Lambda dz\ \Big[K[\tilde{\phi};z,y]B_\perp[\phi,\tilde{\phi};x,z] - K[\tilde{\phi};z,x]B_\perp[\phi,\tilde{\phi};y,z]\Big] \\ &= \int_\Lambda dz\,(\phi(z)-\phi[\tilde{\phi};z])\left[\frac{\delta B_\perp[\phi,\tilde{\phi};x,z]}{\delta\tilde{\phi}(y)} - \frac{\delta B_\perp[\phi,\tilde{\phi};y,z]}{\delta\tilde{\phi}(x)}\right]\end{aligned} \tag{35}$$

We can get a set of *necessary* conditions for the existence of L if we restrict them to the trajectory T where $\phi(x)=\phi[\tilde{\phi};x]$:

$$\text{(C1T):}\ \left.\frac{\delta A[\phi,\tilde{\phi};x]}{\delta\phi(y)}\right|_{\phi=\phi[\tilde{\phi}]} = \left.\frac{\delta A[\phi,\tilde{\phi};y]}{\delta\phi(x)}\right|_{\phi=\phi[\tilde{\phi}]} \tag{36}$$

$$\text{(C2T):}\ \left.\frac{\delta A[\phi,\tilde{\phi};x]}{\delta\tilde{\phi}(y)}\right|_{\phi=\phi[\tilde{\phi}]} = B_\perp[\phi[\tilde{\phi}],\tilde{\phi};y,x] \tag{37}$$

$$\text{(C3T):}\ \int_\Lambda dz\left[K[\tilde{\phi};z,y]\left.\frac{\delta A[\phi,\tilde{\phi};z]}{\delta\tilde{\phi}(x)}\right|_{\phi=\phi[\tilde{\phi}]} - K[\tilde{\phi};z,x]\left.\frac{\delta A[\phi,\tilde{\phi};z]}{\delta\tilde{\phi}(y)}\right|_{\phi=\phi[\tilde{\phi}]}\right] = 0 \tag{38}$$

Please observe that the conditions (C1T) and (C3T) only depend on the functionals A and K, the same ones that we need to get V_0 in equation (31). We are interested in obtaining V_0 and, therefore, we will use only (C1T) and (C3T). (CT2) become just a property that is of no use for our practical purposes. It is out of the scope of this work to attempt to rigorously prove the sufficient conditions (C1), (C2), and (C3) assuming (C1T), (C2T), and (C3T).

(3) THE INFLUENCE OF THE EQUATIONS OF MOTION

We know that ϕ and $\tilde{\phi}$ are functional related on the path that is solution of the Hamilton's equations (13). Let us write such equations using the canonical transformation (18) and substituting $\phi(x)$ by $\phi[\tilde{\phi};x]$ and $\pi(x)$ by $A[\phi,\tilde{\phi};x]$:

$$\partial_t\phi[\tilde{\phi};x,t] = \tilde{R}_1[\tilde{\phi};x,t] \quad , \quad \partial_t A[\phi[\tilde{\phi}],\tilde{\phi};x,t] = \tilde{R}_2[\tilde{\phi};x,t] \tag{39}$$

where

$$\begin{aligned} R_1[\phi,\tilde{\phi};x] &= \left.\frac{\delta H[\phi,\pi]}{\delta\pi(x)}\right|_{\pi=A[\phi,\tilde{\phi};x]} \\ R_2[\phi,\tilde{\phi};x] &= -\left.\frac{\delta H[\phi,\pi]}{\delta\phi(x)}\right|_{\pi=A[\phi,\tilde{\phi};x]}\end{aligned} \tag{40}$$

and $\tilde{R}_{1,2}[\tilde{\phi};x] = R_{1,2}[\phi[\tilde{\phi}],\tilde{\phi};x]$. We can now expand the time derivatives and we get:

$$\int_\Lambda dy\, \frac{\delta\phi[\tilde{\phi};x]}{\delta\tilde{\phi}(y)}\bigg|_{\tilde{\phi}(t)} \partial_t\tilde{\phi}(y,t) = \tilde{R}_1[\tilde{\phi};x,t]$$

$$\int_\Lambda dy\, \left[\frac{\delta A[\phi,\tilde{\phi};x]}{\delta\phi(y)}\bigg|_{\substack{\phi=\phi[\tilde{\phi};t]\\ \tilde{\phi}=\tilde{\phi}(t)}} \tilde{R}_1[\tilde{\phi};y,t] + \frac{\delta A[\phi,\tilde{\phi};x]}{\delta\tilde{\phi}(y)}\bigg|_{\substack{\phi=\phi[\tilde{\phi};t]\\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t\tilde{\phi}(y,t)\right] = \tilde{R}_2[\tilde{\phi};x,t] \quad (41)$$

These equations are combined to disregard their dependence on $\partial_t\tilde{\phi}$:

$$\int_\Lambda dy \int_\Lambda dz\, \frac{\delta A[\phi,\tilde{\phi};x]}{\delta\tilde{\phi}(y)}\bigg|_{\phi=\phi[\tilde{\phi}]} \tilde{R}_1[\tilde{\phi};z]K^{-1}[\tilde{\phi};y,z] = \tilde{R}_2[\tilde{\phi};x] - \int_\Lambda dy\, \frac{\delta A[\phi,\tilde{\phi};x]}{\delta\phi(y)}\bigg|_{\phi=\phi[\tilde{\phi}]} \tilde{R}_1[\tilde{\phi};y] \quad (42)$$

where

$$\int_\Lambda dy\, K[\tilde{\phi};x,y]K^{-1}[\tilde{\phi};y,z] = \delta(x-z) \quad (43)$$

Observe that we have dropped out the time dependence, considering that this relation for the functional $\phi[\tilde{\phi}]$ holds at each point in the path. We can get a more convenient expression where K^{-1} disappears by integrating both sides by $\int_\Lambda dx K[\tilde{\phi};x,z]$ and using the (C3T) property above:

$$(EM): \int_\Lambda dx\, \Bigg[K[\tilde{\phi};x,y]\tilde{R}_2[\tilde{\phi};x] - \tilde{R}_1[\tilde{\phi};x]\Bigg(\frac{\delta A[\phi,\tilde{\phi};x]}{\delta\tilde{\phi}(y)}\bigg|_{\phi=\phi[\tilde{\phi}]} + \int_\Lambda dz\, K[\tilde{\phi};z,y]\frac{\delta A[\phi,\tilde{\phi};z]}{\delta\phi(x)}\bigg|_{\phi=\phi[\tilde{\phi}]}\Bigg)\Bigg] = 0 \quad (44)$$

Finally, the necessary conditions over the A and $\phi[\tilde{\phi}]$ functional are (C1T) (eq.36), (CT2) (eq.37) and (EM) (eq.(44)). To go further, we should propose some functional forms for both functionals. From now on, we are going to restrict ourselves to one-dimensional systems. The application of these ideas to larger dimensions is left for future works.

(4) FUNCTIONAL FORMS FOR A AND $\phi[\tilde{\phi}]$ FOR 1-D SYSTEMS

Let us assume that the local functional $A[\tilde{\phi};x]$ is of the form:

$$A[\phi,\tilde{\phi};x] = a(\phi(x),\phi'(x),\ldots,\phi^{(n)}(x),\tilde{\phi}(x),\tilde{\phi}'(x),\ldots,\tilde{\phi}^{(m)}(x);\phi^*(x)) \quad (45)$$

for given arbitrary integer values of $m,n \geq 0$. We have included an explicit dependence on the stationary state because we know that $\pi^*(x) = A[\phi^*,\tilde{\phi}^*;x] = 0$ and such degree of freedom maybe necessary in some cases. Similarly, at the fix boundaries (if any) $\pi(x) =$

$A[\phi, \tilde{\phi}; x] = 0 \quad \forall x \in \partial\Lambda$. Both properties should be taken into account when defining the a function.

Our second assumption is to take a generic form for the functional $\phi[\tilde{\phi}]$. We consider that there exists an implicit relation of the form:

$$\phi^{(l)}(x) = f(\phi(x), \phi'(x), \ldots, \phi^{(l-1)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \ldots, \tilde{\phi}^{(s)}(x); \phi^*(x)) \tag{46}$$

for given arbitrary values of $l > 0$ and $s \geq 0$. We also study the case

$$\phi(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), \ldots, \tilde{\phi}^{(s)}(x); \phi^*(x)) \tag{47}$$

We expect a priori that $\tilde{\phi}(x)$ is related with ϕ through a non-local functional strongly dependent on boundary conditions.That's is the expected price that we pay to do easily the "time integration" to get the quasi-potential in the $(\tilde{\phi}, \tilde{\pi})$ new variables (see eq.(31)). The form of the non-local dependence is out of our algebraic control. We choose this form of the functional relation between ϕ and $\tilde{\phi}$ that are "algebraically simple" because it may capture the complexity of a non-equilibrium system through this class of analytic relations as it was shown by the work of Derrida et al. [7]. Nevertheless, one may attempt some other more complex possibilities, but, as we will see, with our elections, we are already at the edge of the mathematics we know today that permits us to get some explicit solution to our problem.

However, we face a problem with this strategy. We are looking for a non-linear differential equation of order s whose solution gives us the relation between $\tilde{\phi}$ and ϕ. Such differential equations have s arbitrary constants to be fixed. That is not a problem for periodic boundary conditions because we look for almost everywhere analytic functions. However, for systems whose dynamic is restricted in a fixed interval, the boundary conditions are just the field's values at the ends of it. Therefore we may have a set of arbitrary constants that are free and that define a family of solutions for our problem. In such cases, we think that those non-assigned constants should be fixed by looking for their values that minimize the quasi-potential.

The forms elected for A and $\phi[\tilde{\phi}]$ are generic enough to permit us to fit the functions a and f with the conditions (CT1), (CT3) and (EM).

Let us define:

$$P_n(x) = \sum_{k=0}^{n} \frac{\partial a}{\partial u_k}\bigg|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} \tag{48}$$

$$Q_m(x) = \sum_{k=0}^{m} \frac{\partial a}{\partial v_k}\bigg|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} \tag{49}$$

$$L_l(x) = \sum_{k=0}^{l-1} \frac{\partial f}{\partial u_k}\bigg|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} - \frac{d^l}{dx^l} \quad l > 0 \tag{50}$$

$$S_s(x) = \sum_{k=0}^{s} \frac{\partial f}{\partial v_k}\bigg|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} \tag{51}$$

$L_0 = -1$. Where $a = a(u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_m; \phi^*(x))$ and similarly f. Where the convention is that after any derivative of their arguments we should do the substitution $u_k \to \phi^{(k)}(x)$ and $v_k \to \tilde{\phi}^{(k)}(x)$.

Moreover, we can do a functional derivative with respect to $\tilde{\phi}(y)$ in both sides of eq.(46) or (47) to obtain an equation for the functional $K[\tilde{\phi}; x, y] = \delta\phi[\tilde{\phi}; x]/\delta\tilde{\phi}(y)$:

$$L_l(x)K[\tilde{\phi}; x, y] + S_s(x)\delta(x - y) = 0 \tag{52}$$

Let us rewrite the conditions by using these differential operators:

- (C1T): Equation (36) can we written in this case as $P_n(x)\delta(x-y) = P_n(y)\delta(x-y)$. This happens if and only if the operator $P_n(x)$ is self-adjoint, $P_n^\dagger(x) = P_n(x)$ (see Appendix II for a brief reminder about properties and definitions of self-adjoint linear differential operators). We can prove this by observing that by definition and for any w integrable test function

$$\int dx w(x) P_n(x)\delta(x-y) = \int dx (P_n^\dagger(x) w(x))\delta(x-y) = P_n^\dagger(y) w(y)$$
$$= \int dx w(x) P_n^\dagger(y)\delta(x-y) \Rightarrow P_n(x)\delta(x-y) = P_n^\dagger(y)\delta(x-y) \tag{53}$$

Therefore $P_n(y)\delta(x-y) = P_n^\dagger(y)\delta(x-y)$ qed. That is,

$$(C1T) : P_n(x) = P_n^\dagger(x) \tag{54}$$

- (C3T): Equation (38) becomes:

$$Q_m^\dagger(x) K[\tilde{\phi}; x, y] = Q_m^\dagger(y) K[\tilde{\phi}; y, x] \tag{55}$$

and after using eq.(52) we find:

$$(C3T): Q_m^\dagger(x)(L_l(x))^{-1}S_s(x) = S_s^\dagger(x)(L_l^\dagger(x))^{-1}Q_m(x) \tag{56}$$

that is $Q_m^\dagger(x)(L_l(x))^{-1}S_s(x)$ should be self-adjoint.

