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# Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations

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**Abstract.** We assume that a system at a mesoscopic scale is described by a field  $\phi(x, t)$  that evolves by a Langevin equation with a white noise whose intensity is controlled by a parameter  $1/\sqrt{\Omega}$ . The system stationary state distribution in the small noise limit ( $\Omega \rightarrow \infty$ ) is of the form  $P_{st}[\phi] \simeq \exp(-\Omega V_0[\phi])$ , where  $V_0[\phi]$  is called the *quasipotential*.  $V_0$  is the unknown of a Hamilton–Jacobi equation. Therefore,  $V_0$  can be written as an action computed along a path that is the solution from Hamilton’s equation that typically cannot be solved explicitly. This paper presents a theoretical scheme that builds a suitable canonical transformation that permits us to do such integration by deforming the original path into a straight line and including some weights along with it. We get the functional form of such weights through conditions on the existence and structure of the canonical transformation. We apply the scheme to get the quasipotential algebraically for several one-dimensional nonequilibrium models as the diffusive and reaction–diffusion systems.

**Keywords:** macroscopic fluctuation theory, stationary states, fluctuating hydrodynamics

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## 1. Introduction

Thermodynamics show that many macroscopic properties of systems at equilibrium states are related to each other through a thermodynamic potential. Once we know, for instance, the entropy for a one-component system as a function of energy and mass density, we can deduce many other observables: specific heat, compressibility, pressure, or temperature. However, thermodynamics do not give us the explicit form of a system's thermodynamic potential. The equilibrium statistical mechanics solve this problem by introducing the Gibbs invariant measure for the microscopic degrees of freedom. In our opinion, the elegant part of this connection between the microscopic and macroscopic descriptions is that the Gibbs measure depends on the object that defines the full microscopic system dynamics: the Hamiltonian. Therefore, all of the system's dynamical microscopic details are summed up and contained in the thermodynamic potentials.

We see that systems at equilibrium have a complete set of theories that allow us to address much interesting macroscopic behavior, for instance, phase transitions.

However, systems at equilibrium are not the most common states in nature. Typically, the systems at stationary states contain currents of any type, energy, mass, etc, that appear due to unbalanced boundary conditions and/or the effect of external agents that induce some driving. From the microscopic point of view, few things, but very relevant ones, have changed compared with the equilibrium case: we typically have a system of interacting particles whose dynamics are still Hamiltonian except that we now include on it the dynamical effects from the boundaries and/or the external agents. These apparently small changes break down the theories that apply to systems at equilibrium. First, we do not know how to build a complete macroscopic theory similar to thermodynamics. Nevertheless, there have been many efforts to justify the existence of some intermediate or mesoscopic descriptions as the Boltzmann equation or macroscopic ones, such as the Navier–Stokes equations for fluids [1]. Second, the natural invariant measure defined on the phase space is of no practical use. For instance, we can use the SRB measure at the non-equilibrium attractor when the system is ‘very’ chaotic [2]. It is expected that the volume of the nonequilibrium attractor is zero due to dissipation. Still, it could be assumed to be dense in phase space when the degrees of freedom tend to infinity. However, the attractor’s topological structure typically depends strongly on the overall system dynamic trajectories, and therefore it is unknown [3]. We should compare this complex structure with, for example, the microcanonical measure at equilibrium where it is constant on the ‘*a priori*’ well-known attractor that is the equal energy manifold  $H(x, p) = E$ .

Some of these problems may be circumvented by studying systems with Markovian dynamics. The attractors are compact sets that depend on the physical constraints of the variables. Therefore, many of the complexities associated with the attractor topology go away compared with nonequilibrium particle systems. There have been many efforts to elucidate general properties of such non-equilibrium systems on lattice models: voter model, contact process, exclusion process, etc [4]. For many years the stationary measure typically could only be obtained in a few simple cases, such as the zero-dimensional stochastic models, systems with local detailed balance condition [5] or in the thermodynamic limit of the KMP model for heat conduction [6]. A breakthrough took place by the rigorous derivation of the stationary probability for the one-dimensional boundary driven symmetric simple exclusion model (SSE) by Derrida *et al* [7]. In SSE, each site can be empty or with one particle. The dynamics are very simple: a randomly chosen particle may hop to an empty neighbor’s site with some given probability. However, at the two boundaries, there are exit and incoming probability rates that may be different. Therefore, it may create a net current of particles through the system. When the number of lattice sites,  $N$ , tend to infinity and the density at the boundaries is fixed and given by  $\phi(0) = \phi_0$  and  $\phi(1) = \phi_1$ ,  $\phi_0 > \phi_1$ , Derrida *et al* deduced that the probability to find a density profile  $\eta = \{\eta(x), \forall x \in [0, 1]\}$ , where  $x \in [0, 1]$  and  $\eta(x) \in [0, 1]$ , is given by a large deviation functional:

$$P[\eta] \simeq \exp[-N(V_0[\eta])] \quad (N \rightarrow \infty), \quad (1)$$

Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations where  $V_0[\eta]$  is called *quasipotential* and it is given by:

$$V_0[\eta] = V_0[\phi^*] + G[\eta, \tilde{\eta}] \tag{2}$$

with

$$G[\eta, \tilde{\eta}] = \int_0^1 dx \left[ \eta(x) \log \left[ \frac{\eta(x)}{\tilde{\eta}(x)} \right] + (1 - \eta(x)) \log \left[ \frac{1 - \eta(x)}{1 - \tilde{\eta}(x)} \right] + \log \left[ \frac{\tilde{\eta}'(x)}{\phi_1 - \phi_0} \right] \right], \tag{3}$$

where  $\tilde{\eta}(x)$  is an auxiliary function that is the solution of the second order differential equation

$$\eta(x) = \tilde{\eta}(x) + \tilde{\eta}(x)(1 - \tilde{\eta}(x)) \frac{\tilde{\eta}''(x)}{\tilde{\eta}'(x)^2} \tag{4}$$

with boundary conditions:  $\tilde{\eta}(0, 1) = \phi_{0,1}$  and the stationary profile is given by  $\phi^*(x) = \phi_0 + x(\phi_1 - \phi_0)$ . They also mention that all of the  $\tilde{\eta}(x)$  functions such that

$$\frac{\delta G[\eta, \tilde{\eta}]}{\delta \tilde{\eta}(x)} = 0 \tag{5}$$

are the ones that solve the differential equation (4), which is a bit intriguing. The reader can find at [7] many interesting properties and insights of this explicit quasipotential as, for instance, the existence, uniqueness, and monotonicity of the solutions  $\tilde{\eta}(x)$  from the differential equation (4).

Let us remind that the quasipotential for systems at equilibrium is directly related to the free energy functional that is a *thermodynamic potential*. That makes  $V_0$  a fascinating object to analyze when looking for a non-equilibrium thermodynamic theory (if possible). For instance, let us focus just on the mathematical structure of Derrida’s result. Please observe that the quasipotential seems to be a local functional as it happens at equilibrium. However, the auxiliary field that is the solution of the second-order differential equation depends on the given  $\eta(x)$  in a non-local way and on the boundary conditions. In our opinion, the elegant and inspiring part of this solution is how the non-local behavior that is typical in many non-equilibrium stationary states is mathematically codified.

Later, using similar techniques, Enaud and Derrida [8] obtained the quasipotential for the boundary-driven asymmetric simple exclusion process (ASEP) for the driving field aligned with the density gradient. They obtained a quasipotential’s mathematical form similar to the SSEP case showed above: a local functional depending on an auxiliary field and a second-order differential equation for it. The strategy of studying lattice models with Markovian dynamics was successful, and it gave us an important reference on the structure of the quasipotentials. However, up to our knowledge, these results based on Derrida’s matrix technique for one-dimensional models with exclusion process are the unique ones where the quasipotentials have been exactly derived.

A simplified formal path towards the quasipotential study was already on the stake by using Fokker–Planck descriptions of non-equilibrium situations [5]. In the context

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of non-equilibrium many-body systems, it is assumed that there are a set of macroscopic fields that evolve following known deterministic dynamics and a weak stochastic term (typically white noise) reminiscence of the microscopic fluctuations. Let us mention the pioneering work of Graham and Tél [9], where the general properties of the quasipotential for systems with finite degrees of freedom were studied. For instance, they introduced the possibility of a Lagrangian transition that would imply the non-differentiability of the quasipotential in some regions of configurational space. In fact, this property seems to be natural for many systems at non-equilibrium stationary states. Let us also point out that they also developed a gradient expansion of the quasipotential for the supercritical complex Ginzburg–Landau equation [10]. These works have been applied with great success in several models and fields. Let us remark here just its use in the study of biological systems where the quasipotentials give a complete description of, for instance, the most probable path that a complex network of chemical reactions follows to go from a local minimum to another one [11, 12]. Let us finally mention the seminal works from Donsker and Varadhan [13] that developed the mathematical theory of large deviations (TLD) for Markov processes. TLD gives mathematical support on the existence and properties of the rate function that, in our context, is the quasipotential (see the interesting review by Touchette about TLD in statistical mechanics [14]).

Bertini and co-workers introduced further improvement by formulating the macroscopic fluctuation theory (MFT) [15]. MFT was formulated based on many previous rigorous results connecting microscopic stochastic lattice models with their corresponding macroscopic dynamical equations. Large deviation formulas were also obtained, and thus, the mesoscopic description of such systems. They compiled all of this information to define a theory that is a generalization of the already known fluctuating hydrodynamics [16]. That is, systems described by hydrodynamic continuum fields that evolve following a Langevin-like equation. First, they computed the quasipotential for the zero-range model, and they found that it is local. Moreover, they applied MFT to the continuum mesoscopic version of the SSE model. They found that the quasipotential obtained by Derrida *et al* (equations (2) and (3)) was also the solution of the Hamilton–Jacobi equation that defined the quasipotential in MFT. Bertini *et al* [17] obtained the quasipotential for the mesoscopic version of the Kipnis, Marchioro, Presutti model for heat conduction (KMP) [6]. In fact, they proposed a functional  $G[\eta, \tilde{\eta}]$  inspired by equation (3), and they showed that it was the solution of the corresponding Hamilton–Jacobi equation from MFT. They also found that the quasipotential for the boundary driven ASEP from Enaud and Derrida [8] was also the solution for the MFT [18] and expanded such a result when the drift due to the external field is strong enough, and it points against that due to the density gradient [19]. They explicitly found a Lagrangian transition, that is, the quasipotential has non-differential behavior in this case. All of these results showed that MFT had solid theoretical grounds to describe non-equilibrium systems at the mesoscopic level correctly. Let us mention that the quasipotentials study is just a part of the set of properties of non-equilibrium systems that MFT describes self-consistently. A fascinating review of many aspects that MFT sheds some light on can be found in [20].

We have seen that the exact results from Derrida *et al* using their matrix method and the inspired works from Bertini *et al* by defining MFT have opened the way for a deep

Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations understanding of the quasipotential’s mathematical structure. However, to go beyond this point, new insights that permit us to study more systematically other systems or/and dimensions are needed. This paper focuses on looking for a general algebraic method to obtain the quasipotential from MFT.