- (EM): After some trivial algebra we find that (44) can be written:

$$Q_m^\dagger(x)\tilde{R}_1[\tilde{\phi};x] = S_s^\dagger(x)(L_l^\dagger(x))^{-1}\left(P_n(x)\tilde{R}_1[\tilde{\phi};x] - \tilde{R}_2[\tilde{\phi};x]\right) \tag{57}$$

or, similarly

$$L_l(x)\tilde{R}_1[\tilde{\phi};x] = S_s(x)(Q_m(x))^{-1}\left(P_n(x)\tilde{R}_1[\tilde{\phi};x] - \tilde{R}_2[\tilde{\phi};x]\right) \tag{58}$$

Note that we define the inverse of any differential operator $T(x)$ through its associated Green function:

$$T(x)f(x) = g(x) \Rightarrow f(x) = \int_\Lambda dy\, G(x,y)g(y) \equiv (T(x))^{-1}g(x) \tag{59}$$

where $G(x,y)$ is solution of

$$T(x)G(x,y) = \delta(x,y) \tag{60}$$

At this point, let us remind our goal: we want to find the functions a and f (given n, m, l and s) such that they fulfill the conditions (C1T), (C3T) and (EM). Inverse differential operators' presence makes it almost impossible to find a systematic way to get the unknown functions. For instance, we should first find the Green function associated with such a still unknown operator. We know that it can be done systematically for regular boundary value problems (self-adjoint differential operators) once we know the eigenfunctions and eigenvalues of the operator (see, for instance, Ref.[21]). However, those depend again on the explicit form of the operator. Therefore, from our present knowledge of these issues, it is almost impossible to get algebraically a set of eigenfunctions of our operators L, or Q that are unknown functionals of $\tilde{\phi}$ and its local derivatives. Consequently, we consider only situations where there aren't inverse operators (C3T) and (EM). With this practical problem in mind there are two possibilities:

- (a) $l = 0$: $L_0(x) = -1$. In this case the operator L is just a constant and the conditions are:

$$
\begin{aligned}
\text{(C1T): } & P_n(x) = P_n^\dagger(x) \\
\text{(C3T): } & Q_m^\dagger(x) S_s(x) = S_s^\dagger(x) Q_m(x) \\
\text{(EM): } & Q_m^\dagger(x)\tilde{R}_1[\tilde{\phi};x] = -S_s^\dagger(x)\left(P_n(x)\tilde{R}_1[\tilde{\phi};x] - \tilde{R}_2[\tilde{\phi};x]\right)
\end{aligned} \tag{61}
$$

- (b) $m = 0$: $Q_0(x) = \partial a/\partial u_0|_{u=\phi,v=\tilde{\phi}} \neq 0$. In this case the operator Q is just a function and the conditions are written:

$$
\begin{aligned}
\text{(C1T): } & P_n(x) = P_n^\dagger(x) \\
\text{(C3T): } & S_s(x) Q_0(x)^{-1} L_l^\dagger(x) = L_l(x) Q_0(x)^{-1} S_s^\dagger(x) \\
\text{(EM): } & L_l(x)\tilde{R}_1[\tilde{\phi};x] = S_s(x) Q_0(x)^{-1}\left(P_n(x)\tilde{R}_1[\tilde{\phi};x] - \tilde{R}_2[\tilde{\phi};x]\right)
\end{aligned} \tag{62}
$$

where we have used eq.(58).

(C1T) and (C3T) conditions require that a differential operator being self-adjoint. This happens only when the degree of the operator is even (the highest order derivative). That is, n should be even in cases (a) and (b), $l+s$ should be even for case (a) and $m+s$ even for case (b).

We will systematically consider different scenarios (a) and (b) by applying them to some typical models as the diffusive dynamics and the reaction-diffusion dynamics. In Appendix III it is explained the detailed schemes used to find a and f from the conditions (C1T), (C3T), and (EM) in each case and the analytic derivation of the corresponding quasipotential. In the next section, we define de models we have worked out and the results.

IV. ONE DIMENSIONAL DIFFUSIVE MODEL

Let us assume that our system is defined by a field $\phi(x)$ with $x \in [0,1]$. The Diffusive Model is defined by the Langevin equation:

$$
\partial_t \phi(x,t) + \frac{dj[\phi;x]}{dx} \quad , \quad j[\phi;x] = G[\phi;x] + \sqrt{\chi(\phi(x,t))}\psi(x,t) \tag{63}
$$

where ψ is a uncorrelated white noise and

$$
G[\phi;x] = -D(\phi(x))\frac{d\phi}{dx} + \chi(\phi(x))E \tag{64}
$$

$D(\lambda)$ and $\chi(\lambda)$ are the diffusion and mobility functions respectively and E is a constant driving field. The hamiltonian that define the paths to build the quasi-potential is given by eq.17 (see Ref.[20] and references therein). D and χ are designed in such a way that the system stationary state could be an equilibrium state with respect the quasipotential:

$$V_{eq}[\phi] = V_{eq}[\phi^*] + \int_0^1 dx\ [v_{eq}[\phi; x] - v_{eq}[\phi^*; x]] \tag{65}$$

where

$$v_{eq}[\phi; x] = v(\phi(x)) - 2Ex\phi(x) \tag{66}$$

We can think E being a kind of gravitational force acting over a mass field $\phi(x)$. We can see that $V_{eq}[\phi]$ is the solution of the Hamilton-Jacobi equation

$$H[\phi, \frac{\delta V_{eq}[\phi]}{\delta \phi}] = 0 \tag{67}$$

when

$$D(\lambda) = \frac{1}{2} v''(\lambda)\chi(\lambda) \tag{68}$$

that it is called *Einstein Relation*. The equilibrium state is achieved when applying the appropriate boundary conditions:

$$\left.\frac{\delta V_{eq}[\phi]}{\delta \phi(x)}\right|_{x=0,1} = 0 \quad \Rightarrow \quad v'(\phi_0) = 0 \quad , \quad v'(\phi_1) = 2E \tag{69}$$

where $\phi(i) = \phi_i \quad i = 0, 1$.

Finally, the equilibrium configuration is obtained from the deterministic part of the Langevin equation by asking that the current G equals to zero:

$$-D(\phi^*(x))\frac{d\phi^*(x)}{dx} + \chi(\phi^*(x))E = 0 \tag{70}$$

The solution of this equation is

$$\int_{\phi(0)}^{\phi^*(x)} d\phi\, \frac{D(\phi)}{\chi(\phi)} = Ex \tag{71}$$

and assuming that the Einstein relation holds, it can be written

$$v'(\phi^*(x)) = 2Ex \tag{72}$$

where $\phi^*(i) = \phi_i \quad i = 0, 1$. Observe that the boundary conditions should be $\phi_{0,1}$ (for a given E) to be at an equilibrium state. When we choose any other different set of boundary conditions, the system develops a non-zero current, and the system is in a non-equilibrium stationary state with a quasipotential $V_0[\phi] \neq V_{eq}[\phi]$. The stationary state is then solution of

$$-D(\phi^*(x))\frac{d\phi^*(x)}{dx} + \chi(\phi^*(x))E = J \tag{73}$$

where J is the current that it is determined by the boundary conditions. We can also get non-equilibrium stationary states with periodic boundary conditions and a non-zero driving field E. In this case $\phi^*(x) = \phi^*$ and $J = \chi(\phi^*)E$.

This section aims to show some results after applying the scheme defined above to get V_0 (for technical details, see Appendix III). The scheme is connected to the dynamics through the functions R_1 and R_2 that in this case are:

$$\begin{aligned} R_1[\phi,\tilde{\phi};x] &= -\frac{d}{dx}\left[-D(\phi(x))\phi'(x) + (E + \frac{da(x)}{dx})\chi(\phi(x))\right] \\ R_2[\phi,\tilde{\phi};x] &= -\chi'(\phi(x))\frac{da(x)}{dx}\left(E + \frac{1}{2}\frac{da(x)}{dx}\right) - D(\phi(x))\frac{d^2a(x)}{dx^2} \end{aligned} \tag{74}$$

where $a(x)$ is a function of ϕ, $\tilde{\phi}$ and their derivatives (see eq.(45)).

(i) (n,m,l,s)=(0,0,0,0): $D(\phi) = c\chi'(\phi)$

In this case we find from conditions (C1T), (C3T) and (EM) (see Appendix III) that

$$a(\phi(x),\tilde{\phi}(x);\phi^*(x)) = \int_{\phi^*(x)}^{\phi(x)} du \frac{2D(u)}{\chi(u)} \tag{75}$$

This model is known as the *generalized zeroth range model*. The quasi-potential is given by:

$$V_0[\eta] = V_0[\phi^*] + 2c\int_\Lambda dx \int_{\phi^*(x)}^{\eta(x)} dw \log\frac{\chi(w)}{\chi(\phi^*(x))} \tag{76}$$

where $\phi^*(x)$ is the stationary state that is solution of

$$c\frac{d^2}{dx^2}\chi(\phi^*(x)) = E\frac{d}{dx}\chi(\phi^*(x)) \tag{77}$$

This result was already obtained by Bertini et al.[22] using other techniques (in fact they found another expression that can be transformed into ours by doing the change of variables: $w = SZ'(S)/Z(S)$ where $S = \chi(w)$). For periodic boundary conditions the solution is a constant: $\phi^*(x) = \phi^*$ fixed by the initial condition:

$$\frac{1}{|\Lambda|}\int_\Lambda dx\, \phi(x,0) = \phi^* \tag{78}$$

and the current is $J = -E\chi(\phi^*)$. For fixed boundary conditions at $x = 0$ and $x = 1$, $\phi(0,t) = \phi_0$ and $\phi(1,t) = \phi_1$ respectively, then

$$\chi(\phi^*(x)) = \frac{\chi_1\left(e^{\bar{E}x} - 1\right) + \chi_0\left(e^{\bar{E}} - e^{\bar{E}x}\right)}{e^{\bar{E}} - 1} \tag{79}$$

where $\bar{E} = E/c$ and $\chi_{0,1} = \chi(\phi_{0,1})$. The current is given by

$$J = E\frac{\chi_1 - \chi_0 e^{\bar{E}}}{e^{\bar{E}} - 1} \tag{80}$$

Observe that equilibrium is obtained when $J = 0$, that is, when $\chi_1 = \chi_0 e^{\bar{E}}$ and the equilibrium stationary profile is given by $\chi(\phi_e^*) = \chi_0 e^{\bar{E}x}$. This example shows that the quasipotential does not contain a structural difference between equilibrium or non-equilibrium situations. This happens uniquely in this particular model (see Ref.[18]).

(ii) $(n,m,l,s) = (0,0,0,2)$

We find four cases:

- $\boldsymbol{D(\phi) = D_0 > 0}$ and $\boldsymbol{\chi(\phi) = k_0 + k_1\phi + k_2\phi^2 > 0}$. The quasi-potential is:

$$\begin{aligned}
&V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \left[v(\eta(x)) - v(\tilde{\eta}(x)) - (\eta(x) - \tilde{\eta}(x))v'(\tilde{\eta}(x))\right.\\
&-\frac{2D_0}{k_2}\log\left[\frac{\chi(\tilde{\eta}(x))}{\chi(\phi^*(x))}\right]\\
&+\frac{2D_0}{k_2 E}\left(\left(\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E\right)\log\left[\left|\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E\right|\right] - \frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))}\log\left[\left|\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))}\right|\right]\right.\\
&\left.\left.-\left(\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))} - E\right)\log\left[\left|\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))} - E\right|\right] + \frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))}\log\left[\left|\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))}\right|\right]\right)\right]
\end{aligned} \tag{81}$$

  where the field $\tilde{\eta}(x)$ is solution of the differential equation:

$$\eta(x) = \tilde{\eta}(x) - \frac{\chi(\tilde{\eta}(x))}{k_2\tilde{\eta}'(x)}\frac{E\chi'(\tilde{\eta}(x))\tilde{\eta}'(x) - D_0\tilde{\eta}''(x)}{E\chi(\tilde{\eta}(x)) - D_0\tilde{\eta}'(x)} \tag{82}$$

  with given boundary conditions $\tilde{\phi}(x) = \phi(x)\ \forall x \in \partial\Lambda$.

  We observe that the case $k_2 \to 0$ seems to be singular; however, it is not. In order to do the limit let us assume that k_2 is a perturbative parameter and then we assume that exist a well defined expansion: $\tilde{\eta}(x) = \tilde{\eta}_0(x) + k_2\tilde{\eta}_1(x) + \ldots$. We apply this expansion to eq. (82), and it appears the order k_2^{-1}. Its coefficient should be zero and therefore:

$$-D_0\tilde{\eta}_0''(x) + Ek_1\tilde{\eta}_0'(x) = 0 \Rightarrow \tilde{\eta}_0(x) = \phi^*(x) \tag{83}$$

where $\phi^*(x)$ is now the stationary state when $k_2 = 0$. For instance, $\phi^*(x)$ for the fixed boundary condition case ($\phi^*(0) = \phi_0$ and $\phi^*(1) = \phi_1$) is

$$\phi^*(x) = \frac{J}{k_1 E} + \frac{\phi_1 - \phi_0}{e^{\tilde{E}} - 1} e^{\tilde{E}x} \quad , \quad J = E k_1 \frac{\phi_0 e^{\tilde{E}} - \phi_1}{e^{\tilde{E}} - 1} \tag{84}$$

where $\tilde{E} = Ek_1/D_0$ and J is the current. The order k_2^0 has the form:

$$\eta(x) = -\phi^*(x)\left(1 + \frac{2D_0 \phi^{*\prime}(x)}{k_0 E + J}\right) - \frac{\chi_0(\phi^*(x))}{(k_0 E + J)\phi^{*\prime}(x)} \left(k_1 E \tilde{\eta}_1'(x) - D_0 \tilde{\eta}_1''(x)\right) \tag{85}$$

where $\chi_0(u) = k_0 + k_1 u$. This differential equation for $\tilde{\eta}_1(x)$ should be solved with boundary conditions $\tilde{\eta}_1(x) = 0 \; \forall x \in \partial\Lambda$ because in the expansion we have defined $\tilde{\phi}_0$ that it carries the natural boundary conditions. Finally, with all this information we can expand the quasi-potential around $k_2 = 0$ and we get:

- $\boldsymbol{D(u) = D_0}$ and $\boldsymbol{\chi(u) = k_0 + k_1 u}$.

$$\begin{aligned} V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \Big[& v(\eta(x)) - v(\phi^*(x)) - (\eta(x) - \phi^*(x)) v'(\phi^*(x)) \\ & + \frac{2D_0^2 E}{\chi_0(\phi^*(x))^2} \left(\chi_0(\phi^*(x))\tilde{\eta}_1'(x) - k_1 \phi^{*\prime}(x)\tilde{\eta}_1(x)\right) \log\left[\left|1 - \frac{E\chi_0(\phi^*(x))}{D_0 \phi^{*\prime}(x)}\right|\right] \\ & - \frac{2D_0 k_1 \tilde{\eta}_1(x)}{\chi_0(\phi^*(x))}\Big] \end{aligned} \tag{86}$$

with $\tilde{\eta}_1(x)$ solution of eq.(85). Observe that $\tilde{\phi}_1^*(x) \neq \phi^*(x)$. We can compute $\tilde{\phi}_1^*(x)$ from eq.(85) for the fixed boundary condition case:

$$\begin{aligned} \tilde{\phi}_1^*(x) = \frac{\phi_1 - \phi_0}{k_1} \frac{1}{(e^{\tilde{E}} - 1)^2} \Big[& (\phi_1 - \phi_0)\left(e^{\tilde{E}x} - 1\right)\left(e^{\tilde{E}} - e^{\tilde{E}x}\right) \\ & + 2\tilde{E}(\phi_0 e^{\tilde{E}} - \phi_1)\left[e^{\tilde{E}x}(1 - x) + \frac{e^{\tilde{E}x} - e^{\tilde{E}}}{e^{\tilde{E}} - 1}\right]\Big] \end{aligned} \tag{87}$$

that in the limit $\tilde{E} \to 0$ is reduced to

$$\tilde{\phi}_1^*(x) = \frac{2}{k_1}(\phi_1 - \phi_0)^2 x(1 - x) \tag{88}$$

- $\boldsymbol{D(\phi) = \bar{D}_0/(\bar{\Lambda}_0 + \phi)^2}$, $\boldsymbol{\chi(\phi) = k_0 + k_1\phi}$. The quasi-potential is:

$$\begin{aligned} V_0[\eta] = V_0[\phi^*] + \int_\lambda dx \Bigg[& v(\eta(x)) - v(\phi^*) - (\eta(x) + \bar{\Lambda}_0)v'(\tilde{\eta}(x)) + (\phi^*(x) + \bar{\Lambda}_0)v'(\phi^*(x)) \\ & + \frac{\bar{D}_0}{a_0} v''(\phi^*(x))\phi^{*\prime}(x) \log \left[\left| a_1 + \frac{2a_0}{v''(\phi^*(x))\phi^{*\prime}(x)} \right| \right] \\ & - \frac{\bar{D}_0}{a_0} v''(\tilde{\eta}(x))\tilde{\eta}'(x) \log \left[\left| a_1 + \frac{2a_0}{v''(\tilde{\eta}(x))\tilde{\eta}'(x)} \right| \right] \\ & - \frac{2\bar{D}_0}{a_1} \log \left[\left| \frac{2a_0 + a_1 v''(\tilde{\eta}(x))\tilde{\eta}'(x)}{2a_0 + a_1 v''(\phi^*(x))\phi^{*\prime}(x)} \right| \right] \Bigg] \end{aligned} \tag{89}$$

where $\tilde{\eta}(x)$ is solution of the differential equation:

$$\begin{aligned} \eta(x) = & -\bar{\Lambda}_0 \\ & + \frac{\bar{D}_0 \left[(k_1 D(\tilde{\eta}(x)) - D'(\tilde{\eta}(x))\chi(\tilde{\eta}(x)))\tilde{\eta}'(x)^2 - D(\tilde{\eta}(x))\chi(\tilde{\eta}(x))\tilde{\eta}''(x) \right]}{\tilde{\eta}'(x) D(\tilde{\eta}(x)) \left[a_0 \chi(\tilde{\eta}(x)) - a_1 D(\tilde{\eta}(x))\tilde{\eta}'(x) \right]} \end{aligned} \tag{90}$$

for any given $\eta(x)$ field and boundary conditions. J is the constant current at the stationary state: $J = E\chi(\phi^*) - D(\phi^*)\phi^{*\prime}(x)$ and it is fixed by the boundary conditions. $a_0 = k_0 E - J - k_1 E \bar{\Lambda}_0$, $a_1 = k_0 - k_1 \bar{\Lambda}_0$ and $v(u) = 2\int du \int du D(u)/\chi(u)$.

(iii) $\boldsymbol{(n, m, l, s) = (0, 0, 1, 1)}$: $\boldsymbol{\chi(\phi) = \chi_0}$

It applies when $\chi(\phi) = \chi_0$ for any $D(\phi)$. The quasi-potential is then:

$$V_0[\eta] = V_0[\phi^*] + \frac{2}{\chi_0} \int_\Lambda dx \int_{\phi^*(x)}^{\eta(x)} du \int_{\phi^*(x)}^{u} dv\, D(v) \tag{91}$$

where $\phi^*(x)$ is solution of

$$D(\phi^*(x))\phi^{*\prime}(x) = \chi_0 E - J \tag{92}$$

V. ONE DIMENSIONAL REACTION-DIFFUSION MODELS

We first study the reaction-diffusion model whose Langevin equation is given by (7) with

$$F[\phi; x] = g(\phi)\phi''(x) + w(\phi(x)) \tag{93}$$

In this case the functions R_1 and R_2 are:

$$\begin{aligned} R_1[\phi, \tilde{\phi}; x] &= g(\phi)\phi''(x) + w(\phi(x)) + a(x)h^2(\phi(x)) \\ R_2[\phi, \tilde{\phi}; x] &= -a(x) \left[g'(\phi(x))\phi''(x) + w'(\phi(x)) + a(x)h(\phi(x))h'(\phi(x)) \right] \\ & \quad - (a(x)g(\phi(x)))'' \end{aligned} \tag{94}$$

where a is a function of ϕ, $\tilde{\phi}$ and their derivatives (see eq.(45)). We have found the quasipotential when:

(i) $(n,m,l,s) = (0,0,0,0)$: $g(\phi) = \bar{g}(\phi)(\phi - \phi^*)^{-\alpha}$, $\alpha > 0$

and it is given by:

$$V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \int_{\phi^*}^{\eta(x)} \frac{du}{g(u)} \tag{95}$$

where ϕ^* is a constant solution (assumed to be unique) of $w(\phi^*) = 0$.

Another interesting model we have studied is the Poissonian Reaction-Diffusion Dynamics. This mesoscopic model is deduced from a stochastic markovian lattice model in which there is a competition between conservative exchange dynamics and a spin-flip one (see ref.[23]). In the fast rate limit for the exchange dynamics and after some time and space rescaling, one obtains the deterministic equation:

$$\frac{\partial \phi_D(x,t)}{\partial t} = \frac{\partial^2 \phi_D(x,t)}{\partial x^2} + b(\phi_D(x,t)) - d(\phi_D(x,t)) \tag{96}$$

where $\phi_D(x)$ represents a normalized density: $0 \leq \phi_D(x) \leq 1$ and b and d functions are directly related with the microscopic spin-flip dynamics. Moreover, the structure of the mesoscopic noise is represented by the hamiltonian:

$$\begin{aligned} H[\phi,\pi] = \int_\Lambda dx \Big[&\pi(x)\phi''(x) + \pi'(x)\phi(x)(1-\phi(x)) \\ &- b(\phi(x))(1-e^{\pi(x)}) - d(\phi(x))(1-e^{-\pi(x)}) \Big] \end{aligned} \tag{97}$$

Observe that this hamiltonian is not quadratic in π that is related to the Poissonian structure of the underlying noise.

Let us assume only periodic boundary conditions. In this case the stationary state is a constant solution of $b(\phi^*) = d(\phi^*)$ that it is assumed to be unique.

The R_1 and R_2 functionals are now:

$$\begin{aligned} R_1[\phi,\tilde{\phi};x] &= \phi''(x) - 2(a'(x)\phi(x)(1-\phi(x)))' + b(\phi(x))e^{a(x)} - d(\phi(x))e^{-a(x)} \\ R_2[\phi,\tilde{\phi};x] &= -a''(x) - a'(x)^2(1-2\phi(x)) + b'(\phi(x))(1-e^{a(x)}) \\ &\quad + d'(\phi(x))(1-e^{-a(x)}) \end{aligned} \tag{98}$$

We have found the quasipotential corresponding to the case:

(ii) $(n,m,l,s) = (0,0,0,0)$: $b(\phi) = \phi^*(1-\phi)h(\phi)$, $d(\phi) = (1-\phi^*)\phi h(\phi)$, $h(\phi) > 0$

$$V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \,\left[\eta(x)\log\left[\frac{\eta(x)}{\phi^*}\right] + (1-\eta(x))\log\left[\frac{1-\eta(x)}{1-\phi^*}\right]\right] \tag{99}$$

This case was already studied by Gabrielli et al (1997) [18, 24]

CONCLUSIONS

The mesoscopic description of non-equilibrium systems given by the MFT [18] is a solid background to study their generic properties. Even though MFT is mathematically simpler than its microscopic original description, it is still difficult to extract precise information from our actual analytical tools. In this paper, we intended to build a method to get the stationary measure represented by the quasi-potential at the small noise limit. Formally, the quasi-potential is obtained by a time integral of some variables along a path defined by a Hamiltonian that depends on the studied system. We proposed to do a canonical transformation in such a way that such path is, in some way, deformed into a straight line that makes it possible to do explicitly such integration on the new variables. However, we pay the price of getting compatibility conditions that select the transformation having such property. Finally, we had to deal with how to get the required functionals from such conditions. We propose in this paper an algebraic technique that assumes generic functional structures for the unknown items, and it sees the conditions as identities that they should fulfill at each point. In this way, the coefficients of higher-order derivatives should be zero. Therefore, order by order, we manage to completely determine the functional structures that are compatible with the conditions that guarantee the existence of the canonical transformation. We explicitly apply this method to already well-known cases, and we discover new solutions that may be of general interest.