As we will see, in the MFT context, the quasipotential is the solution of a Hamilton–Jacobi equation of the form:

$$H \left[ \eta, \frac{\delta V_0[\eta]}{\delta \eta} \right] \equiv \int_{\Lambda} dx \bar{h} \left[ \eta(x), \frac{\delta V_0[\eta]}{\delta \eta(x)} \right] = 0, \quad (6)$$

where  $\bar{h}$  is a local functional on the arguments. Formally, this equation is solved by using the method of characteristics [21]. That is,  $H[\eta, \pi]$  is assumed to be a Hamiltonian that defines a dynamical system, where  $\pi(x)$  is the conjugate field to  $\phi(x)$ . The quasipotential  $V_0[\eta]$  is then given by:

$$V_0[\eta] = V_0[\phi^*] + \int_{-\infty}^0 dt \int_{\Lambda} dx \pi(x, t) \partial_t \phi(x, t), \quad (7)$$

where the fields  $(\phi(x, t), \pi(x, t))$  are the solution of the Hamilton’s equations associated to the Hamiltonian  $H[\phi, \pi]$ , where  $(\phi(x, -\infty), \pi(x, -\infty)) = (\phi^*(x), 0)$  and  $(\phi(x, 0), \pi(x, 0)) = (\eta(x), \pi(x))$ .

Except for trivial cases, it is unknown how to solve the Hamilton equations to get the trajectory  $(\phi(t), \pi(t))$ . Therefore, we are unable to compute the time integral to get  $V_0$ . At this point, the need to solve the Hamilton equations induced us to ask the following question: is it possible to define a canonical transformation  $(\phi, \pi) \rightarrow (\tilde{\phi}, \tilde{\pi})$  such that, in the new variables, we can make the time-integral get  $V_0$ ?

In this paper, we explore this idea by using a type 1 canonical transformation defined by the generator of the transformation  $L[\phi, \tilde{\phi}]$ . It is impossible to write down an  $L$  generator such that in the new variables, we could do the time integrals explicitly. Therefore, we first assume the existence of a map  $\phi(x) = \phi[\tilde{\phi}; x]$  between the Hamiltonian paths at each  $t$ . We show that under this assumption, the quasipotential can be obtained by a parametric integral that connects, by a straight line, the stationary state  $\phi^*$  with  $\eta$ . However, the integral is now weighted by two unknown functionals. We assume explicit analytical forms for the unknown functionals. We determine them from a set of compatibility conditions to fulfill to be part of a well-defined canonical transformation. As we will see, the interesting part of this method is that we do not need to solve any differential equation. This scheme is imperfect because we should also restrict the original dynamical model to some concrete forms for each trial functionals form. Moreover, the pair, model, and elected functionals do not always have a solution, and therefore this is a digging-like method to find some gold nuggets. Nevertheless, we reproduce all of the known quasipotentials with this method, and we discover some new ones.

We present all of these results in the following manner. In section 2, we define the Langevin dynamics of the system and fast review how to get the quasipotential and some concepts that we will use. Moreover, we define the one-dimensional models that we study explicitly in the paper. In section 3, we do the canonical transformation and see how the quasipotential’s formal solution is affected. We introduce the necessity for the map between  $\phi$  and  $\tilde{\phi}$ . We derive the quasipotential that appears to depend on two



Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations unknown functionals. We find the conditions they should follow to be part of a well-defined canonical transformation. Finally, as an example, we get the quasipotential for the zero-range model by using our method. Section 4 generalizes the method by defining some general functional forms for the unknown functionals and expressing their compatibility conditions on operational form. Section 5 is devoted to obtaining quasipotentials using our method for the one-dimensional diffusive system. In section 6, we derive some quasipotentials for one-dimensional reaction–diffusion models.

## 2. The quasipotential for a Langevin description of mesoscopic systems and models studied

We assume that our systems at a mesoscopic level of description are characterized by a unique scalar field  $\phi(x, t) \in IR$ , where  $x \in \Lambda \subset IR^d$ ,  $d$  is the spatial dimension and  $t$  is the time. We have initially restricted ourselves to this case in this paper for the sake of simplicity. Still, one can straightforwardly generalize all of the results below to systems described by vector fields. The system dynamics is given by a mesoscopic Langevin equation with a white noise. For instance, in the case of reaction dynamics (RD) it is:

$$\partial_t \phi(x, t) = F[\phi(t); x] + h[\phi(t); x] \xi(x, t), \tag{8}$$

where we use in this paper the notation  $W[\phi, \psi, \dots; x]$  to indicate a local functional that may depend on the fields:  $\phi(x)$ ,  $\psi(x)$ , their first derivatives with respect to the arguments  $x$ :  $\nabla_x \phi(x)$ ,  $\nabla_x \psi(x)$ , higher derivatives and even they may have more complex structures as local integrals of the fields over domains around  $x$ . Moreover, a parametric dependence on the fields, for instance the time  $t$ , is written as  $W[\phi(t); x]$  meaning that  $W$  is a local functional that depend on  $\phi(x, t)$ , their derivatives on  $x$  or any other  $x$ -functional dependence. Finally,  $\xi(x, t)$  is an uncorrelated Gaussian random field:

$$\begin{aligned} \langle \xi(x, t) \rangle &= 0 \\ \langle \xi(x, t) \xi(x', t') \rangle &= \frac{1}{\Omega} \delta(x - x') \delta(t - t') \end{aligned} \tag{9}$$

and we follow Ito’s scheme. The dynamics become deterministic when  $\Omega \rightarrow \infty$ :

$$\partial_t \phi_D(x, t) = F[\phi_D(t); x]. \tag{10}$$

We assume along this paper that the deterministic dynamics have a unique stationary state and that it is locally stable:

$$F[\phi^*; x] = 0, \quad \phi^*(x) = \lim_{t \rightarrow \infty} \phi_D(x, t) \tag{11}$$

for almost any initial state  $\phi_D(x, 0) = \phi_0(x) \in \Lambda$ . Our system may have periodic boundary conditions ( $\phi(x + L) = \phi(x)$  with  $L$  being the vector defining the basic cell), fixed boundary conditions ( $\phi(x, t) = f_0(x), \forall x \in \partial\Lambda$ ) or a mixture of both.

When the noise intensity is very small, the stationary probability distribution is of the form:

$$P_{st}[\eta] \simeq \exp[-\Omega V_0[\eta]] \quad (\Omega \rightarrow \infty), \quad (12)$$

where  $V_0[\eta]$  is the so-called *quasipotential*. It is well known [9, 22] that  $V_0$  is the solution of the *Hamilton–Jacobi equation*:

$$H \left[ \phi, \frac{\delta V_0[\phi]}{\delta \phi} \right] \equiv \int_{\Lambda} dx \frac{\delta V_0[\phi]}{\delta \phi(x)} \left[ F[\phi; x] + \frac{1}{2} \frac{\delta V_0[\phi]}{\delta \phi(x)} h[\phi; x]^2 \right] = 0 \quad (13)$$

with boundary conditions:

$$\left. \frac{\delta V_0[\phi]}{\delta \phi(x)} \right|_{x \in \partial \Lambda} = 0. \quad (14)$$

The formal solution can be obtained from the expression:

$$V_0[\eta] = V_0[\phi^*] + \int_{-\infty}^0 dt \int_{\Lambda} dx \pi(x, t) \partial_t \phi(x, t), \quad (15)$$

where the fields  $(\pi(x, t), \phi(x, t))$  are the solution of the Hamilton’s equations:

$$\begin{aligned} \partial_t \phi(x, t) &= \frac{\delta H[\phi(t), \pi(t)]}{\delta \pi(x, t)} \\ \partial_t \pi(x, t) &= -\frac{\delta H[\phi(t), \pi(t)]}{\delta \phi(x, t)}, \end{aligned} \quad (16)$$

where the Hamiltonian  $H$  is defined by equation (13). Hamilton’s equations should be solved with the system’s spatial boundary conditions and with the time boundaries:  $(\phi(x, -\infty), \pi(x, -)) = (\phi^*(x), 0)$  and  $(\phi(x, 0), \pi(x, 0)) = (\eta(x), \pi(x)) \forall x \in \Lambda$ .

Let us point out some properties that we will use below:

- $\phi^*(x)$  is the absolute minimum of the quasi potential:

$$\left. \frac{\delta V_0[\phi]}{\delta \phi(x)} \right|_{\phi(x)=\phi^*(x)} = 0 \quad \forall x \in \Lambda. \quad (17)$$

That is so because in the strict limit  $\Omega \rightarrow \infty$  we should get the stationary deterministic solution (11). In other words:

$$P_{st}[\eta] = \prod_{x \in \Lambda} \delta(\eta(x) - \phi^*(x)). \quad (18)$$

- $H[\phi^*, 0] = 0$  by construction and therefore,  $H[\phi(t), \pi(t)] = 0$ .
- $\pi(x, t) = \delta V_0[\eta] / \delta \eta(x) |_{\eta=\phi(t)}$  from equation (15).
- Notice that for fixed boundary conditions:  $(\phi(x, t), \pi(x, t)) = (\phi(x), 0) \forall x \in \partial \Lambda, t$ , where we have included  $\pi(x, t) |_{x \in \partial \Lambda} = 0$ . This condition reflects that the boundary is thought of as an equilibrium thermal bath having the property  $\partial V_B(\phi) / \partial \phi(x) = 0$ , with  $V_B$  an equilibrium potential. That is,  $\pi_B(x) |_{x \in \partial \Lambda} = 0$  and,



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by continuity  $\pi_B(x) = \pi(x) \forall x \in \partial\Lambda$ . This choice, of course, affects the nature of the fluctuations about the system’s stationary state, but it has the advantage that it permits us to have systems at equilibrium. Then, just by changing the boundaries, we can create nonequilibrium stationary states.

- There can be more than one path solution of Hamilton’s equations that go from  $(\phi^*, 0)$  up to  $(\eta, \pi_n)$ , where  $n$  could change for each of the paths. Then it is implicitly understood that one should take in equation (15) the path that minimizes the value of  $V_0$ .
- For diffusive dynamics (DD) everything is equal except for the Hamiltonian (13) that in this case is:

$$H[\phi, \pi] = \int_{\Lambda} dx \nabla\pi(x) \cdot \left[ G[\phi; x] + \frac{1}{2}\chi[\phi; x]\nabla\pi(x) \right], \tag{19}$$

where  $G$  is the determinist part of the current and  $\chi$  is related with the noise intensity.

The above definitions and properties are well known in the literature, and that is why we pass through them fast. We refer the readers to [22] for the details about how the above expressions are derived for systems with RD and DD and several comments about the properties of the stationary state.