We think that this method may be developed and improved further in several ways. For instance, it could be interesting to apply it to higher dimensional systems, at least, initially, for simple cases as the zeroth range model or even the SSE. On the other hand, in the paper, we focused on transformations whose compatibility conditions do not contain differential operators' inverse. For those cases, we handle the structure of the compatibility conditions easily. We think that there is a vast work to be done dealing with the more generic cases where such inverse operators appear. Probably they carry stronger non-local properties necessary to describe the behavior of more complex systems. In this paper, we only dealt with canonical transformations of type I, and the study of other types may imply new quasi-potential structures. Finally, it could be interesting to set up a systematic perturbation theory similar to the Bouchet et al. [25] but associated with the canonical transformation we have presented.

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APPENDIX I

Let $R[\phi;x]$ be a given local functional on ϕ. We know that

$$\frac{\delta B[\phi]}{\delta\phi(x)}=R[\phi;x] \tag{100}$$

and let us assume that $B[\phi]$ exists, that is

$$\frac{\delta R[\phi;x]}{\delta\phi(y)}=\frac{\delta R[\phi;y]}{\delta\phi(x)} \tag{101}$$

Then we can show that

$$B[\phi]=B[\phi^*]+\int_0^1 d\lambda\,\int_\Lambda dx\,(\phi(x)-\phi^*(x))R[\phi(\lambda);x] \tag{102}$$

where

$$\phi(x;\lambda) = \phi^*(x) + \lambda(\phi(x) - \phi^*(x)) \tag{103}$$

Demonstration: Let us show that the derivative of (102) is (100).

$$\frac{\delta B[\phi]}{\delta\phi(y)} = \int_0^1 d\lambda \int_\Lambda dx \left[R[\phi(\lambda);x]\delta(x-y) + (\phi(x) - \phi^*(x))\frac{\delta R[\phi(\lambda);x]}{\delta\phi(y)}\right] \tag{104}$$

We can use the relations:

$$\begin{aligned}\frac{\delta R[\phi(\lambda);x]}{\delta\phi(y)} &= \lambda\frac{\delta R[\phi;x]}{\delta\phi(y)}\bigg|_{\phi=\phi(\lambda)} \\ \frac{dR[\phi(\lambda);x]}{d\lambda} &= \int_\Lambda dy\, \frac{\delta R[\phi;y]}{\delta\phi(x)}\bigg|_{\phi=\phi(\lambda)}(\phi(y) - \phi^*(y))\end{aligned} \tag{105}$$

where we have made use of (101). Then we get

$$\frac{\delta B[\phi]}{\delta\phi(y)} = \int_0^1 d\lambda \left[R[\phi(\lambda);y] + \lambda\frac{dR[\phi(\lambda);y]}{d\lambda}\right] = R[\phi(\lambda);y] \quad \text{c.q.d.} \tag{106}$$

APPENDIX II: SELF ADJOINT CONDITIONS FOR A N-DIFFERENTIAL OPERATOR IN ONE DIMENSION

Let's L be a linear differential operator of n-th order:

$$L = \sum_{k=0}^{n} a_k(x)\frac{d^k}{dx^k} \tag{107}$$

where $a_k(x) \in \mathcal{R} \quad \forall x \in \Lambda \subset \mathcal{R}$. We define the inner product for two real analytic functions $u(x)$, $v(x)$:

$$\langle u, v\rangle = \int_\Lambda dx\, u(x)v(x) \tag{108}$$

The adjoint of L, $L^\dagger$, is then defined by

$$\langle L^\dagger u, v\rangle = \langle u, Lv\rangle \tag{109}$$

Therefore

$$L^\dagger = \sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}a_k(x) \tag{110}$$

where there are assumed that the set of real functions $v(x)$, $u(x)$, where L and $L^\dagger$ respectively apply have boundary conditions such that

$$\sum_{k=1}^{n}\sum_{l=0}^{k-1}(-1)^l\frac{d^l}{dx^l}(a_k(x)u(x))\frac{d^{k-1-l}}{dx^{k-1-l}}v(x)\bigg|_{\partial\Lambda} = 0 \tag{111}$$

L is called *self-adjoint* if $L = L^\dagger$ and the set of boundary conditions for the $v(x)$ and $u(x)$ functions coincide and fulfills eq.(111). Therefore, the coefficients of L such that $L = L^\dagger$ should be related by

$$a_k(x) = \sum_{l=k}^{n} (-1)^l \binom{l}{k} \frac{d^{l-k}}{dx^{l-k}} a_l(x) \quad k = 1, \ldots, n \tag{112}$$

Observe that only the operators L with n even can be self-adjoint. That can be shown by applying relation (112) to the case $k = n$.

One realizes that not all the n-relations defined by (112) are independent. In fact, we can show that the independent set of relations that define a self-adjoint operator is given by:

$$a_{2l+1}(x) = \frac{1}{(2l+1)!} \sum_{s=0}^{m-l-1} (2(s+l+1))!\, c_s \frac{d^{2s+1}}{dx^{2s+1}} a_{2l+2s+2}(x) \tag{113}$$

where $l = 0, \ldots, m-1$ and $n = 2m$. The c's are a set of numbers generated by the recurrence:

$$c_l = \frac{1}{2(2l+1)!} - \frac{1}{2} \sum_{k=0}^{l-1} \frac{c_k}{(2l-2k)!} \quad , \quad l > 0 \quad , \quad c_0 = 1/2 \tag{114}$$

For instance: $c_1 = -1/24$, $c_2 = 1/240$, $c_3 = -17/40320$, $c_4 = 31/725760$, ... Curiously enough we find that c's follow another relation:

$$\frac{1}{(2l+2)!} = \sum_{k=0}^{l} \frac{c_k}{(2l-2k+1)!} \tag{115}$$

We have computed the first one hundred values of c's and found that they alternate signs and their modulus decrease exponentialy fast: $|c_l| \simeq 1.82 \exp[-2.29 l]$.

As an example,for $n = 2$ the condition for self-adjointness is:

$$a_1(x) = \frac{da_2(x)}{dx} \tag{116}$$

and for $n = 4$ we have two conditions:

$$a_1(x) = \frac{da_2(x)}{dx} - \frac{d^3 a_4(x)}{dx^3} \quad , \quad a_3(x) = 2\frac{da_4(x)}{dx} \tag{117}$$

Let us define the Green functions G and $G^\dagger$ solutions of the equations:

$$LG(x, x_0) = \delta(x - x_0) \quad , \quad L^\dagger G^\dagger(x, x_0) = \delta(x - x_0) \tag{118}$$

<u>*Proposition:*</u> L is self-adjoint if and only if $G(x_1, x_2) = G(x_2, x_1)$

To prove the proposition let us choose $u(x) = G^\dagger(x, x_2)$ and $v(x) = G(x, x_1)$ for the inner product in eq. (108). Then we get:

$$G^\dagger(x_1, x_2) = G(x_2, x_1) \tag{119}$$

If L is self-adjoint then $G^\dagger(x_1, x_2) = G(x_1, x_2)$ and using eq.(119) we prove the right implication. If we assume $G(x_1, x_2) = G(x_2, x_1)$ we see from (119) that $G^\dagger(x_1, x_2) = G(x_1, x_2)$. Assuming that there is an unique solution for each operator we get $L^\dagger = L$.

APPENDIX III: METHOD TO OBTAIN V_0

III.1: Allowed operators and Computation strategy

We saw that in cases (a) and (b) (Eqs. (61) and (62) respectively) we should choose a set of values (n, m, l, s) that define the form of operators P, Q,L and S respectively. From condition (C1T), we know that n should be even to fulfill the condition that P is self-adjoint. Similarly, (C3T) in case (a) implies $m + s$ should be also even and in case (b) $s + l$ should also be even. We can do a little better by keeping a trace of the larger derivative of $\tilde{\phi}(x)$ for each condition. In this case, we should explicitly define the dynamics. We have done such computation for $l = 0$ (case (a)) and the Diffusive and the Reaction-Diffusion dynamics defined in the main text. We have found that there are the consistency relations:

$$n = 0 \Rightarrow m \leq s \quad , \quad n > 0 \Rightarrow m = s + n \tag{120}$$

That is, we can attempt the following cases:

where at each column we fix $n = 0$ and then $n = 2$ and so on. We have excluded the coincident values between case (a) and case (b). Once we choose one of these values (n, m, l, s), we have the structure of the operators and a part of the functional form of a and f:

$$a(x) = a(\phi(x), \phi'(x), \ldots, \phi^{(n)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \ldots, \tilde{\phi}^{(m)}(x); \phi^*(x)) \tag{121}$$

$$\phi^{(l)}(x) = f(\phi(x), \phi'(x), \ldots, \phi^{(l-1)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \ldots, \tilde{\phi}^{(s)}(x); \phi^*(x)) \tag{122}$$

The method intends to fix the mathematical form of such functions. The main idea is that once we assume the form of a and f, the conditions (C1T), (C3T), and (EM) should become identities. For instance, eq.(122) is a differential equation when we give $\phi(x)$ and

(n,m,l,s)	
Case (a) (l=0)	Case (b) (m=0)
(0,0,0,0)	**(0,0,1,1)**
(0,1,0,1)	**(0,0,2,2)**
(0,0,0,2)	**(0,0,1,3)**
(0,0,0,4)	**(0,0,3,1)**
(0,1,0,3)	...
(0,2,0,2)	**(2,0,1,1)**
...	**(2,0,2,0)**
(2,2,0,0)	...
...	

the boundary conditions. No other new differential equation relating ϕ and $\tilde{\phi}$ should appear when checking that the conditions apply. Let us give one example.

Let f be of the form: $\phi(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), \tilde{\phi}''(x))$. Let us assume that we can isolate the highest derivative $\tilde{\phi}''(x) = F_2(\phi(x), \tilde{\phi}(x), \tilde{\phi}'(x))$. Then $\tilde{\phi}^{(3)}(x) = F_3(\phi(x), \phi'(x), \tilde{\phi}(x), \tilde{\phi}'(x))$, and in general, $\tilde{\phi}^{(k)}(x) = F_k(\phi(x), \phi'(x), \ldots, \phi^{(k-2)}(x), \tilde{\phi}(x), \tilde{\phi}'(x))$. On the other hand, $\phi(x)$ is assumed to be an analytic function. Therefore, for any given point in the domain, $\bar{x} \in \Lambda$, we can reconstruct $\phi(x)$ just by giving all their derivatives at such point. However, $\phi(x)$ is arbitrary, and we are free to choose all the derivatives of $\phi(x)$ at $\bar{x}$. Therefore, we may reasonably assume that $\tilde{\phi}^{(n)}(\bar{x})$ for $n \geq 2$ may get arbitrary and independent values because of its dependence on the derivatives of $\phi(\bar{x})$.

Having this in mind, we see that the conditions (C1T), (C3T), and (EM) are just relations where there are derivatives of ϕ and $\tilde{\phi}$ of a different order. They should be correct *for any ϕ-field* and therefore, *for any value of the derivatives of* $\tilde{\phi}(x)$ of degree greater or equal to two.

Therefore, once we substitute ϕ by $f(\tilde{\phi}, \tilde{\phi}'(x), \tilde{\phi}''(x))$ in any of the conditions, we get a polynomial expression in the derivatives $\tilde{\phi}^{(n)}$ with $n \geq 2$ whose coefficients are functions with $\tilde{\phi}(x)$, $\tilde{\phi}'(x)$ and $\tilde{\phi}''(x)$ (in this example). Each of these high order derivatives may have arbitrary values and each of their coefficients should be identically equal to zero.

The coefficients that we equal to zero contains, typically derivatives of f and a, and they also contain functions that depend on the dynamics. Therefore, f and a may depend on the dynamics, and, sometimes, only a particular dynamics can make zero a coefficient.

This scheme is done orderly from higher to small order in the polynomial of the derivatives of $\tilde{\phi}$ for each condition. Once we determine some property of the unknown functions, we include it in the conditions, and we redo the computations to get the remaining high-order derivatives' next coefficient. This method has been applied successfully using algebraic programs like Mathematica. It permits us to do long computations without errors. That is very important because we are dealing with identities, and any small mistake during the algebraic trivial but lengthy evaluation implies that the conditions (C1T), (C3T), and (EM) are never fulfilled.

We should also take into account that $\pi(x) = a(x)$. Therefore it is mandatory that:

$$\begin{aligned} &a(\phi^*(x), (\phi^*)'(x), \ldots, (\phi^*)^{(n)}(x), (\tilde{\phi})^*(x), (\tilde{\phi}^*)'(x), \ldots, (\tilde{\phi}^*)^{(m)}(x); \phi^*(x)) = 0 \\ &a(\phi(x), \phi'(x), \ldots, \phi^{(n)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \ldots, \tilde{\phi}^{(m)}(x); \phi^*(x)) = 0 \quad \forall x \in \partial\Lambda \end{aligned} \tag{123}$$

where $\tilde{\phi}^*(x)$ is the stationary state in the new variables that is related with the original stationary state $\phi^*(x)$ through eq.(122):

$$(\phi^*)^{(l)}(x) = f(\phi^*(x), (\phi^*)'(x), \ldots, (\phi^*)^{(l-1)}(x), \tilde{\phi}^*(x), (\tilde{\phi}^*)'(x), \ldots, (\tilde{\phi}^*)^{(s)}(x); \phi^*(x)) \tag{124}$$

Observe that the second condition in eq. (123) only applies when the field's values are fixed at the system's boundaries. These conditions helps us to determine a and f functions.

About the stationary state there are two possibilities: (i) $\tilde{\phi}^*(x) = \phi^*(x)$ or (ii) $\tilde{\phi}^*(x) \neq \phi^*(x)$ and this affects to the boundary conditions for the $\tilde{\phi}(x)$ fields that are necessary to solve the differential equation (122). The (i) case is the most convenient (but more restrictive) because implies that both fields have the same boundary conditions $\tilde{\phi}(x) = \phi(x)\, \forall x \in \partial\Lambda$. Case (ii) needs that eq.(124) to be solved explicitly and then fix some of the s constants by given the boundary values for $\phi(x)$ and the conditions that $\tilde{\phi}(x)$ do not evolve at the boundaries: $\tilde{R}_{1,2}[\tilde{\phi}; x] = 0\, \forall x \in \partial\Lambda$ (see eq.(39) and definitions below it).

At this point we illustrate the overall method in detail for some particular non-trivial cases.

III.2. Diffusive Dynamics and $(n, m, l, s) = (0, 0, 0, 0)$

In this case we choose $a = a(\phi(x), \tilde{\phi}(x); \phi^*(x))$ and $\phi(x) = f(\tilde{\phi}(x), \phi^*(x))$. The corresponding operators (eqs. (48,49,50,51)) have the form:

$$
\begin{aligned}
P_0(x) &= \frac{\partial a(u_0, v_0)}{\partial u_0}\bigg|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \\
Q_0(x) &= \frac{\partial a(u_0, v_0)}{\partial v_0}\bigg|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \\
L_0(x) &= -1 \\
S_0(x) &= \frac{\partial f(u_0, v_0)}{\partial v_0}\bigg|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}}
\end{aligned}
\tag{125}
$$

where we have included a specific dependence on the stationary state for the a function. In this case, conditions (C1T) and (C3T) are fulfilled by construction.

We work then on condition (EM) where we look for the higher derivatives of $\tilde{\phi}(x)$ and make their coefficients equal to zero:

- Condition (EM), order $\tilde{\phi}^{(2)}$: There are two possibilities but only one is relevant:

$$
-2D(u_0)\frac{\partial f(v_0)}{\partial v_0} + \chi(u_0)\left(\frac{\partial a(u_0, v_0)}{\partial v_0} + \frac{\partial f(v_0)}{\partial v_0}\frac{\partial a(u_0, v_0)}{\partial u_0}\right) = 0 \tag{126}
$$

  This condition implies that a does not contain an explicit dependence on v_0 (along the trajectory).This fact, together with the property that a is zero at the stationary state, allows us to find:

$$
a(u_0, v_0; u_s) = \int_{us}^{u_0} du\, \frac{2D(u)}{\chi(u)} \tag{127}
$$

  where $u_s \to \phi^*(x)$.

That is enough to fulfill the condition (EM) completely. We observe that f is not determined. However, we will see that it is not necessary to get the quasi-potential.

In this case, we can use the relation $H[\phi, \pi] = 0$ along the Hamilton trajectory that connects the stationary point $(\phi^*, 0)$ with the given (η, π) to get an "extra" condition that we use to determine which D and χ can be applied.

We know that for diffusive systems

$$
H[\phi, \pi] = \int_\Lambda dx\, \pi'(x)\left[-D(\phi(x))\phi'(x) + \chi(\phi(x))(E + \frac{1}{2}\pi'(x))\right] \tag{128}
$$

We can write the derivative of $\phi(x)$ given by eq.(127) in two distinct forms :

$$\pi'(x) = \frac{2}{\chi(\phi(x))}\left[\left(\int_{\phi^*(x)}^{\phi(x)} du\, D(u)\right)' - \left(\frac{\chi(\phi(x))}{\chi(\phi^*(x))} - 1\right)\left(\int^{\phi^*(x)} du D(u)\right)'\right.$$
$$\left. = \frac{2D(\phi(x))}{\chi(\phi(x))}\phi'(x) - \frac{2D(\phi^*(x))}{\chi(\phi^*(x))}\phi^{*'}(x)\right] \tag{129}$$

and we substitute the first form by the first derivative of π that appears from the left in the hamiltonian and the second in the second one. After some simple algebra we get

$$\int_\Lambda dx\, \frac{J\phi^{*'}(x)}{\chi(\phi^*(x)^2}\left[-\chi'(\phi^*(x))\int_{\phi^*(x)}^{\phi(x)} du\, D(u) + D(\phi^*(x))(\chi(\phi(x)) - \chi(\phi^*(x)))\right] = 0 \tag{130}$$

where J is the current at the stationary state: $-D(\phi^*(x))\phi^{*'}(x) + \chi(\phi^*(x))E = J$. This equation was obtained by Bertini et al. when discussing the conditions of locality for quasipotentials in Diffusive Systems [18]. The relation (130) should be true for any stationary state ϕ^* that can be obtained by, for instance, fixing the fields at the boundaries of Λ and for any field $\phi(x)$. Therefore

$$\chi'(\phi^*(x))\int_{\phi^*(x)}^{\phi(x)} du\, D(u) = D(\phi^*(x))(\chi(\phi(x)) - \chi(\phi^*(x))) \Rightarrow D(\phi) = c\chi(\phi) \tag{131}$$

The systems with such relation between the diffusion and the mobility are called *generalized zeroth range models.*

Once we know the form of a and f and the models that can be described by this forms we compute the quasipotential by using eq.(31):

$$V[\eta] = V[\phi^*] + \int_0^1 d\lambda \int_\Lambda dx\, (\tilde\eta(x) - \phi^*(x)) \int_\Lambda dy\, a(\phi[\tilde\phi(\lambda); y])\frac{\delta\phi[\tilde\phi(\lambda); y]}{\delta\tilde\phi(x,\lambda)} \tag{132}$$

where $\phi[\tilde\phi; x] = f(\tilde\phi(x); \phi^*(x))$ and $\tilde\phi(x;\lambda) = \phi^*(x) + \lambda(\tilde\eta(x) - \phi^*(x))$. We do easily the functional derivative: $\delta\phi[\tilde\phi(\lambda); y]/\delta\tilde\phi(x,\lambda) = \partial f(v_0; \phi^*(x))/\partial v_0|_{v_0=\tilde\phi(x;\lambda)}\delta(x-y)$. Then

$$V[\eta] = V[\phi^*] + 2c\int_\Lambda dx \int_0^1 d\lambda\, (\tilde\eta(x) - \phi^*(x))\frac{\partial f(v_0; \phi^*(x))}{\partial v_0}\bigg|_{v_0=\tilde\phi(x;\lambda)}$$
$$\log\frac{\chi(f(\tilde\phi(x;\lambda); \phi^*(x)))}{\chi(\phi^*(x))} \tag{133}$$

where we have used eq.(127) and $D(u) = c\chi'(u)$. We do the change of variables $\lambda \to f(\tilde\phi(x;\lambda); \phi^*(x))$ at each x-value and we get:

$$V[\eta] = V[\phi^*] + 2c\int_\Lambda dx \int_{\phi^*(x)}^{\eta(x)} dw \log\frac{\chi(w)}{\chi(\phi^*(x))} \tag{134}$$

III.3. Diffusive Dynamics and $(n,m,l,s)=(0,0,0,2)$

In this case we choose $a=a(\phi(x),\tilde{\phi}(x))$ and $\phi(x)=f(\tilde{\phi}(x),\tilde{\phi}'(x),\tilde{\phi}''(x))$. The operators (eqs. (48,49,50,51)) have the form:

$$\begin{aligned}
P_0(x) &= \left.\frac{\partial a(u_0,v_0)}{\partial u_0}\right|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \\
Q_0(x) &= \left.\frac{\partial a(u_0,v_0)}{\partial v_0}\right|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \\
L_0(x) &= -1 \\
S_2(x) &= \left.\frac{\partial f(u_0,\underline{v})}{\partial v_0}\right|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} + \left.\frac{\partial f(u_0,\underline{v})}{\partial v_1}\right|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} \frac{d}{dx} + \left.\frac{\partial f(u_0,\underline{v})}{\partial v_2}\right|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} \frac{d^2}{dx^2}
\end{aligned} \tag{135}$$

where $\underline{v}=(v_0,v_1,v_2)$ and $\underline{\tilde{\phi}}(x)=(\tilde{\phi}(x),\tilde{\phi}'(x),\tilde{\phi}''(x))$. The sufficient conditions given by the cases (a) and (b) (61,62) are equivalent in this case. Condition (C1T) ($P_0(x)$ to be selfadjoint) is fulfilled trivially. Condition (C3T) ($Q_0^\dagger(x)S_2(x)$ to be selfadjoint) is just

$$\left.\frac{\partial a(u_0,v_0)}{\partial v_0}\right|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \left.\frac{\partial f(u_0,\underline{v})}{\partial v_1}\right|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} = \frac{d}{dx}\left[\left.\frac{\partial a(u_0,v_0)}{\partial v_0}\right|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \left.\frac{\partial f(u_0,\underline{v})}{\partial v_2}\right|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}}\right] \tag{136}$$

Finally, the (EM) condition is written from eq. (61) using the expressions for R_1 and R_2 given by equations (74). We express conditions (C3T) and (EM) as a function of $\tilde{\phi}(x)$ and their derivatives knowing that under our assumption $\phi(x)=f(\tilde{\phi}(x),\tilde{\phi}'(x),\tilde{\phi}''(x))$. Both conditions are assumed to be identities that (as we explained above) are fulfill for any value of $\tilde{\phi}^{(n)}(x)$ for $n\le 2$. The conditions have a polynomial structure on high derivatives and therefore their coefficients should be zero. That gives us conditions on the functional forms of our unknowns: a and f. In this case we followed this line of reasoning:

- (1) Condition (EM), order $\tilde{\phi}^{(6)}$:

$$\frac{\partial a(u_0,v_0)}{\partial u_0}\left[\frac{\partial f(u_0,\underline{v})}{\partial v_2}\right]^2\left(2D(u_0)-\chi(u_0)\frac{\partial a(u_0,v_0)}{\partial u_0}\right)=0 \tag{137}$$

  where to simplify the notation: $u_0\to f(\tilde{\phi}(x),\tilde{\phi}'(x),\tilde{\phi}''(x))$ and $\underline{v}\to(\tilde{\phi}(x),\tilde{\phi}'(x),\tilde{\phi}''(x))$ and it is understood that one first do all the derivatives and afterwards we make the substitutions. We see that there are three possibilities that should be analyzed. First, $\partial a/\partial u_0\neq 0$ because the contrary would imply that a only depends on v_0 that is against our initial assumption on a. Similarly $\partial f/\partial v_2 \neq 0$ by construction. Therefore from the last factor we find that

$$a(u_0, v_0) = \int du_0 \frac{2D(u_0)}{\chi(u_0)} + \tilde{a}(v_0) \tag{138}$$

We include this relation into our conditions and go to the next non-zero order.