That is, the problem of finding  $V_0$  is formally solved. However, it is almost impossible at the practical level to obtain the solutions from Hamilton’s equations (16). This paper is devoted to building a strategy to be able to explicitly make the time-integral in equation (15). From now on, we are going to restrict ourselves to one-dimensional systems. The application of these ideas to larger dimensions is left for future works. Below, we explicitly apply the method to get  $V_0$  to the following one dimensional systems:

- Diffusive model:

This one dimensional model is defined by a field  $\phi(x, t)$  with  $x \in [0, 1]$  that evolves by the Langevin equation:

$$\partial_t\phi(x, t) + \frac{dj[\phi(t); x]}{dx}, \quad j[\phi(t); x] = G[\phi(t); x] + \sqrt{\chi(\phi(x, t))}\psi(x, t), \tag{20}$$

where  $\psi$  is an uncorrelated white noise and

$$G[\phi; x] = -D(\phi(x))\frac{d\phi}{dx} + \chi(\phi(x))E \tag{21}$$

$D(\lambda)$  and  $\chi(\lambda)$  are the diffusion and mobility functions respectively and  $E$  is a constant driving field. The Hamiltonian that defines the paths to build the quasipotential is given by equation (19) (see [22] and references therein):

$$H[\phi, \pi] = \int_{\Lambda} dx \frac{d\pi(x)}{dx} \left[ -D(\phi(x))\frac{d\phi(x)}{dx} + \chi(\phi(x)) \left( E + \frac{1}{2} \frac{d\pi(x)}{dx} \right) \right] \tag{22}$$

$D$  and  $\chi$  are designed in such a way that the system stationary state could be an equilibrium state with respect the potential:

$$V_{\text{eq}}[\phi] = V_{\text{eq}}[\phi^*] + \int_0^1 dx [v_{\text{eq}}[\phi; x] - v_{\text{eq}}[\phi^*; x]], \quad (23)$$

where

$$v_{\text{eq}}[\phi; x] = v(\phi(x)) - 2Ex\phi(x). \quad (24)$$

We can think of  $E$  as being a kind of gravitational force acting over a mass field  $\phi(x)$ . We can see that  $V_{\text{eq}}[\phi]$  is the solution of the Hamilton–Jacobi equation

$$H \left[ \phi, \frac{\delta V_{\text{eq}}[\phi]}{\delta \phi} \right] = 0 \quad (25)$$

when

$$D(\lambda) = \frac{1}{2}v''(\lambda)\chi(\lambda) \quad (26)$$

that it is called the *Einstein relation*. The equilibrium state is achieved when applying the appropriate boundary conditions:

$$\left. \frac{\delta V_{\text{eq}}[\phi]}{\delta \phi(x)} \right|_{x=0,1} = 0 \Rightarrow v'(\phi_0) = 0, \quad v'(\phi_1) = 2E, \quad (27)$$

where  $\phi(i) = \phi_i; i = 0, 1$ .

Finally, the equilibrium configuration is obtained from the deterministic part of the Langevin equation by asking that the current  $G$  equals to zero:

$$-D(\phi^*(x))\frac{d\phi^*(x)}{dx} + \chi(\phi^*(x))E = 0. \quad (28)$$

The solution of this equation is

$$\int_{\phi(0)}^{\phi^*(x)} d\phi \frac{D(\phi)}{\chi(\phi)} = Ex \quad (29)$$

and assuming that the Einstein relation holds, it can be written

$$v'(\phi^*(x)) = 2Ex, \quad (30)$$

where  $\phi^*(i) = \phi_i; i = 0, 1$ . Observe that the boundary conditions should be  $\phi_{0,1}$  (for a given  $E$ ) to be at an equilibrium state. When we choose any other different set of boundary conditions, the system develops a non-zero current, and the system is in a non-equilibrium stationary state with a quasipotential  $V_0[\phi] \neq V_{\text{eq}}[\phi]$ . The stationary state is then solution of

$$-D(\phi^*(x))\frac{d\phi^*(x)}{dx} + \chi(\phi^*(x))E = J, \tag{31}$$

where  $J$  is the current that it is determined by the boundary conditions. We can also get non-equilibrium stationary states with periodic boundary conditions and a non-zero driving field  $E$ . In this case  $\phi^*(x) = \phi^*$  and  $J = \chi(\phi^*)E$ .

- Reaction–diffusive models:

We study the one-dimensional reaction–diffusion model whose Langevin equation is given by (8) with

$$F[\phi; x] = g(\phi)\phi''(x) + w(\phi(x)) \tag{32}$$

with the Hamiltonian given by equation (13).

Another interesting model we have studied is the Poissonian reaction–diffusion dynamics. This mesoscopic model is deduced from a stochastic Markovian lattice model in which there is a competition between conservative exchange dynamics and a spin-flip one (see [24]). In the fast rate limit for the exchange dynamics and after some time and space rescaling, one obtains the deterministic equation:

$$\frac{\partial\phi_D(x, t)}{\partial t} = \frac{\partial^2\phi_D(x, t)}{\partial x^2} + b(\phi_D(x, t)) - d(\phi_D(x, t)), \tag{33}$$

where  $\phi_D(x)$  represents a normalized density:  $0 \leq \phi_D(x) \leq 1$  and  $b$  and  $d$  functions are directly related with the microscopic spin-flip dynamics. Moreover, the structure of the mesoscopic noise is represented by the Hamiltonian:

$$H[\phi, \pi] = \int_{\Lambda} dx [\pi(x)\phi''(x) + \pi'(x)\phi(x)(1 - \phi(x)) - b(\phi(x))(1 - e^{\pi(x)}) - d(\phi(x))(1 - e^{-\pi(x)})]. \tag{34}$$

Observe that this Hamiltonian is not quadratic in  $\pi$  as it was in equation (13). That is related to the Poissonian structure of the underlying noise. We will assume periodic boundary conditions. In this case, the stationary state is a constant solution of  $b(\phi^*) = d(\phi^*)$  that it is assumed to be unique.

### 3. A method to solve Hamilton–Jacobi equations

We want to get explicit solutions for the Hamilton–Jacobi equation (13) by using the formal solution (15) as a starting point. The main idea of our method is to find a canonical transformation under which we can explicitly do a time integral in (15). We will see that we can deform the integrating path to be, effectively, a straight line. We can do that under some (assumed mild) assumptions, such as the existence of a one-to-one transform between the original path and the new one coming from the canonical transformation. However, the transformed action integral (15) is weighted by two functionals. One is the canonical transformation’s functional derivative, and the other depends on the one-to-one transform. The key point of the method is to get both unknown functionals.

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We show that those functionals can be algebraically obtained by using the conditions about the existence of the canonical transformation and the use of the Hamiltonian equation of motion for the original and canonical transformed systems. Once we get such weights, we can straightforwardly compute the quasipotential. Let us develop the full strategy step by step.

### 3.1. The canonical transformation

Let us build a general *type 1* canonical transformation on a generic field Hamiltonian  $H(\phi, \pi)$  through the generator  $L[\phi, \tilde{\phi}]$ :

$$(\phi, \pi) \rightarrow (\tilde{\phi}, \tilde{\pi}) : \pi(x) = \frac{\delta L[\phi, \tilde{\phi}]}{\delta \phi(x)} \equiv A[\phi, \tilde{\phi}; x], \tilde{\pi}(x) = -\frac{\delta L[\phi, \tilde{\phi}]}{\delta \tilde{\phi}(x)} \equiv B[\phi, \tilde{\phi}; x]. \tag{35}$$

These equations define a one-to-one relationship between the two sets of variables during the system’s evolution under the Hamiltonian  $H$ . The quasipotential (15) is written in the new variables:

$$V_0[\eta] = V_0[\phi^*] + \int_{-\infty}^0 d\tau \int_{\Lambda} dx \left. \frac{\delta L[\phi, \tilde{\phi}]}{\delta \phi(x)} \right|_{\substack{\phi=\phi(\tau) \\ \tilde{\phi}=\tilde{\phi}(\tau)}} \partial_{\tau} \phi(x, \tau). \tag{36}$$

$L$  does not depend on  $t$  explicitly, and therefore we can use the relation:

$$\partial_t L[\phi(t), \tilde{\phi}(t)] = \int_{\Lambda} dx \left[ \left. \frac{\delta L[\phi, \tilde{\phi}]}{\delta \phi(x)} \right|_{\substack{\phi=\phi(t) \\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t \phi(x, t) + \left. \frac{\delta L[\phi, \tilde{\phi}]}{\delta \tilde{\phi}(x)} \right|_{\substack{\phi=\phi(t) \\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t \tilde{\phi}(x, t) \right] \tag{37}$$

to get

$$V_0[\eta] = V_0[\phi^*] + L[\eta, \tilde{\eta}] - L[\phi^*, \tilde{\phi}^*] + \int_{-\infty}^0 d\tau \int_{\Lambda} dx \tilde{\pi}(x, \tau) \partial_{\tau} \tilde{\phi}(x, \tau), \tag{38}$$

where the fields  $\tilde{\eta}$  and  $\tilde{\phi}^*$  are the canonical transformed  $\eta$  and  $\phi^*$ , respectively.

At this point, it could look like that we have not gained too much because we still have to do time integral to get the quasipotential. However, we have the possibility to design a convenient form for  $(\tilde{\phi}, \tilde{\pi})$  so that the integral in equation (15) can be done. Therefore, our next step is to look for the necessary assumptions to find such optimal transformation.

### 3.2. $V_0$ 's convenient form

In the transformed variables  $(\tilde{\phi}, \tilde{\pi})$ , we can define its quasipotential by equation (15),  $\tilde{V}_0[\tilde{\phi}]$ . Therefore,

$$\tilde{\pi}(x)|_{\tilde{T}} = \left. \frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} \right|_{\tilde{T}}, \tag{39}$$

where  $\tilde{T}$  represents any pair  $(\tilde{\phi}(x, t), \tilde{\pi}(x, t))$  that are the solution of Hamilton's equations (16) with the canonical transformed Hamiltonian:  $\tilde{H}(\tilde{\phi}, \tilde{\pi}) = H(\phi, \pi)$  and the corresponding boundary conditions. This implies two relevant properties:

- There exists a functional relation between the paths  $\phi(t)$  and  $\tilde{\phi}(t)$  solutions of the respective Hamilton's equation of motion:

$$\phi(x, t) = \phi[\tilde{\phi}(t); x]. \tag{40}$$

This can be seen by restricting the canonical transformation (35) to the trajectories and using equation (39):

$$\left. \frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} \right|_{\tilde{T}} = - \left. \frac{\delta L[\phi, \tilde{\phi}]}{\delta \tilde{\phi}(x)} \right|_{T, \tilde{T}}. \tag{41}$$

Therefore, from this implicit equation we assume that equation (40) exists at each  $t$  from the original and transformed Hamiltonian trajectories.

- $V_0$  in equation (15) can be written:

$$V_0[\eta] = V_0[\phi^*] + L[\eta, \tilde{\eta}] - L[\phi^*, \tilde{\phi}^*] + \tilde{V}_0[\tilde{\eta}] - \tilde{V}_0[\tilde{\phi}^*]. \tag{42}$$

Observe from equation (42) that the quasipotential is linearly related to the generator of the canonical transformation. This relation permits us to explain the existence of the extremal property (5) observed by Derrida *et al* in [7] (let us mention that a similar property was observed by Bertini and co-workers when studying the quasipotential associated with a model of heat flow [17]): from equation (42), we define  $G[\phi, \tilde{\phi}] = V_0[\phi^*] + L[\phi, \tilde{\phi}] + \tilde{V}_0[\tilde{\phi}] - \tilde{V}_0[\tilde{\phi}^*]$ . Then, the condition  $\delta G[\phi, \tilde{\phi}]/\delta \tilde{\phi}(x) = 0$  on  $T, \tilde{T}$ , is just equation (41) that relates the variables  $\phi$  with the transformed ones  $\tilde{\phi}$  and therefore  $\phi(x) = \phi[\tilde{\phi}; x]$  is the extremal solution of (5). This result implies that the auxiliary field defined in Derrida *et al*'s paper [7] is just the canonical transformed  $\phi$ -field.