- (2) Condition (C3T), order $\tilde{\phi}^{(3)}$:

$$\frac{\partial \tilde{a}(v_0)}{\partial v_0} \frac{\partial^2 f}{\partial v_2^2} = 0 \tag{139}$$

As before, $\partial \tilde{a}(v_0)/\partial v_0 \neq 0$ by construction and therefore f should be a linear function of v_2:

$$f(v_0, v_1, v_2) = f_0(v_0, v_1) + v_2 f_1(v_0, v_1) \tag{140}$$

By using this relation, we rewrite the full (C3T) condition as:

$$v_1 \frac{\partial}{\partial v_0} (\tilde{a}'(v_0) f_1(v_0, v_1)) = \tilde{a}'(v_0) \frac{\partial f_0(v_0, v_1)}{\partial v_1} \tag{141}$$

We use some of these relations into the condition (EM) and move on to the next non-zero order.

- (3) Conditions (EM), order $\tilde{\phi}^{(4)}$:

$$-4D(u_0) - 2u_0 D'(u_0) - g_0(v_0, v_1) D'(u_0) + g_1(v_0, v_1) \chi''(v_0) = 0 \tag{142}$$

where

$$\begin{aligned} g_0(v_0, v_1) &= 2\tilde{a}'(v_0)^{-1} \left(v_1^2 \tilde{a}''(v_0) f_1(v_0, v_1) - f_0(v_0, v_1) \tilde{a}'(v_0)\right) \\ g_1(v_0, v_1) &= v_1 f_1(v_0, v_1) \left(2E + v_1 \tilde{a}'(v_0)\right) \end{aligned} \tag{143}$$

because u_0 is arbitrary and independent on v_0 and v_1 we have three possible scenarios:

$$\begin{aligned} &\text{(a) } D(u_0) = D_0 \\ &\text{(b) } \chi''(u_0) = 0 \\ &\text{(c) } D'(u_0) \neq 0 \quad , \quad \chi''(u_0) \neq 0 \end{aligned} \tag{144}$$

Observe that we have not considered the possibility that $D'(u_0) = c_0 \chi''(u_0)$ because it is studied in the $(n, m, l, s) = (0, 0, 0, 0)$ case.

– (a) Equation (142) implies:

$$g_1(v_0,v_1)=\Lambda_1 \quad , \quad \chi''(u_0)=\frac{4D_0}{\Lambda_1} \tag{145}$$

and therefore this case applies only when $D(u_0)=D_0$, $\chi(u_0)=k_0+k_1u_0+k_2u_0^2$. Then from (145) we find:

$$f_1(v_0,v_1)=\frac{2D_0}{k_2v_1}\frac{1}{2E+v_1\tilde{a}'(v_0)} \tag{146}$$

We use now eq.(141) to get the form of $f_0(v_0,v_1)$:

$$f_0(v_0,v_1)=f_{00}(v_0)-\frac{4D_0E\tilde{a}''(v_0)}{k_2\tilde{a}'(v_0)^2}\frac{1}{2E+v_1\tilde{a}'(v_0)} \tag{147}$$

where $f_{00}(v_0)$ is an arbitrary function to be determined.

– (b) In this case $\chi(u_0)=k_0+k_1u_0$ and equation (142) implies:

$$\frac{D'(u_0)}{D(u_0)}=\frac{-4}{\Lambda_0+2u_0}\Rightarrow D(u_0)=\frac{D_0}{(\Lambda_0+2u_0)^2} \tag{148}$$

and

$$g_0(v_0,v_1)=\Lambda_0\Rightarrow f_0(v_0,v_1)=-\frac{\Lambda_0}{2}+\tilde{a}'(v_0)^{-1}v_1^2\tilde{a}''(v_0)f_1(v_0,v_1) \tag{149}$$

We again use the remaining of the condition (C3T), eq.(141), to find $f_1(v_0,v_1)$:

$$f_1(v_0,v_1)=\tilde{a}'(v_0)f_{11}(v_1\tilde{a}'(v_0)) \tag{150}$$

where $f_{11}(\beta)$ is and arbitrary function to be determined.

– (c) In this case we get the conditions:

$$\begin{aligned}
&g_0(v_0,v_1)=\Lambda_0\Rightarrow f_0(v_0,v_1)=-\frac{\Lambda_0}{2}+\tilde{a}'(v_0)^{-1}v_1^2\tilde{a}''(v_0)f_1(v_0,v_1)\\
&g_1(v_0,v_1)=\Lambda_1\Rightarrow f_1(v_0,v_1)=\frac{\Lambda_1}{v_1}\frac{1}{2E+v_1\tilde{a}'(v_0)}\\
&-4D(u_0)-2u_0D'(u_0)-\Lambda_0D'(u_0)+\Lambda_1\chi''(u_0)=0
\end{aligned}$$

One can easily check that the condition (141) is fulfilled by these $f_0(v_0,v_1)$ and $f_1(v_0,v_1)$ functions.

- Conditions (EM), order $\tilde{\phi}^{(2)}$ (order $\tilde{\phi}^{(3)}$ are zero in all three cases (a), (b) and (c)). At this point we determine the remaining unknowns observing the properties of $a(u_0,v_0)$, the stationary state and the boundary conditions (see for instance eqs. (123,124)). In particular we first assume that $\phi^*(x) = \tilde{\phi}^*(x)$. This choice is convenient because implies that the boundary conditions for the transformed field $\tilde{\phi}(x)$ are the same as the ones for $\phi(x)$ and therefore the second property in (123) is immediately fulfilled. Then:

$$a(\phi^*(x), \phi^*(x)) = 0 \Rightarrow \tilde{a}(\phi^*(x)) = \bar{a}(\phi^*(x)) \quad , \quad \bar{a}(w) = -\int^w dv \, \frac{2D(v)}{\chi(v)} \tag{151}$$

where we have used (138). There are two possibilities: (a) $\tilde{a}(v_0) = \bar{a}(\phi^*(x))$ or (b) $\tilde{a}(v_0) = \bar{a}(v_0)$ $\forall v_0$. The first case contradicts our a's initial choice where we assumed that there weren't any explcit x-dependence on it. Therefore

$$a(u_0, v_0) = 2\int_{v_0}^{u_0} du \frac{D(u)}{\chi(u)} \tag{152}$$

The second piece of information that we should use is the form of the stationary state. For Diffusive Dynamics the stationary state is solution of the differential equation:

$$\frac{d}{dx}\left[-D(\phi^*(x))\frac{d\phi^*(x)}{dx} + E\chi(\phi^*(x))\right] = 0 \tag{153}$$

with the corresponding boundary conditions. We also know that f relates ϕ and $\tilde{\phi}$ and, in particular we can apply this relation to their stationary states that should be equal due to the form of a. Therefore eq.(122) can be written:

$$\phi^*(x) = f_0(\phi^*(x), \phi^{*'}(x)) + \phi^{*''}(x) f_1(\phi^*(x), \phi^{*'}(x)) \tag{154}$$

Obviously both differential equations should have the same solutions given the boundary conditions. Just by eliminating $\phi^{*''}(x)$ from both equations we get a relation between f_0 and f_1 at the stationary state that help us to determine the missing parts of f. We use all these properties to finish the computation in all three cases:

- (a) $f_{00}(v_0) = v_0$ and therefore:

$$f(v_0, v_1, v_2) = v_0 - \frac{\chi(v_0)}{k_2 v_1}\frac{E\chi'(v_0)v_1 - D_0 v_2}{E\chi(v_0) - D_0 v_1} \tag{155}$$

- (b) In this case $\chi(v_0) = k_0 + k_1 v_0$ that makes that the stationary state has the nice property that $D(\phi^*(x))\phi^{*'}(x) = Ek_1\phi^*(x) + Ek_0 - J$ where J is the stationary current. That makes possible to determine the unknown function $f_{11}(\beta)$:

$$f_{11}(\beta) = \frac{2\bar{D}_0}{\beta(2J - (\beta + 2E)(k_0 - k_1\bar{\Lambda}_0))} \tag{156}$$

where $\bar{D}_0 = D_0/4$ and $\bar{\Lambda}_0 = \Lambda_0/2$ so $D(u) = \bar{D}_0/(\bar{\Lambda}_0 - u)^2$. Finally we get:

$$\begin{aligned} & f(v_0, v_1, v_2) = \\ & -\bar{\Lambda}_0 + \bar{D}_0 \frac{(D(v_0)\chi'(v_0) - D'(v_0)\chi(v_0))v_1^2 - D(v_0)\chi(v_0)v_2}{v_1 D(v_0)((k_0 E - J - k_1 E\bar{\Lambda}_0)\chi(v_0) - (k_0 - k_1\bar{\Lambda}_0)D(v_0)v_1)} \end{aligned} \tag{157}$$

– (c) This order only is accomplished for situations already studied in (a) or (b).

We see that in the case $(n, m, l, s) = (0, 0, 0, 2)$ we have obtained in an algebraic manner the precise form of a and f functions such that the conditions (C1T), (C3T) and (EM) are fulfilled for some forms of D and χ. At this point we can compute the quasi-potential for each case.

(a) $\boldsymbol{D(u) = D_0}$, $\boldsymbol{\chi(u) = k_0 + k_1 u + k_2 u^2}$

We found:

$$\begin{aligned} A[\phi, \tilde{\phi}; x] &= 2D_0 \int_{\tilde{\phi}(x)}^{\phi(x)} \frac{du}{\chi(u)} \equiv a(\phi(x), \tilde{\phi}(x)) \\ \phi[\tilde{\phi}; x] &= \tilde{\phi}(x) - \frac{\chi(\tilde{\phi}(x))}{k_2 \tilde{\phi}'(x)} \frac{E\chi'(\tilde{\phi}(x))\tilde{\phi}'(x) - D_0\tilde{\phi}''(x)}{E\chi(\tilde{\phi}(x)) - D_0\tilde{\phi}'(x)} \end{aligned} \tag{158}$$

we introduce these expressions into eq.(31) that it can be written as:

$$\begin{aligned} V_0[\eta] &= V_0[\phi^*] + I_1 - I_2 \\ I_1 &= \int_0^1 d\lambda \int_\Lambda dx\, (\tilde{\eta}(x) - \phi^*(x)) \frac{\delta}{\delta\tilde{\phi}(x;\lambda)} \left[\int_\Lambda dy \int du_0\, a(u_0, v_0) \right]_{\substack{u_0 = \phi[\tilde{\phi}(\lambda); y] \\ v_0 = \tilde{\phi}(y;\lambda)}} \\ I_2 &= \int_0^1 d\lambda \int_\Lambda dx\, (\tilde{\eta}(x) - \phi^*(x)) \int du_0 \left. \frac{\partial a(u_0, v_0)}{\partial v_0} \right|_{\substack{u_0 = \phi[\tilde{\phi}(\lambda); x] \\ v_0 = \tilde{\phi}(x;\lambda)}} \end{aligned} \tag{159}$$

where $\tilde{\phi}(x;\lambda) = \phi^*(x) + \lambda(\tilde{\eta}(x) - \phi^*(x))$ and we have used the relation:

$$\begin{aligned} \int_\Lambda dy\, a(\phi[\tilde{\phi}; y], \tilde{\phi}(y)) \frac{\delta\phi[\tilde{\phi}; y]}{\delta\tilde{\phi}(x)} &= \frac{\delta}{\delta\tilde{\phi}(x)} \int_\Lambda dy \int du\, a(u, v) \Big|_{\substack{u = \phi[\tilde{\phi}; y] \\ v = \tilde{\phi}(y)}} \\ &\quad - \int du \left. \frac{\partial a(u, v)}{\partial v} \right|_{\substack{u = \phi[\tilde{\phi}; y] \\ v = \tilde{\phi}(y)}} \end{aligned} \tag{160}$$

The first integral in (159) is just:

$$\begin{aligned} I_1 &= \int_0^1 d\lambda \frac{d}{d\lambda} \left[\int_\Lambda dy \int du_0\, a(u_0, v_0) \right]_{\substack{u_0 = \phi[\tilde{\phi}(\lambda); y] \\ v_0 = \tilde{\phi}(y;\lambda)}} \\ &= \int_\Lambda dy\, [v(\eta(y)) - \eta(y)v'(\eta(y)) - v(\phi^*(y)) + \phi^*(y)v'(\phi^*(y))] \end{aligned} \tag{161}$$

where $v(u_0) = 2D_0 \int du_0 \int du_0/\chi(u_0)$.