These two properties allow us to get a more convenient form to compute the quasipotential algebraically. Let us define the *restricted* transformation  $\tilde{L}$  by substituting  $\phi$  by its relation with  $\tilde{\phi}$  along  $T$  in (40):

$$\tilde{L}[\tilde{\phi}] = L[\phi[\tilde{\phi}], \tilde{\phi}] \tag{43}$$

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$$\frac{\delta \tilde{L}[\tilde{\phi}]}{\delta \tilde{\phi}(x)} = -\frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} + \int_{\Lambda} dy A[\phi[\tilde{\phi}], \tilde{\phi}; y] K[\tilde{\phi}; y, x], \quad (44)$$

where

$$K[\tilde{\phi}; x, y] = \frac{\delta \phi[\tilde{\phi}; x]}{\delta \tilde{\phi}(y)}, \quad A[\phi, \tilde{\phi}; x] = \frac{\delta L[\phi, \tilde{\phi}]}{\delta \phi(x)}. \quad (45)$$

Assuming that  $\tilde{L}[\tilde{\phi}]$  exists (we will address this issue below), we know that (see appendix A):

$$\begin{aligned} \tilde{L}[\tilde{\eta}] &= \tilde{L}[\tilde{\phi}^*] - \tilde{V}_0[\tilde{\eta}] + \tilde{V}_0[\tilde{\phi}^*] + \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \tilde{\phi}^*(x)) \\ &\quad + \int_{\Lambda} dy A[\phi[\tilde{\phi}(\lambda)], \tilde{\phi}(\lambda); y] K[\tilde{\phi}(\lambda), y, x], \end{aligned} \quad (46)$$

where

$$\tilde{\phi}(x, \lambda) = \tilde{\phi}^*(x) + \lambda(\tilde{\eta}(x) - \tilde{\phi}^*(x)) \quad (47)$$

and the quasipotential (42) can be written on its *convenient form*:

$$V_0[\eta] = V_0[\phi^*] + \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \tilde{\phi}^*(x)) \int_{\Lambda} dy A[\phi[\tilde{\phi}(\lambda)], \tilde{\phi}(\lambda); y] K[\tilde{\phi}(\lambda); y, x]. \quad (48)$$

We see that the time integral in (15) has been ‘deformed’ by a straight path connecting the stationary state  $(\tilde{\phi}^*, 0)$  and the target state  $(\tilde{\eta}, \tilde{\pi})$ . Moreover, to get  $V_0$ , we do not need to know the full canonical transformation but the functionals  $A$  and  $K$  defined in equation (45). In expression (48) is hidden a practical problem: for a given canonical transformation  $L$ , we easily get the functional  $A$  (it is just a functional derivative of  $L$ ), but we cannot obtain the functional relation between  $\phi$  and  $\tilde{\phi}$  (equation (40)) because we should solve the Hamilton equations explicitly and get the paths to build the map. In conclusion, it seems that we are stuck with the same problem that we initially had. We cannot follow the natural but somehow impossible scheme of deriving the full canonical transformation and afterward get the functionals  $A$  and  $\phi[\tilde{\phi}; x]$ .

However, the relation (48) can be useful if we change our point of view. Let us assume that the functionals  $A$  and  $\phi[\tilde{\phi}; x]$  are given: what are their conditions to guarantee that they come from a well-defined canonical transformation? That would happen whenever (1)  $A$  fulfills the conditions for the existence of  $L$  and therefore for  $\tilde{L}$ , and (2)  $\phi(x, t) = \phi[\tilde{\phi}(t); x]$  is compatible with Hamilton’s equations of motion for  $\phi$  and  $\tilde{\phi}$ . Therefore, our starting point will initially assume a particular family of functional forms for  $A$  and  $\phi[\tilde{\phi}; x]$ . Then, we will apply those minimal conditions to determine their detailed functional structure and select the form of the functionals defining the Langevin equation. We will show that this strategy completely determines the functional  $A$  and  $\phi[\tilde{\phi}; x]$ . Moreover, we also can do all of those steps in a systematic algebraic manner.



### 3.3. Minimal conditions on $A$ and $\phi[\tilde{\phi}; x]$ functionals

We know that giving the two functionals  $A[\phi, \tilde{\phi}; x]$  and  $B[\phi, \tilde{\phi}; x]$  in equation (35), they may be considered as the first derivatives of  $L[\phi, \tilde{\phi}]$  if and only if their doubled crossed derivatives are independent on the applied order. That is:

$$\begin{aligned}
 \text{(C1): } & \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} = \frac{\delta A[\phi, \tilde{\phi}; y]}{\delta \phi(x)} \\
 \text{(C2): } & \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = \frac{\delta B[\phi, \tilde{\phi}; y]}{\delta \phi(x)} \\
 \text{(C3): } & \frac{\delta B[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = \frac{\delta B[\phi, \tilde{\phi}; y]}{\delta \phi(x)}.
 \end{aligned}
 \tag{49}$$

On the other hand, equation (41) implies that  $B[\phi, \tilde{\phi}; x]$  should be at least of the form:

$$B[\phi, \tilde{\phi}; x] = \frac{\delta \tilde{V}_0[\tilde{\phi}]}{\delta \tilde{\phi}(x)} + \int_{\Lambda} dy (\phi(y) - \phi[\tilde{\phi}; y]) B_{\perp}[\phi, \tilde{\phi}; x, y].
 \tag{50}$$

Using this expression for  $B$  we can write the conditions (C2) and (C3) in the following form:

$$\text{(C2')} : \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = B_{\perp}[\phi, \tilde{\phi}; y, x] + \int_{\Lambda} dz (\phi(z) - \phi[\tilde{\phi}; z]) \frac{\delta B_{\perp}[\phi, \tilde{\phi}; y, z]}{\delta \phi(x)}
 \tag{51}$$

$$\begin{aligned}
 \text{(C3')} : & \int_{\Lambda} dz \left[ K[\tilde{\phi}; z, y] B_{\perp}[\phi, \tilde{\phi}; x, z] - K[\tilde{\phi}; z, x] B_{\perp}[\phi, \tilde{\phi}; y, z] \right] \\
 & = \int_{\Lambda} dz (\phi(z) - \phi[\tilde{\phi}; z]) \left[ \frac{\delta B_{\perp}[\phi, \tilde{\phi}; x, z]}{\delta \tilde{\phi}(y)} - \frac{\delta B_{\perp}[\phi, \tilde{\phi}; y, z]}{\delta \tilde{\phi}(x)} \right].
 \end{aligned}
 \tag{52}$$

We can get a set of *necessary* conditions for the existence of  $L$  if we restrict them to the trajectory  $T$  where  $\phi(x) = \phi[\tilde{\phi}; x]$ :

$$\text{(C1T): } \left. \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} \right|_{\phi=\phi[\tilde{\phi}]} = \left. \frac{\delta A[\phi, \tilde{\phi}; y]}{\delta \phi(x)} \right|_{\phi=\phi[\tilde{\phi}]}
 \tag{53}$$

$$\text{(C2T): } \left. \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \right|_{\phi=\phi[\tilde{\phi}]} = B_{\perp}[\phi[\tilde{\phi}], \tilde{\phi}; y, x]
 \tag{54}$$

$$(C3T): \int_{\Lambda} dz \left[ K[\tilde{\phi}; z, y] \frac{\delta A[\phi, \tilde{\phi}; z]}{\delta \tilde{\phi}(x)} \Big|_{\phi=\phi[\tilde{\phi}]} - K[\tilde{\phi}; z, x] \frac{\delta A[\phi, \tilde{\phi}; z]}{\delta \tilde{\phi}(y)} \Big|_{\phi=\phi[\tilde{\phi}]} \right] = 0. \tag{55}$$

Please observe that conditions (C1T) and (C3T) only depend on the functionals  $A$  and  $K$ , the same ones that we need to get  $V_0$  in equation (48). We are interested in obtaining  $V_0$  and, therefore, we will use only (C1T) and (C3T). (C2T) becomes just a property that is of no use for our practical purposes. It is out of the scope of this work to attempt to rigorously prove the sufficient conditions (C1), (C2), and (C3) assuming (C1T), (C2T), and (C3T).

The condition on the  $\phi[\tilde{\phi}; x]$  functional is obtained by using the Hamilton’s equations of motion. We know that  $\phi$  follow a path that is the solution of Hamilton’s equations (16). Let us rewrite such equations using the canonical transformation (35) and substituting  $\phi(x)$  by  $\phi[\tilde{\phi}; x]$  and  $\pi(x)$  by  $A[\phi, \tilde{\phi}; x]$ :

$$\partial_t \phi[\tilde{\phi}(t); x] = \tilde{R}_1[\tilde{\phi}(t); x], \quad \partial_t A[\phi[\tilde{\phi}(t)], \tilde{\phi}(t); x] = \tilde{R}_2[\tilde{\phi}(t); x], \tag{56}$$

where

$$\begin{aligned} R_1[\phi, \tilde{\phi}; x] &= \frac{\delta H[\phi, \pi]}{\delta \pi(x)} \Big|_{\pi=A[\phi, \tilde{\phi}; x]} \\ R_2[\phi, \tilde{\phi}; x] &= - \frac{\delta H[\phi, \pi]}{\delta \phi(x)} \Big|_{\pi=A[\phi, \tilde{\phi}; x]} \end{aligned} \tag{57}$$

and  $\tilde{R}_{1,2}[\tilde{\phi}; x] = R_{1,2}[\phi[\tilde{\phi}], \tilde{\phi}; x]$ . We can now expand the time derivatives and we get:

$$\begin{aligned} \int_{\Lambda} dy \frac{\delta \phi[\tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \Big|_{\tilde{\phi}(t)} \partial_t \tilde{\phi}(y, t) &= \tilde{R}_1[\tilde{\phi}(t); x] \\ \int_{\Lambda} dy \left[ \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} \Big|_{\substack{\phi=\phi[\tilde{\phi}(t)] \\ \tilde{\phi}=\tilde{\phi}(t)}} \tilde{R}_1[\tilde{\phi}(t); y] + \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \Big|_{\substack{\phi=\phi[\tilde{\phi}(t)] \\ \tilde{\phi}=\tilde{\phi}(t)}} \partial_t \tilde{\phi}(y, t) \right] &= \tilde{R}_2[\tilde{\phi}(t); x]. \end{aligned} \tag{58}$$

These equations are combined to disregard their dependence on  $\partial_t \tilde{\phi}$ :

$$\begin{aligned} \int_{\Lambda} dy \int_{\Lambda} dz \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \Big|_{\phi=\phi[\tilde{\phi}]} \tilde{R}_1[\tilde{\phi}; z] K^{-1}[\tilde{\phi}; y, z] \\ = \tilde{R}_2[\tilde{\phi}; x] - \int_{\Lambda} dy \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} \Big|_{\phi=\phi[\tilde{\phi}]} \tilde{R}_1[\tilde{\phi}; y], \end{aligned} \tag{59}$$

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$$\int_{\Lambda} dy K[\tilde{\phi}; x, y] K^{-1}[\tilde{\phi}; y, z] = \delta(x - z). \quad (60)$$

Observe that we have dropped out the time dependence, considering that this relation for the functional  $\phi[\tilde{\phi}]$  holds along each point in the path. We can get a more convenient expression where  $K^{-1}$  disappears by integrating both sides by  $\int_{\Lambda} dx K[\tilde{\phi}; x, z]$  and using the (C3T) property above:

$$\begin{aligned} \text{(EM)} : \int_{\Lambda} dx & \left[ K[\tilde{\phi}; x, y] \tilde{R}_2[\tilde{\phi}; x] \right. \\ & \left. - \tilde{R}_1[\tilde{\phi}; x] \left( \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} \Big|_{\phi=\phi[\tilde{\phi}]} + \int_{\Lambda} dz K[\tilde{\phi}; z, y] \frac{\delta A[\phi, \tilde{\phi}; z]}{\delta \phi(x)} \Big|_{\phi=\phi[\tilde{\phi}]} \right) \right] = 0. \end{aligned} \quad (61)$$

(EM) is the minimal condition on the functionals  $A$  and  $\phi[\tilde{\phi}; x]$  to make them compatible with the Hamilton’s equations of motion.

Therefore, the set of necessary conditions over the  $A$  and  $\phi[\tilde{\phi}]$  functionals that we are to consider here are (C1T) (equation (53)), (C2T) (equation (54)) and (EM) (equation (61)). As we already said, to go further, we will propose some functional forms for both functionals. Let us do a particular example to illustrate how these conditions are enough to define the functionals  $A$  and  $\phi[\tilde{\phi}]$  and how we get the corresponding quasipotential.

### 3.4. The full method at work: an example

Let us choose as an example the one-dimensional diffusive model defined in section 2 by the Langevin equation (20) with the determinist part given by (21). Observe that the functions  $D(\phi)$  and  $\chi(\phi)$  are unspecified. The initial step for this method is to choose a concrete family of functionals for  $A$  and  $\phi[\tilde{\phi}; x]$ . In this example, we elect the simplest functional forms:

$$\begin{aligned} A[\phi, \tilde{\phi}; x] &= a(\phi(x), \tilde{\phi}(x), \phi^*(x)) \\ \phi(x) &= f(\tilde{\phi}(x), \phi^*(x)) \end{aligned}, \quad (62)$$

where  $a$  and  $f$  are functions to be determined and  $\phi^*(x)$  is the deterministic stationary state. We have included an explicit dependence on the stationary state because we know that  $\pi^*(x) = A[\phi^*, \tilde{\phi}^*; x] = 0$  and such a degree of freedom is necessary. We also should take into account that, at the fixed boundaries (if any),  $\pi(x) = A[\phi, \tilde{\phi}; x] = 0 \forall x \in \partial\Lambda$ . Let us remark that equation (62) is an arbitrary choice. We could have included any derivative of  $\phi(x)$  or  $\tilde{\phi}(x)$  on the arguments of both functions and/or any other more complex functional non-local structure. We will comment below about the choices that we are able to handle. Let us stress that *a priori*, there is no guarantee that a given

Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations family of functionals is going to be compatible with the set of conditions (C1T), (C3T), and (EM) above. Election (62) implies:

$$\begin{aligned} \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \phi(y)} &= a^{(1,0,0)}(u_0, v_0, u_0^*) \delta(x - y) \\ \frac{\delta A[\phi, \tilde{\phi}; x]}{\delta \tilde{\phi}(y)} &= a^{(0,1,0)}(u_0, v_0, u_0^*) \delta(x - y) \\ K[\tilde{\phi}; x, y] &= \frac{\delta \phi[\tilde{\phi}; x]}{\delta \tilde{\phi}(y)} = f^{(1,0)}(v_0, u_0^*) \delta(x - y), \end{aligned} \tag{63}$$

where  $a^{(n,m,l)}(u_0, v_0, u_0^*) \equiv \partial^{n+m+l} a(n, m, l) / \partial u_0^n \partial v_0^m \partial u_0^{*l}$ . In order to simplify notation, we use the convention that  $u_k \leftrightarrow \phi^{(k)}(x) = d^k \phi(x) / dx^k$ ,  $v_k \leftrightarrow \tilde{\phi}^{(k)}(x)$  and  $u_k^* \leftrightarrow \phi^{*(k)}(x)$ . The minimal conditions to have well-defined canonical transformations are, in this case, given by:

- (C1T):  $\delta A[\phi, \tilde{\phi}; x] / \delta \phi(y)$  is symmetric under the  $x, y$  exchange. Therefore, this condition is always fulfilled for this choice.
- (C3T):  $\delta A[\phi, \tilde{\phi}; x] / \delta \tilde{\phi}(y)$  is also symmetric under the  $x, y$  exchange. Therefore, this condition is always fulfilled for this choice.
- (EM): after we substitute equation (63) into the condition (61) we get:

$$f^{(1,0)}(v_0, u_0^*) \tilde{R}_2[\tilde{\phi}; x] = [a^{(0,1,0)}(u_0, v_0, u_0^*) + a^{(1,0,0)}(u_0, v_0, u_0^*) f^{(1,0)}(v_0, u_0^*)] \tilde{R}_1[\tilde{\phi}; x], \tag{64}$$

where  $u_0 = f(v_0, u_0^*)$ . Let us remind that  $\tilde{R}_{1,2}[\tilde{\phi}; x] = R_{1,2}[\phi[\tilde{\phi}], \tilde{\phi}; x]$  and  $R_{1,2}[\phi, \tilde{\phi}; x]$  are defined by equation (57) with the Hamiltonian (22). They have the form:

$$\begin{aligned} R_1[\phi, \tilde{\phi}; x] &= -\frac{d}{dx} \left[ -D(\phi(x)) \phi'(x) + \left( E + \frac{da(x)}{dx} \right) \chi(\phi(x)) \right] \\ R_2[\phi, \tilde{\phi}; x] &= -\chi'(\phi(x)) \frac{da(x)}{dx} \left( E + \frac{1}{2} \frac{da(x)}{dx} \right) - D(\phi(x)) \frac{d^2 a(x)}{dx^2}, \end{aligned} \tag{65}$$

where  $a(x) \equiv a(\phi(x), \tilde{\phi}(x); \phi^*(x))$ .

After doing all of the functional substitutions and spatial derivatives, the unique non-trivial condition (EM) is a differential equation of the form:

$$c_3(v_0, u_0^*) v_2 + c_2(v_0, u_0^*, u_1^*) v_1^2 + c_1(v_0, u_0^*, u_1^*) v_1 + c_0(v_0, u_0^*, u_1^*) = 0, \tag{66}$$

where we are using our simplified notation  $v_k \leftrightarrow \tilde{\phi}^{(k)}(x) = d^k \tilde{\phi}(x) / dx^k$ . For instance,

$$\begin{aligned} c_3(v_0, u_0^*) &= (f^{(1,0)}(v_0, u_0^*) a^{(1,0,0)}(u_0, v_0, u_0^*) + a^{(0,1,0)}(u_0, v_0, u_0^*)) \\ &\quad \times (\chi(u_0) (f^{(1,0)}(v_0, u_0^*) a^{(1,0,0)}(u_0, v_0, u_0^*) + a^{(0,1,0)}(u_0, v_0, u_0^*)) \\ &\quad - 2D(u_0) f^{(1,0)}(v_0, u_0^*)), \end{aligned} \tag{67}$$

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where we remind that  $u_0 = f(v_0, u_0^*)$ .  $c_0$ ,  $c_1$  and  $c_2$  have large similar expressions that we do not show for simplicity. We know that we are assuming that the canonical transformation relates  $\phi(x)$  and  $\tilde{\phi}(x)$  through our choice  $\phi(x) = f(\tilde{\phi}(x), \phi^*(x))$  for any  $\phi(x)$  (compatible with the boundary conditions). However, the differential equation (66) is in itself an apparent contradiction because its existence would imply that we can canonically transform only its finite number of solutions. This is not the proper interpretation. Equation (66) is not a differential equation because it should be correct for any  $\phi(x)$  ( $\tilde{\phi}(x)$ ) function. In fact, for any given and arbitrary point  $x \in \Lambda$ , the field  $\tilde{\phi}(x)$  and its derivatives at such a point,  $v_0, v_1, \dots, v_k$ , have arbitrary values by varying  $\phi(x)$ . Therefore, we have to consider equation (66) as a polynomial formed by independent variables  $v_0, v_1, \dots$ . The polynomial can only be zero for any of  $v_k$ -values when each coefficient  $c_0, c_1, \dots$  is identically equal to zero. This gives us a set of conditions that fix the functional forms for  $f(v_0, u_0^*)$  and  $a(u_0, v_0, u_0^*)$ . We see that the complexity of the coefficients decreases with the derivative order. Therefore, it is convenient to first extract the information from  $c_3(v_0, u_0^*) = 0$ , substitute any condition obtained into the polynomial expression (66) and afterward we study  $c_2(v_0, u_0^*, u_1^*) = 0$ , and so on.

The condition  $c_3(v_0, u_0^*) = 0$  have two factors that should be studied separately:

- (1)  $f^{(1,0)}(v_0, u_0^*)a^{(1,0,0)}(u_0, v_0, u_0^*) + a^{(0,1,0)}(u_0, v_0, u_0^*) = 0$ .

This condition implies the functional relation:

$$f^{(1,0)}(v_0, u_0^*) = -\frac{a^{(0,1,0)}(f(v_0, u_0^*), v_0, u_0^*)}{a^{(1,0,0)}(f(v_0, u_0^*), v_0, u_0^*)}. \tag{68}$$

When we use this relation back into equation (66), we get

$$f^{(0,1)}(v_0, u_0^*) = -\frac{a^{(0,0,1)}(f(v_0, u_0^*), v_0, u_0^*)}{a^{(1,0,0)}(f(v_0, u_0^*), v_0, u_0^*)}. \tag{69}$$

Both relations imply that:

$$\frac{da(\phi(x), \tilde{\phi}(x), \phi^*(x))}{dx} = 0 \Rightarrow \pi(x) = 0 \quad \forall x \in \Lambda. \tag{70}$$

That is a particular trivial solution of our conditions corresponding to being at the stationary state.

- (2)  $\chi(u_0) (f^{(1,0)}(v_0, u_0^*)a^{(1,0,0)}(u_0, v_0, u_0^*) + a^{(0,1,0)}(u_0, v_0, u_0^*)) - 2D(u_0)f^{(1,0)}(v_0, u_0^*) = 0$ .

This relation can be written as:

$$\frac{\partial}{\partial v_0} \left[ a(f(v_0, u_0^*), v_0, u_0^*) - \int^{f(v_0, u_0^*)} du \frac{2D(u)}{\chi(u)} \right] = 0 \tag{71}$$

that implies

$$a(f(v_0, u_0^*), v_0, u_0^*) = \int^{f(v_0, u_0^*)} du \frac{2D(u)}{\chi(u)} + h(u_0^*), \tag{72}$$

where  $h(u)$  is a function to be determined. Observe that  $a(u_0, v_0, u_0^*)$  depends explicitly on  $v_0$  just through  $u_0 = f(v_0, u_0^*)$ . We use the fact that  $a(u_0^*, u_0^*) = 0$  to determine

$h(u_0^*)$ . The final expression is then

$$a(u_0, u_0^*) = \int_{u_0^*}^{u_0} du \frac{2D(u)}{\chi(u)}. \tag{73}$$

This expression for  $a$  reduces the polynomial (66) to:

$$\frac{2u_1^* f^{(1,0)}(v_0, u_0^*) (u_1^* D(u_0^*) - E\chi(u_0^*)) (D(u_0^*)\chi'(u_0) - D(u_0)\chi'(u_0^*))}{\chi(u_0^*)^2} = 0. \tag{74}$$

We see that the unique non-trivial condition (the one that gives us a quasipotential different from the equilibrium one) is obtained when

$$D(u_0^*)\chi'(u_0) = D(u_0)\chi'(u_0^*) \Rightarrow D(u) = c\chi'(u). \tag{75}$$

The diffusive model having this relation between the diffusion and the mobility is called a *zero-range model*.