The second integral needs a little more work. First, it can be written:

$$I_2 = -2D_0 \int_0^1 d\lambda \int_\Lambda dx\, (\tilde{\eta}(x) - \phi^*(x)) \frac{\phi[\tilde{\phi}(\lambda); x]}{\chi(\tilde{\phi}(x;\lambda))} \tag{162}$$

This integral can be separated into two pieces: one without derivatives of $\tilde{\phi}(x;\lambda)$ and the other with its derivatives:

$$\begin{aligned}
I_2 &= I_{21} + I_{22} \\
I_{21} &= \int_0^1 d\lambda \int_\Lambda dx\, \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x;\lambda))} \left[\tilde{\phi}(x;\lambda) - \frac{1}{k_2}\chi'(\tilde{\phi}(x;\lambda))\right] \\
I_{22} &= \frac{D_0}{k_2} \int_0^1 d\lambda \int_\Lambda dx\, (\tilde{\eta}(x) - \phi^*(x)) \frac{1}{u(x;\lambda)\chi(\tilde{\phi}(x;\lambda))} \frac{u'(x;\lambda)}{E - D_0 u(x;\lambda)}
\end{aligned} \tag{163}$$

where $u(x;\lambda) = \tilde{\phi}'(x;\lambda)/\chi(\tilde{\phi}(x;\lambda))$. I_{21} can be evaluated by making the change of variables $\lambda \to \tilde{\phi}(x;\lambda)$ at each x and we get:

$$\begin{aligned}
I_{21} = \frac{1}{2D_0} \int_\Lambda dx \Big[&\tilde{\eta}(x) v'(\tilde{\eta}(x)) - v(\tilde{\eta}(x)) - \frac{2D_0}{k_2} \log \chi(\tilde{\eta}(x)) \\
&- \phi^*(x) v'(\phi^*(x)) + \frac{2D_0}{k_2} \log \chi(\phi^*(x)) \Big]
\end{aligned} \tag{164}$$

where $\tilde{\eta}(x)$ is solution of the differential equation:

$$\eta(x) = \tilde{\eta}(x) - \frac{\chi(\tilde{\eta}(x))}{k_2 \tilde{\eta}'(x)} \frac{E\chi'(\tilde{\eta}(x))\tilde{\eta}'(x) - D_0 \tilde{\eta}''(x)}{E\chi(\tilde{\eta}(x)) - D_0 \tilde{\eta}'(x)} \tag{165}$$

for a given $\eta(x)$ field. Integral I_{22} can be written:

$$I_{22} = \frac{D_0}{k_2 E} \int_0^1 d\lambda \int_\Lambda dx\, \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x;\lambda))} \frac{d}{dx} \log\left[\frac{|D_0 u(x;\lambda)|}{|E - D_0 u(x;\lambda)|}\right] \tag{166}$$

and after integrating by parts on x:

$$I_{22} = -\frac{D_0}{k_2 E} \int_0^1 d\lambda \int_\Lambda dx\, \frac{d}{dx} \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x;\lambda))} \log\left[\frac{|D_0 u(x;\lambda)|}{|E - D_0 u(x;\lambda)|}\right] \tag{167}$$

we use now the relation:

$$\frac{du(x;\lambda)}{d\lambda} = \frac{d}{dx} \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x;\lambda))} \tag{168}$$

to get

$$\begin{aligned}
I_{22} &= -\frac{D_0}{k_2 E} \int_\Lambda dx \int_0^1 d\lambda \frac{du(x;\lambda)}{d\lambda} \log\left[\frac{|D_0 u(x;\lambda)|}{|E - D_0 u(x;\lambda)|}\right] \\
&= -\frac{D_0}{k_2 E} \int_\Lambda dx \int_{u(x;0)}^{u(x;1)} du \log\left[\frac{|D_0 u|}{|E - D_0 u|}\right]
\end{aligned} \tag{169}$$

where $u(x,0) = \phi^{*\prime}(x)/\chi(\phi^*(x))$ and $u(x,1) = \eta'(x)/\chi(\eta(x))$. After doing the integral of the logarithm and putting together all the pieces I_1, I_{21} and I_{22} we get the final expression for V_0:

$$
\begin{aligned}
V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \Bigg[& v(\eta(x)) - v(\tilde{\eta}(x)) - (\eta(x) - \tilde{\eta}(x))v'(\tilde{\eta}(x)) - \frac{2D_0}{k_2}\log\frac{\chi(\tilde{\eta}(x))}{\chi(\phi^*(x))} \\
+ \frac{2D_0}{k_2 E}\Bigg(& \left(\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E\right)\log\left|\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E\right| - \frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))}\log\left|\frac{D_0\tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))}\right| \\
- & \left(\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))} - E\right)\log\left|\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))} - E\right| + \frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))}\log\left|\frac{D_0\phi^{*\prime}(x)}{\chi(\phi^*(x))}\right|\Bigg)\Bigg]
\end{aligned}
\tag{170}
$$

(b) $D(u) = \bar{D}_0/(\bar{\Lambda}_0 + u)^2$, $\chi(u) = k_0 + k_1 u$

We found in this case:

$$
\begin{aligned}
A[\phi, \tilde{\phi}; x] &= 2\int_{\tilde{\phi}(x)}^{\phi(x)} \frac{duD(u)}{\chi(u)} \equiv a(\phi(x), \tilde{\phi}(x)) \\
\phi[\tilde{\phi}; x] &= -\bar{\Lambda}_0 \\
&+ \frac{\bar{D}_0\left[(k_1 D(\tilde{\phi}(x)) - D'(\tilde{\phi}(x))\chi(\tilde{\phi}(x)))\tilde{\phi}'(x)^2 - D(\tilde{\phi}(x))\chi(\tilde{\phi}(x))\tilde{\phi}''(x)\right]}{\tilde{\phi}'(x)D(\tilde{\phi}(x))\left[a_0\chi(\tilde{\phi}(x)) - a_1 D(\tilde{\phi}(x))\tilde{\phi}'(x)\right]}
\end{aligned}
\tag{171}
$$

where $a_0 = k_0 E - J - k_1 E\bar{\Lambda}_0$ and $a_1 = k_0 - k_1\bar{\Lambda}_0$. We use these expressions in Eq. (31), and initially, we follow similar steps as in the above case to get:

$$
\begin{aligned}
V_0[\eta] = V_0[\phi^*] &+ \int_\Lambda dx\ [v(\eta(x)) - v(\phi^*) - \eta(x)v'(\tilde{\eta}(x)) + \phi^*(x)v'(\phi^*(x))] \\
&+ \int_0^1 d\lambda \int_\Lambda dx(\tilde{\eta}(x) - \phi^*(x))v''(\tilde{\phi}(x;\lambda))\phi[\tilde{\phi}; x]
\end{aligned}
\tag{172}
$$

where we remind that $v(u) = 2\int du \int duD(u)/\chi(u)$. After substituting $\phi[\tilde{\phi}; y]$ from its expression we can decompose the last integral into two pieces:

$$
\int_0^1 d\lambda \int_\Lambda dx(\tilde{\eta}(x) - \phi^*(x))v''(\tilde{\phi}(x;\lambda))\phi[\tilde{\phi}; x] = -\bar{\Lambda}_0 I_{11} - \bar{D}_0 I_{12} \tag{173}
$$

where

$$
\begin{aligned}
I_{11} &= \int_0^1 d\lambda \int_\Lambda dx(\tilde{\eta}(x) - \phi^*(x))v''(\tilde{\phi}(x;\lambda)) \\
I_{12} &= \int_0^1 d\lambda \int_\Lambda dx(\tilde{\eta}(x) - \phi^*(x))v''(\tilde{\phi}(x;\lambda))\frac{u'(x;\lambda)}{u(x;\lambda)(a_0 + a_1 u(x;\lambda))}
\end{aligned}
\tag{174}
$$

where $u(x,\lambda) = v''(\tilde{\phi}(x;\lambda))\tilde{\phi}'(x;\lambda)/2$. The integral I_{11} is straightforward evaluated using the change of variables $\lambda \to \tilde{\phi}(x;\lambda)$ at each x. Then:

$$I_{11} = \int_\Lambda dx\ [v'(\tilde{\eta}(x)) - v'(\phi^*(x))] \tag{175}$$

The integral I_{12} is done by first rewritting $u'/(u(a_0+a_1u)) = (\log|u/(a_0+a_1u)|)'/a_0$. Second, we integrate by parts where the surface term is zero due to the boundary conditions and third, we use the relation:

$$\frac{du(x;\lambda)}{d\lambda} = \frac{1}{2}\frac{d}{dx}\left[v''(\tilde{\phi}(x;\lambda)(\tilde{\eta}(x) - \phi^*(x))\right] \tag{176}$$

Finally we put all the terms together and we find:

$$\begin{aligned} V_0[\eta] = V_0[\phi^*] + \int_\lambda dx \Bigg[& v(\eta(x)) - v(\phi^*) - (\eta(x)+\bar{\Lambda}_0)v'(\tilde{\eta}(x)) + (\phi^*(x)+\bar{\Lambda}_0)v'(\phi^*(x)) \\ &+ \frac{\bar{D}_0}{a_0}v''(\phi^*(x))\phi^{*\prime}(x)\log\left[\left|a_1 + \frac{2a_0}{v''(\phi^*(x))\phi^{*\prime}(x)}\right|\right] \\ &- \frac{\bar{D}_0}{a_0}v''(\tilde{\eta}(x))\tilde{\eta}'(x)\log\left[\left|a_1 + \frac{2a_0}{v''(\tilde{\eta}(x))\tilde{\eta}'(x)}\right|\right] \\ &- \frac{2\bar{D}_0}{a_1}\log\left[\left|\frac{2a_0 + a_1v''(\tilde{\eta}(x))\tilde{\eta}'(x)}{2a_0 + a_1v''(\phi^*(x))\phi^{*\prime}(x)}\right|\right]\Bigg] \end{aligned} \tag{177}$$

where $\tilde{\eta}(x)$ is solution of the differential equation:

$$\begin{aligned} \eta(x) = &-\bar{\Lambda}_0 \\ &+ \frac{\bar{D}_0\left[(k_1D(\tilde{\eta}(x)) - D'(\tilde{\eta}(x))\chi(\tilde{\eta}(x)))\tilde{\eta}'(x)^2 - D(\tilde{\eta}(x))\chi(\tilde{\eta}(x))\tilde{\eta}''(x)\right]}{\tilde{\eta}'(x)D(\tilde{\eta}(x))\left[a_0\chi(\tilde{\eta}(x)) - a_1D(\tilde{\eta}(x))\tilde{\eta}'(x)\right]} \end{aligned} \tag{178}$$

for any given $\eta(x)$ field.