Let us remark that from conditions (C1T), (C3T) and (EM), we have determined completely  $a(u_0, v_0, u_0^*)$  and a restricted family of diffusive systems (the zero-range model) compatible with the initial assumptions on the functionals  $A$  and  $\phi[\tilde{\phi}; x]$ . However, it seems that we do not know the precise form of  $f(v_0, u_0^*)$ . We will see that, in this case, the quasipotential can be computed for *any* well-defined  $f$ .

Once we know the form of  $a$  and the models that can be described by this functional forms, we compute the quasipotential by using equation (48) that in this case can be written:

$$V_0[\eta] = V_0[\phi^*] + \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) \int_{\Lambda} dy a(\phi[\tilde{\phi}(\lambda); y]) \frac{\delta\phi[\tilde{\phi}(\lambda); y]}{\delta\tilde{\phi}(x, \lambda)}, \tag{76}$$

where  $\phi[\tilde{\phi}; x] = f(\tilde{\phi}(x); \phi^*(x))$  and  $\tilde{\phi}(x; \lambda) = \phi^*(x) + \lambda(\tilde{\eta}(x) - \phi^*(x))$ . We easily have the functional derivative:  $\delta\phi[\tilde{\phi}(\lambda); y]/\delta\tilde{\phi}(x, \lambda) = \partial f(v_0; \phi^*(x))/\partial v_0|_{v_0=\tilde{\phi}(x; \lambda)} \delta(x - y)$ . Then

$$V_0[\eta] = V_0[\phi^*] + 2c \int_{\Lambda} dx \int_0^1 d\lambda (\tilde{\eta}(x) - \phi^*(x)) \left. \frac{\partial f(v_0; \phi^*(x))}{\partial v_0} \right|_{v_0=\tilde{\phi}(x; \lambda)} \times \log \frac{\chi(f(\tilde{\phi}(x; \lambda); \phi^*(x)))}{\chi(\phi^*(x))}, \tag{77}$$

where we have used equation (48) and  $D(u) = c\chi'(u)$ . We make the change of variables  $\lambda \rightarrow f(\tilde{\phi}(x; \lambda); \phi^*(x))$  at each  $x$ -value and we get:



$$V_0[\eta] = V_0[\phi^*] + 2c \int_{\Lambda} dx \int_{\phi^*(x)}^{\eta(x)} dw \log \frac{\chi(w)}{\chi(\phi^*(x))}, \quad (78)$$

where  $\phi^*(x)$  is the stationary state that in this case is solution of

$$c \frac{d^2}{dx^2} \chi(\phi^*(x)) = E \frac{d}{dx} \chi(\phi^*(x)). \quad (79)$$

For periodic boundary conditions, the solution is a constant:  $\phi^*(x) = \phi^*$  fixed by the initial condition:

$$\frac{1}{|\Lambda|} \int_{\Lambda} dx \phi(x, 0) = \phi^* \quad (80)$$

and the current is  $J = -E\chi(\phi^*)$ . For fixed boundary conditions at  $x = 0$  and  $x = 1$ ,  $\phi(0, t) = \phi_0$  and  $\phi(1, t) = \phi_1$  respectively, then

$$\chi(\phi^*(x)) = \frac{\chi_1 (e^{\bar{E}x} - 1) + \chi_0 (e^{\bar{E}} - e^{\bar{E}x})}{e^{\bar{E}} - 1}, \quad (81)$$

where  $\bar{E} = E/c$  and  $\chi_{0,1} = \chi(\phi_{0,1})$ . The current is given by

$$J = E \frac{\chi_1 - \chi_0 e^{\bar{E}}}{e^{\bar{E}} - 1}. \quad (82)$$

Observe that equilibrium is obtained when  $J = 0$ , that is, when  $\chi_1 = \chi_0 e^{\bar{E}}$  and the equilibrium stationary profile is given by  $\chi(\phi_{eq}^*) = \chi_0 e^{\bar{E}x}$ . The quasipotential becomes, in this case, the *equilibrium potential* that we could have derived directly from the Einstein relation (26) together with  $D(\phi) = c\chi'(\phi)$ . We see that there is no difference between equilibrium and non-equilibrium in the mathematical structure of the quasipotential. This happens uniquely for this particular model (see [20]).

In [15], Bertini *et al* obtained the quasipotential for  $c = 1$ . We recover their expression by assuming:  $\chi(u) = S(u)$ ,  $D(u) = cS'(u)$  and making the change of variables in equation (78):  $w = sZ'(s)/Z(s)$ , where  $Z(s)$  is such that  $S(w) = s$ :

$$V_0[\eta] = V_0[\phi^*] + 2c \int_{\Lambda} dx \left[ \eta(x) \log \frac{S(\eta(x))}{S(\phi^*(x))} - \log \frac{Z(S(\eta(x)))}{Z(S(\phi^*(x)))} \right]. \quad (83)$$

In this explicit example, we have shown how our method works. First, we gave a Langevin equation. Second, we defined a particular family of functionals for  $A[\phi, \tilde{\phi}; x]$  and  $\phi[\tilde{\phi}; x]$ . Third, we used the conditions (C1T), (C3T), and (EM) to fix the detailed structure of the unknown functionals. We were forced during that derivation to introduce some conditions on the Langevin equation:  $D(\phi) = c\chi'(\phi)$ . Finally, we computed the quasipotential. We will see that this scheme applies to all examples we have worked out in the paper. Let us mention that there is no *a priori* guarantee that the scheme should always give a solution. Sometimes, we have found that some conditions are never fulfilled for a Langevin equation and a given family of functionals  $A$  and  $\phi[\tilde{\phi}; x]$ .

#### 4. Generic functional form for $A$ and $\phi[\tilde{\phi}]$ in 1D systems

In this section, we restrict ourselves to one-dimensional models and we generalize the choice of the  $A$  and  $\phi[\tilde{\phi}]$  functionals we have used in the example above. We study the mathematical form of the conditions (C1T), (C3T) and (EM) for such generalization. We will see that we can algebraically handle only a subfamily of such a general choice. Finally, we will discuss some properties of such subfamilies.

##### 4.1. A family of functional forms

Let us assume that the local functional  $A[\tilde{\phi}; x]$  is of the form:

$$A[\phi, \tilde{\phi}; x] = a(\phi(x), \phi'(x), \dots, \phi^{(n)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(m)}(x), \phi^*(x)) \quad (84)$$

for given arbitrary integer values of  $m, n \geq 0$ , where, as we already saw in section 3.4, we have included an explicit dependence on the stationary state because we know that  $\pi^*(x) = A[\phi^*, \tilde{\phi}^*; x] = 0$  and such degree of freedom maybe necessary in some cases. Similarly, at the fix boundaries (if any)  $\pi(x) = A[\phi, \tilde{\phi}; x] = 0 \forall x \in \partial\Lambda$ . Both properties should be taken into account when defining the  $a$  function.

Our second choice is to take a generic form for the functional  $\phi[\tilde{\phi}]$ . We consider that there exists an implicit relation of the form:

$$\phi^{(l)}(x) = f(\phi(x), \phi'(x), \dots, \phi^{(l-1)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(s)}(x), \phi^*(x)) \quad (85)$$

for given arbitrary values of  $l > 0$  and  $s \geq 0$ . We also study the case

$$\phi(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(s)}(x), \phi^*(x)). \quad (86)$$

We choose this form for the functional relation between  $\phi$  and  $\tilde{\phi}$  because it is ‘algebraically simple’ and it may capture the complexity of a non-equilibrium system, as was shown by the work of Derrida *et al* [7] that we briefly explained in the introduction. Nevertheless, one may attempt some other more complex possibilities, but, as we will see, with our election, we are already at the edge of today’s mathematics that permits us to get explicit solutions to our problem. Observe that in equation (85) we are looking for a non-linear differential equation of order  $s$  whose solution gives us the relation between  $\tilde{\phi}$  and  $\phi$ . Such differential equations have typically  $s$  arbitrary constants. Some of them can be fixed by the boundary conditions. If there are constants still free after applying the boundary conditions, they should be fixed by looking at the ones that minimize the quasipotential.

##### 4.2. Re-writing the conditions (C1T), (C3T) and (EM)

The elected local functionals (84) and (85) permit us to rewrite with the conditions (CT1), (CT3) and (EM) in a more compact form. Let us define the differential operators:

$$P_n(x) = \sum_{k=0}^n \frac{\partial a}{\partial u_k} \Big|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} \tag{87}$$

$$Q_m(x) = \sum_{k=0}^m \frac{\partial a}{\partial v_k} \Big|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} \tag{88}$$

$$L_l(x) = \sum_{k=0}^{l-1} \frac{\partial f}{\partial u_k} \Big|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k} - \frac{d^l}{dx^l} \quad l > 0 \tag{89}$$

$$S_s(x) = \sum_{k=0}^s \frac{\partial f}{\partial v_k} \Big|_{\substack{u=\phi \\ v=\tilde{\phi}}} \frac{d^k}{dx^k}. \tag{90}$$

$L_0 = -1$ , where  $a = a(u_0, u_1, \dots, u_n, v_0, v_1, \dots, v, u_0^*)$  and similarly  $f$ , where the convention is that after any derivative of their arguments we should do the substitution  $u_k \rightarrow \phi^{(k)}(x)$  and  $v_k \rightarrow \tilde{\phi}^{(k)}(x)$ .

Moreover, we can do a functional derivative with respect to  $\tilde{\phi}(y)$  in both sides of equation (85) or (86) to obtain an equation for the functional  $K[\tilde{\phi}; x, y] = \delta\phi[\tilde{\phi}; x] / \delta\tilde{\phi}(y)$ :

$$L_l(x)K[\tilde{\phi}; x, y] + S_s(x)\delta(x - y) = 0. \tag{91}$$

Let us rewrite the conditions by using these differential operators:

- (C1T): equation (53) can be written in this case as  $P_n(x)\delta(x - y) = P_n(y)\delta(x - y)$ . This happens if and only if the operator  $P_n(x)$  is self-adjoint,  $P_n^\dagger(x) = P_n(x)$  (see appendix B for a brief reminder about properties and definitions of self-adjoint linear differential operators). We can prove this by observing that by definition and for any  $w$  integrable test function

$$\begin{aligned} \int dx w(x) P_n(x) \delta(x - y) &= \int dx (P_n^\dagger(x) w(x)) \delta(x - y) = P_n^\dagger(y) w(y) \\ &= \int dx w(x) P_n^\dagger(y) \delta(x - y) \Rightarrow P_n(x) \delta(x - y) = P_n^\dagger(y) \delta(x - y). \end{aligned} \tag{92}$$

That is,

$$(C1T) : P_n(x) = P_n^\dagger(x). \tag{93}$$

- (C3T): equation (55) becomes:

$$Q_m^\dagger(x)K[\tilde{\phi}; x, y] = Q_m^\dagger(y)K[\tilde{\phi}; y, x] \tag{94}$$

and after using equation (91) we find:

$$(C3T) : Q_m^\dagger(x)(L_l(x))^{-1}S_s(x) = S_s^\dagger(x)(L_l^\dagger(x))^{-1}Q_m(x) \tag{95}$$

that is  $Q_m^\dagger(x)(L_l(x))^{-1}S_s(x)$  should be self-adjoint.

- (EM): after some trivial algebra, we find that (61) can be written:

$$Q_m^\dagger(x)\tilde{R}_1[\tilde{\phi}; x] = S_s^\dagger(x)(L_l^\dagger(x))^{-1} \left( P_n(x)\tilde{R}_1[\tilde{\phi}; x] - \tilde{R}_2[\tilde{\phi}; x] \right) \quad (96)$$

or, similarly

$$L_l(x)\tilde{R}_1[\tilde{\phi}; x] = S_s(x)(Q_m(x))^{-1} \left( P_n(x)\tilde{R}_1[\tilde{\phi}; x] - \tilde{R}_2[\tilde{\phi}; x] \right). \quad (97)$$

Note that we define the inverse of any differential operator  $T(x)$  through its associated Green function:

$$T(x)f(x) = g(x) \Rightarrow f(x) = \int_{\Lambda} dy G(x, y)g(y) \equiv (T(x))^{-1}g(x), \quad (98)$$

where  $G(x, y)$  is the solution of

$$T(x)G(x, y) = \delta(x, y). \quad (99)$$

At this point, let us repeat our goal: we want to find the functions  $a$  and  $f$  (given  $n$ ,  $m$ ,  $l$  and  $s$ ) such that they fulfill the conditions (C1T), (C3T) and (EM). The presence of inverse differential operators makes it almost impossible to find a systematic algebraic way to get the unknown functions. For instance, we should first find the Green function associated with such a still unknown operator. We know that it can be done systematically for regular boundary value problems (self-adjoint differential operators) once we know the eigenfunctions and eigenvalues of the operator (see, for instance, [23]). However, those depend again on the explicit form of the operator. Therefore, from our present knowledge on these issues, it is almost impossible to get algebraically a set of eigenfunctions of our operators  $L$ , or  $Q$  that are unknown functionals of  $\tilde{\phi}$  and its local derivatives. Consequently, we are going to consider in this paper only situations where there are not inverse operators in (C3T) and (EM). There are two possibilities that are compatible with such practical limitations:

- (a)  $l = 0$ :  $L_0(x) = -1$ . In this case the operator  $L$  is just a constant and the conditions are:

$$\begin{aligned} \text{(C1T): } P_n(x) &= P_n^\dagger(x) \\ \text{(C3T): } Q_m^\dagger(x)S_s(x) &= S_s^\dagger(x)Q_m(x) \\ \text{(EM): } Q_m^\dagger(x)\tilde{R}_1[\tilde{\phi}; x] &= -S_s^\dagger(x) \left( P_n(x)\tilde{R}_1[\tilde{\phi}; x] - \tilde{R}_2[\tilde{\phi}; x] \right). \end{aligned} \quad (100)$$

- (b)  $m = 0$ :  $Q_0(x) = \partial a / \partial u_0|_{u=\phi, v=\tilde{\phi}} \neq 0$ . In this case the operator  $Q$  is just a function and the conditions are written:

$$\begin{aligned} \text{(C1T): } P_n(x) &= P_n^\dagger(x) \\ \text{(C3T): } S_s(x)Q_0(x)^{-1}L_l^\dagger(x) &= L_l(x)Q_0(x)^{-1}S_s^\dagger(x) \\ \text{(EM): } L_l(x)\tilde{R}_1[\tilde{\phi}; x] &= S_s(x)Q_0(x)^{-1} \left( P_n(x)\tilde{R}_1[\tilde{\phi}; x] - \tilde{R}_2[\tilde{\phi}; x] \right), \end{aligned} \quad (101)$$

where we have used equation (97).

In section 5, we will systematically consider different scenarios (a) and (b) by applying them to some typical models as the DD and the reaction–diffusion dynamics.

### 4.3. Allowed operators and computation strategy

We saw that in cases (a) and (b) (equations (100) and (101), respectively) we should choose a set of values  $(n, m, l, s)$  that define the form of operators  $P, Q, L$  and  $S$ , respectively. From condition (C1T), we know that  $n$  should be even to fulfill the condition that  $P$  is self-adjoint. Similarly, (C3T) in case (a) implies  $m + s$  should be even, and in case (b),  $s + l$  should also be even. We can do a little better by keeping a trace of the larger derivative of  $\tilde{\phi}(x)$  for each condition. In this case, we should explicitly define the dynamics. We have done such computation for  $l = 0$  (case (a)) and the diffusive and the reaction–diffusion dynamics defined in section 2. We found that for such models the operators should follow the consistency relations:

$$n = 0 \Rightarrow m \leq s, \quad n > 0 \Rightarrow m = s + n. \tag{102}$$

(n, m, l, s)	
Case (a) (l = 0)	Case (b) (m = 0)
(0, 0, 0, 0)	(0, 0, 1, 1)
(0, 1, 0, 1)	(0, 0, 2, 2)
(0, 0, 0, 2)	(0, 0, 1, 3)
(0, 0, 0, 4)	(0, 0, 3, 1)
(0, 1, 0, 3)	...
(0, 2, 0, 2)	(2, 0, 1, 1)
...	(2, 0, 2, 0)
(2, 2, 0, 0)	...
...	

That is, we can attempt the following cases: where at each column we fix  $n = 0$  and then  $n = 2$  and so on. We have excluded the coincident values between case (a) and case (b). Once we choose one of these permitted values  $(n, m, l, s)$ , the method is intended to fix the mathematical form of the functions  $a$  (84) and  $f$  (85). The main idea is that once we assume the form of  $a$  and  $f$ , the conditions (C1T), (C3T), and (EM) should become identities. In general, these conditions are differential equations. For instance, let us assume that  $f$  is of the form:  $\tilde{\phi}(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), \tilde{\phi}''(x))$ . Let us assume that we can isolate the highest derivative  $\tilde{\phi}''(x) = F_2(\phi(x), \phi(x), \phi'(x))$ . Then  $\tilde{\phi}^{(3)}(x) = F_3(\phi(x), \phi'(x), \tilde{\phi}(x), \tilde{\phi}'(x))$ , and in general,  $\tilde{\phi}^{(k)}(x) = F_k(\phi(x), \phi'(x), \dots, \phi^{(k-2)}(x), \tilde{\phi}(x), \tilde{\phi}'(x))$ . On the other hand,  $\phi(x)$  is assumed to be an analytic function. Therefore, for any given point in the domain,  $\bar{x} \in \Lambda$ , we can reconstruct  $\phi(x)$  just by giving all their derivatives at such point. However,  $\phi(x)$  is arbitrary, and we are free to choose all of the derivatives of  $\phi(x)$  at  $\bar{x}$ .

Therefore, we may reasonably assume that  $\tilde{\phi}^{(k)}(\bar{x})$  for  $k \geq 2$  may get arbitrary and independent values because of their dependence on the derivatives of  $\phi(\bar{x})$ . Having this in mind, we see that the conditions (C1T), (C3T), and (EM) are just polynomials where there are derivatives of  $\phi$  and  $\tilde{\phi}$  of a different order. They should be correct *for any  $\phi$ -field* and therefore, *for any value of the derivatives of  $\tilde{\phi}(x)$*  of a degree greater or equal to two. Therefore, once we substitute  $\phi$  by  $f(\tilde{\phi}, \tilde{\phi}'(x), \tilde{\phi}''(x))$  in any of the conditions, we get a polynomial expression in the derivatives  $\tilde{\phi}^{(n)}$  with  $n \geq 2$  whose coefficients are functions with  $\tilde{\phi}(x)$ ,  $\tilde{\phi}'(x)$  and  $\tilde{\phi}''(x)$  (in this example). Each of these high order derivatives may have arbitrary values and each of their coefficients should be identically equal to zero. The coefficients that we equal to zero contains, typically derivatives of  $f$  and  $a$ , and they also contain functions that depend on the dynamics. Therefore,  $f$  and  $a$  may depend on the dynamics, and, sometimes, only particular dynamics can make zero a coefficient.

This scheme is done orderly from higher to small order in the polynomial of the derivatives of  $\tilde{\phi}$  for each condition. Once we determine some properties of the unknown functions, we include them in the conditions, and we redo the computations to get the remaining high-order derivatives' next coefficient. This method has been applied successfully using algebraic programs like Mathematica. It permits us to do long computations without errors. That is very important because we are dealing with identities, and any small mistake during the algebraic trivial but lengthy evaluation implies that conditions (C1T), (C3T), and (EM) are never fulfilled.

We should also take into account that  $\pi(x) = a(x)$ . Therefore, it is mandatory that:

$$\begin{aligned} a(\phi^*(x), (\phi^*)'(x), \dots, (\phi^*)^{(n)}(x), (\tilde{\phi}^*)'(x), (\tilde{\phi}^*)''(x), \dots, (\tilde{\phi}^*)^{(m)}(x); \phi^*(x)) &= 0 \\ a(\phi(x), \phi'(x), \dots, \phi^{(n)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(m)}(x); \phi^*(x)) &= 0 \quad \forall x \in \partial\Lambda, \end{aligned} \tag{103}$$

where  $\tilde{\phi}^*(x)$  is the stationary state in the new variables that is related with the original stationary state  $\phi^*(x)$  through equation (85):

$$(\phi^*)^{(l)}(x) = f(\phi^*(x), (\phi^*)'(x), \dots, (\phi^*)^{(l-1)}(x), \tilde{\phi}^*(x), (\tilde{\phi}^*)'(x), \dots, (\tilde{\phi}^*)^{(s)}(x); \phi^*(x)). \tag{104}$$

Observe that the second condition in equation (103) only applies when the field's values are fixed at the system's boundaries. These conditions help us to determine  $a$  and  $f$  functions.

About the stationary state there are two possibilities: (i)  $\tilde{\phi}^*(x) = \phi^*(x)$  or (ii)  $\tilde{\phi}^*(x) \neq \phi^*(x)$  and this affects the boundary conditions for the  $\tilde{\phi}(x)$  fields that are necessary to solve the differential equation (85). The (i) case is the most convenient (but more restrictive) because it implies that both fields have the same boundary conditions  $\tilde{\phi}(x) = \phi(x) \forall x \in \partial\Lambda$ . Case (ii) needs equation (104) to be solved explicitly and then fix some of the  $s$  constants given by the boundary values for  $\phi(x)$  and the conditions that  $\tilde{\phi}(x)$  do not evolve at the boundaries:  $\tilde{R}_{1,2}[\tilde{\phi}; x] = 0 \forall x \in \partial\Lambda$  (see equation (56) and the definitions below it).



### 5. Quasipotentials for one dimensional diffusive models

We show in this section the quasipotentials obtained by electing some concrete values  $(n, m, l, s)$  that define the differential operators on the conditions (C1T), (C3T) and (EM) in equations (100) and (101). We have kept some fine details on their derivations and we refer to the explicit example on section 3.4 to fill the gaps.

(i)  $(n, m, l, s) = (0, 0, 0, 0)$ .

This case corresponds to the explicit example we studied in section 3.4 (zero-range model:  $D(\phi) = c\chi'(\phi)$ ) where we got its quasipotential. Observe that the corresponding operators (equations (87)–(90)) have the form:

$$\begin{aligned} P_0(x) &= a^{(1,0,0)}(u_0, v_0, u_0^*) \\ Q_0(x) &= a^{(0,1,0)}(u_0, v_0, u_0^*) \\ L_0(x) &= -1 \\ S_0(x) &= f^{(1,0)}(v_0, u_0^*) \end{aligned} \tag{105}$$

and one can check that the general conditions (100) are reduced to the ones obtained in section 3.4.