III.4. Diffusive Dynamics and $(n, m, l, s) = (0, 0, 1, 1)$

In this case we choose $a = a(\phi(x), \tilde{\phi}(x))$ and $\phi(x) = f(\phi(x), \tilde{\phi}(x), \tilde{\phi}'(x))$. The corresponding operators (eqs. (48,49,50,51)) have the form:

$$
\begin{aligned}
P_0(x) &= \frac{\partial a(u_0, v_0)}{\partial u_0}\bigg|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} \\
Q_0(x) &= \frac{\partial a(u_0, v_0)}{\partial v_0}\bigg|_{\substack{u_0=\phi(x)\\ v_0=\tilde{\phi}(x)}} \\
L_1(x) &= \frac{\partial f(u_0, v_0, v_1)}{\partial u_0}\bigg|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} - \frac{d}{dx} \\
S_1(x) &= \frac{\partial f(u_0, v_0, v_1)}{\partial v_0}\bigg|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} + \frac{\partial f(u_0, v_0, v_1)}{\partial v_1}\bigg|_{\substack{u_0=\phi(x)\\ \underline{v}=\underline{\tilde{\phi}}(x)}} \frac{d}{dx}
\end{aligned}
\tag{179}
$$

The conditon (C1T) is fulfilled by construction. Condition (C3T) is to ask the second order differential operator $S_1(x)(\partial_{v_0} a)^{-1} L_1(x)^\dagger$ to be self-adjoint. The strategy to unveil the form of the funtions a and f is:

- (1) Condition (C3T), order $\tilde{\phi}^{(2)}$:

$$
\frac{\partial^2 f(u_0, v_0, v_1)}{\partial v_1^2} = 0 \Rightarrow f(u_0, v_0, v_1) = f_0(u_0, v_0) + v_1 f_1(u_0, v_0) \tag{180}
$$

- (2) Rest of conditon (C3T):

$$
f_0(u_0, v_0)\frac{\partial f_1(u_0, v_0)}{\partial u_0} = \frac{\partial f_0(u_0, v_0)}{\partial v_0} + f_1(u_0, v_0)\frac{\partial f_0(u_0, v_0)}{\partial u_0} \tag{181}
$$

- (3) Condition (EM), order $\tilde{\phi}^{(3)}$: we found two possibilities,

$$
\begin{aligned}
f_1(u_0, v_0) &= -\frac{\frac{\partial a(u_0, v_0)}{\partial v_0}}{\frac{\partial a(u_0, v_0)}{\partial u_0}} \\
f_1(u_0, v_0) &= \frac{\chi(u_0)\frac{\partial a(u_0, v_0)}{\partial v_0}}{2D(u_0) - \chi(u_0)\frac{\partial a(u_0, v_0)}{\partial u_0}}
\end{aligned}
$$

One can show that the first condition drive us to the case $(n, m, l, s) = (0, 0, 0, 0)$.

- (4) Condition (EM), order $\tilde{\phi}^{(2)}$: we get a partial differential equation on $f_0(u_0, v_0)$ that, together eq.(181) implies that a solution is found for:

$$\begin{aligned} a(u_0, v_0) &= \tilde{a}(u_0) - \tilde{a}(v_0) \\ f_0(u_0, v_0) &= c_0 \frac{\chi(u_0)}{2D(u_0) - \chi(u_0)\tilde{a}'(u_0)} \\ f_1(u_0, v_0) &= -\frac{\tilde{a}'(v_0)\chi(u_0)}{2D(u_0) - \chi(u_0)\tilde{a}'(u_0)} \end{aligned} \tag{182}$$

  This implies that (C3T) is fulfilled.

- (5) Rest of condition (EM):

$$c_0(c_0 - 2E)\chi(u_0)^2(c_0 - v1\tilde{a}'(v_0))\chi''(u_0) = 0 \tag{183}$$

  There are three possibilities: $c_0 = 0$, $c_0 = 2E$ or $\chi''(u_0) = 0$. In order to elucidate which one of these is the correct, we apply our equation to the stationary state. First we know that $\phi^*(x) = \tilde{\phi}^*(x)$ because $a(\phi^*(x), \tilde{\phi}^*(x)) = 0$. Then, we apply this to our equation $u_1 = f(u_0, v_0, v_1)$ and we find that $c_0 = 2\phi^{*'}(x)D(\phi^*(x))/\chi(\phi^*(x))$. By other hand we know that the stationary state is solution of $\phi^{*'}(x)D(\phi^*(x))/\chi(\phi^*(x)) = E - J/\chi(\phi^*(x))$ that implies $\chi(u_0) = \chi_0$. This is coherent with one of the conditions in eq.(183) and (EM) is fulfilled. Finally, we can write

$$\phi'(x)[2D(\phi(x)) - \chi_0\tilde{a}'(\phi(x))] = 2(E\chi_0 - J) - \chi_0\tilde{\phi}'(x)\tilde{a}'(\tilde{\phi}(x)) \tag{184}$$

  that it can be integrated:

$$\tilde{a}(\phi(x)) - \tilde{a}(\tilde{\phi}(x)) = \frac{2}{\chi_0}\int_{\phi^*(x)}^{\phi(x)} du D(u) \tag{185}$$

  The quasipotential can be calculated following similar steps as in the above cases. We know that V_0 can be written as:

$$V_0[\eta] = V_0[\phi^*] + I_1 - I_2 \tag{186}$$

  where

$$\begin{aligned} I_1 &= \int_\Lambda dx \left[\int_{\phi^*(x)}^{\eta(x)} du\, \tilde{a}(u) - \eta(x)\tilde{a}(\tilde{\eta}(x)) + \phi^*(x)\tilde{a}(\phi^*(x))\right] \\ I_2 &= \int_0^1 d\lambda \int_\Lambda dx\ (\eta(x) - \phi^*(x))\,\phi(x;\lambda)\tilde{a}'(\tilde{\phi}(x;\lambda)) \end{aligned} \tag{187}$$

I_2 can be simplified by using eq.(185) with $\tilde{\phi}(x;\lambda)$ and $\phi(x;\lambda)$ and after doing a λ derivative in it we get the relation:

$$\tilde{a}'(\tilde{\phi}(x;\lambda))\,(\eta(x)-\phi^*(x))=\frac{d\phi(x;\lambda)}{d\lambda}\left(\tilde{a}'(\phi(x;\lambda))-\frac{2}{\chi_0}D(\phi(x;\lambda))\right) \tag{188}$$

Finally, we do the change of variables $\lambda \to \phi(x;\lambda)$ at each x and we find:

$$I_2=\int_\Lambda dx\int_{\phi^*(x)}^{\eta(x)}du\,u\left[\tilde{a}'(u)-\frac{2}{\chi_0}D(u)\right] \tag{189}$$

Putting together I_1 and I_2 we get:

$$V_0[\eta]=V_0[\eta^*]+\frac{2}{\chi_0}\int_\Lambda dx\int_{\phi^*(x)}^{\eta(x)}du\int_{\phi^*(x)}^{u}dvD(v) \tag{190}$$

for any $D(u)$ and $\chi(u)=\chi_0$.

III.5. Reaction-Diffusion Dynamics and $(n,m,l,s)=(0,0,0,0)$

As in the case of Diffusion Dynamics, conditions (C1T) and (C3T) are fulfilled by construction. The condition (EM) is analysed order by order on the derivatives of $\tilde{\phi}$:

- Condition (EM), order $\tilde{\phi}^{(2)}$:

$$a(u_0,v_0;u_s)=\frac{C}{g(u_0)} \tag{191}$$

This is a singular case because at the stationary state $a=0$ and therefore $g(u_s)^{-1}=0$. Let us assume that

$$g(u)=\bar{g}(u)(u-u_s)^{-\alpha}\quad \alpha>0 \tag{192}$$

This is coherent whenever the stationary state is a constant: $\phi^*(x)=\phi^*$ (because we are not considering in this paper local functions of g, w or h). Observe that with this election the deterministic evolution equation is:

$$\dot{\phi}_D(x,t)=\bar{g}(\phi_D(x,t))(\phi_D(x,t)-\phi^*)^{-\alpha}\phi_D''(x,t)+w(\phi_D(x,t)) \tag{193}$$

Near the equilibrium, $\phi(x,t)=\phi^*+\epsilon(x,t)$ the dominant terms of this equation for very small values of ϵ can be written:

$$\partial_t\epsilon(x,t)=w'(\phi^*)\epsilon(x,t)+\epsilon^{-\alpha}\epsilon(x,t)^{-\alpha}\bar{g}(\phi^*)\partial_{xx}^2\epsilon(x,t) \tag{194}$$

where ϕ^* is solution of $w(\phi^*) = 0$. The diffusion term is singular but it goes very fast to zero because it behaves as if the system had almost an infinite diffusivity and, therefore, it homogenizes any initial profile very fast in such a way that the spatial second derivative becomes zero. The reaction term makes ϵ to evolve exponentially fast towards zero whenever $w'(\phi^*) < 0$. Therefore this singular system is well behaved near the stationary solution.

- The rest of the (EM) condition is fulfilled when:

$$h^2(u) = -2g(u)w(u) \tag{195}$$

One can check that with the form of a and h, the hamiltonian $H[\phi, \pi] = 0$ along the trajectory.

The quasipotential can we written as

$$V_0[\eta] = V_0[\phi^*] + C \int_0^1 d\lambda \int_\Lambda dx \, \frac{(\tilde{\eta}(x) - \phi^*(x))}{g(f(\tilde{\phi}(x;\lambda)))} \frac{\partial f(v;\phi^*)}{\partial v}\bigg|_{v=\tilde{\phi}(x;\lambda)} \tag{196}$$

where $\phi[\tilde{\phi}; x] = f(\tilde{\phi}(x); \phi^*(x))$ and $\tilde{\phi}(x;\lambda) = \phi^*(x) + \lambda(\tilde{\eta}(x) - \phi^*(x))$. We do the change of variable $\lambda \to \tilde{\phi}(x;\lambda)$ at each x point and we get

$$V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \int_{\phi^*}^{\eta(x)} \frac{du}{g(u)} \tag{197}$$

III.6. Poissonian Reaction-Diffusion Dynamics and $(n, m, l, s) = (0, 0, 0, 0)$

Conditions (CT1) and (CT3) are fulfilled by construction. The first non trivial order in (EM) condition is order $\tilde{\phi}^{(2)}$ where there are two possibilities but only one is interesting:

$$f'(v_0) - f(v_0)(1 - f(v_0)) \frac{da(f(v_0), v_0)}{dv_0} = 0 \Rightarrow a(u_0, v_0) = \log \frac{u_0}{1 - u_0} + C \tag{198}$$

where C is a constant that is fixed by the condition: $a(\phi^*, \phi^*) = 0$: $C = \log[(1 - \phi^*)/\phi^*]$. We use this result and we get the rest of the (EM) condition:

$$\begin{aligned} &u_0^2(1-\phi^*)^2 b(u_0) - \phi^{*2}(1-u_0)^2 d(u_0) \\ &= u_0(1-u_0)(u_0 - \phi^*)\left[\phi^*(1-u_0)d'(u_0) - u_0(1-\phi^*)b'(u_0)\right] \end{aligned} \tag{199}$$

The solution of this equation is:

$$b(u) = \phi^*(1-u)h(u) \quad , \quad d(u) = (1-\phi^*)uh(u) \tag{200}$$

where $h(u)$ is a positive function.

Once we know $a(u_0, v_0)$ and the functions $b(u)$ and $d(u)$ that fulfills the conditions we can compute the quasipotential. We follow almost the same steps as we did for the Diffusion case for $(n, m, l, s) = (0, 0, 0, 0)$ and we get

$$V_0[\eta] = V_0[\phi^*] + \int_\Lambda dx \left[\eta(x) \log \left[\frac{\eta(x)}{\phi^*} \right] + (1 - \eta(x)) \log \left[\frac{1 - \eta(x)}{1 - \phi^*} \right] \right] \tag{201}$$