(ii)  $(n, m, l, s) = (0, 0, 0, 2)$ .

In this case, we choose  $a = a(\phi(x), \tilde{\phi}(x))$  and  $\phi(x) = f(\tilde{\phi}(x), \tilde{\phi}'(x), \tilde{\phi}''(x))$ . The operators (equations (87)–(90)) have the form:

$$\begin{aligned} P_0(x) &= a^{(1,0)}(u_0, v_0) \\ Q_0(x) &= a^{(0,1)}(u_0, v_0) \\ L_0(x) &= -1 \\ S_2(x) &= f^{(1,0,0)}(\underline{v}) + f^{(0,1,0)}(\underline{v})\frac{d}{dx} + f^{(0,0,1)}(\underline{v})\frac{d^2}{dx^2}, \end{aligned} \tag{106}$$

where  $\underline{v} = (v_0, v_1, v_2)$ ,  $u_k \equiv \phi^{(k)}(x)$ ,  $v_k \equiv \tilde{\phi}^{(k)}(x)$ . The sufficient conditions given by the cases (a) and (b) (100), (101) are equivalent in this case. They are:

- (C1T):  $P_0(x)$  is self-adjoint by construction, therefore this condition is fulfilled.
- (C3T):  $Q_0^\dagger(x)S_2(x)$  should be self-adjoint. This condition shows that applying the relations for self-adjointness (see appendix B) to such a second order differential operator implies:

$$a^{(0,1)}(u_0, v_0)f^{(0,1,0)}(\underline{v}) = \frac{d}{dx} [a^{(0,1)}(u_0, v_0)f^{(0,0,1)}(\underline{v})]. \tag{107}$$

- (EM): this condition is written from equation (100) using the expressions for  $R_1$  and  $R_2$  given by equation (65). It is a long expression that we do not write explicitly but it is of the form of a polynomial in  $v_6, v_5, v_4, v_3$ :

$$c_0(u_0, \underline{v})v_6 + c_1(u_0, \underline{v})v_5 + c_2(u_0, \underline{v})v_4 + c_3(u_0, \underline{v})v_3^2 + \dots = 0, \tag{108}$$

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where  $u_0 = f(\underline{v})$ . Both conditions are assumed to be identities that (as we explained above) are fulfilled for any value of  $v_n$  for  $n \leq 2$ . Therefore, the polynomial structure on high derivatives imply that their coefficients should be zero. That gives us conditions on the functional forms of our unknowns:  $a$  and  $f$ . It is convenient to be solving such conditions in an optimal way. We followed this line of reasoning:

- (1) Coefficient of  $v_6$  from (EM) equal to zero:

$$a^{(1,0)}(u_0, v_0) f^{(0,0,1)}(\underline{v})^2 (2D(u_0) - \chi(u_0) a^{(1,0)}(u_0, v_0)) = 0. \quad (109)$$

We know that  $a^{(1,0)}(u_0, v_0) \neq 0$  because the contrary would imply that  $a$  only depends on  $v_0$  that is against our initial assumption on  $a$ . Similarly,  $f^{(0,0,1)}(\underline{v}) \neq 0$  by construction. Therefore, from the last factor we find that

$$a(u_0, v_0) = \int du_0 \frac{2D(u_0)}{\chi(u_0)} + \tilde{a}(v_0). \quad (110)$$

We include this relation into conditions (CT3) and (EM) and we go to the next higher non-zero order.

- (2) Coefficient of  $v_3$  from (C3T) equal to zero:

$$\tilde{a}^{(1)}(v_0) f^{(0,0,2)}(\underline{v}) = 0 \quad (111)$$

$\tilde{a}^{(1)}(v_0) f^{(0,0,2)}$  by construction and therefore  $f$  should be a linear function of  $v_2$ :

$$f(\underline{v}) = f_0(v_0, v_1) + v_2 f_1(v_0, v_1). \quad (112)$$

By using this relation, we rewrite the full (C3T) condition as:

$$v_1 \frac{\partial}{\partial v_0} (\tilde{a}'(v_0) f_1(v_0, v_1)) = \tilde{a}'(v_0) \frac{\partial f_0(v_0, v_1)}{\partial v_1}. \quad (113)$$

We use these relations into the condition (EM) and move on to the next non-zero order.

- (3) Coefficient of  $v_4$  from (EM) equal to zero:

$$-4D(u_0) - 2u_0 D'(u_0) - g_0(v_0, v_1) D'(u_0) + g_1(v_0, v_1) \chi''(v_0) = 0, \quad (114)$$

where

$$g_0(v_0, v_1) = 2\tilde{a}'(v_0)^{-1} (v_1^2 \tilde{a}''(v_0) f_1(v_0, v_1) - f_0(v_0, v_1) \tilde{a}'(v_0)) \quad (115)$$

$$g_1(v_0, v_1) = v_1 f_1(v_0, v_1) (2E + v_1 \tilde{a}'(v_0)) \quad (116)$$

$u_0$  is arbitrary and independent on  $v_0$  and  $v_1$ , therefore equation (114) has three possible scenarios:

$$\begin{aligned} \text{Case (a)} \quad & D(u_0) = D_0, \quad g_1(v_0, v_1) = \Lambda_1 = cte \\ \text{Case (b)} \quad & \chi''(u_0) = 0, \quad g_0(v_0, v_1) = \Lambda_0 = cte \\ \text{Case (c)} \quad & g_0(v_0, v_1) = \Lambda_0 = cte, \quad g_1(v_0, v_1) = \Lambda_1 = cte. \end{aligned} \quad (117)$$

Observe that we have not considered the possibility that  $D'(u_0) = c_0\chi''(u_0)$ , because it is studied in the  $(n, m, l, s) = (0, 0, 0, 0)$  case. At this point, we should study each case, separately:

- \* Case (a):  $D(u_0) = D_0, g_1(v_0, v_1) = \Lambda_1 = cte$   
Equation (114) implies:

$$\chi''(u_0) = \frac{4D_0}{\Lambda_1} \tag{118}$$

and therefore this case applies only when  $D(u_0) = D_0, \chi(u_0) = k_0 + k_1u_0 + k_2u_0^2$  with  $\Lambda_1 = 2D_0/k_2$ . Equation (116) implies:

$$f_1(v_0, v_1) = \frac{2D_0}{k_2v_1} \frac{1}{2E + v_1\tilde{a}'(v_0)}. \tag{119}$$

We use the remainder of condition (CT3), equation (113), to get the form of  $f_0(v_0, v_1)$ :

$$f_0(v_0, v_1) = f_{00}(v_0) - \frac{4D_0E\tilde{a}''(v_0)}{k_2\tilde{a}'(v_0)^2} \frac{1}{2E + v_1\tilde{a}'(v_0)}, \tag{120}$$

where  $f_{00}(v_0)$  is an arbitrary function to be determined.

- \* Case (b):  $\chi''(u_0) = 0, g_0(v_0, v_1) = \Lambda_0 = cte$   
In this case,  $\chi(u_0) = k_0 + k_1u_0$  and equation (114) implies:

$$\frac{D'(u_0)}{D(u_0)} = \frac{-4}{\Lambda_0 + 2u_0} \Rightarrow D(u_0) = \frac{D_0}{(\Lambda_0 + 2u_0)^2} \tag{121}$$

and

$$g_0(v_0, v_1) = \Lambda_0 \Rightarrow f_0(v_0, v_1) = -\frac{\Lambda_0}{2} + \tilde{a}'(v_0)^{-1}v_1^2\tilde{a}''(v_0)f_1(v_0, v_1). \tag{122}$$

We again use the remaining of the condition (C3T), equation (113), to find  $f_1(v_0, v_1)$ :

$$f_1(v_0, v_1) = \tilde{a}'(v_0)f_{11}(v_1\tilde{a}'(v_0)), \tag{123}$$

where  $f_{11}(\beta)$  is an arbitrary function to be determined.

- \* Case (c):  $g_0(v_0, v_1) = \Lambda_0 = cte, g_1(v_0, v_1) = \Lambda_1 = cte$   
From the definitions of  $g_0$  and  $g_1$  in equations (115) and (116) respectively, we get the conditions:

$$\begin{aligned} g_0(v_0, v_1) = \Lambda_0 &\Rightarrow f_0(v_0, v_1) = -\frac{\Lambda_0}{2} + \tilde{a}'(v_0)^{-1}v_1^2\tilde{a}''(v_0)f_1(v_0, v_1) \\ g_1(v_0, v_1) = \Lambda_1 &\Rightarrow f_1(v_0, v_1) = \frac{\Lambda_1}{v_1} \frac{1}{2E + v_1\tilde{a}'(v_0)} - 4D(u_0) - 2u_0D'(u_0) \\ &\quad - \Lambda_0D'(u_0) + \Lambda_1\chi''(u_0) = 0. \end{aligned}$$

One can easily check that the condition (113) is fulfilled by these  $f_0(v_0, v_1)$  and  $f_1(v_0, v_1)$  functions.

- We now assume that  $\phi^*(x) = \tilde{\phi}^*(x)$ . This choice is convenient because implies that the boundary conditions for the transformed field  $\tilde{\phi}(x)$  are the same as the ones for  $\phi(x)$  and therefore the second property in (103) is immediately fulfilled. Then:

$$a(\phi^*(x), \phi^*(x)) = 0 \Rightarrow \tilde{a}(\phi^*(x)) = \bar{a}(\phi^*(x)), \quad - \int^w dv \frac{2D(v)}{\chi(v)}, \quad (124)$$

where we have used (110). There are two possibilities: (a)  $\tilde{a}(v_0) = \bar{a}(\phi^*(x)) \forall v_0$  or (b)  $\tilde{a}(v_0) = \bar{a}(v_0) \forall v_0$ . The first case contradicts our  $a$ 's initial choice where we assumed that there were not any explicit  $x$ -dependence on it. Therefore,

$$a(u_0, v_0) = 2 \int_{v_0}^{u_0} du \frac{D(u)}{\chi(u)}. \quad (125)$$

- Another piece of information is obtained from the differential equation that defines the stationary state, that for this diffusive system is given by:

$$\frac{d}{dx} \left[ -D(\phi^*(x)) \frac{d\phi^*(x)}{dx} + E\chi(\phi^*(x)) \right] = 0 \quad (126)$$

with the corresponding boundary conditions. We also know from equation (112) that:

$$\phi^*(x) = f_0(\phi^*(x), \phi^{*'}(x)) + \phi^{*''}(x) f_1(\phi^*(x), \phi^{*'}(x)). \quad (127)$$

Obviously both differential equations should have the same solutions given the boundary conditions. Just by eliminating  $\phi^{*''}(x)$  from both equations we get a relation between  $f_0$  and  $f_1$  at the stationary state that helps us to determine the missing parts of  $f$  in all three cases:

- \* Case (a):  $f_{00}(v_0) = v_0$  and therefore:

$$f(v_0, v_1, v_2) = v_0 - \frac{\chi(v_0)}{k_2 v_1} \frac{E\chi'(v_0)v_1 - D_0 v_2}{E\chi(v_0) - D_0 v_1}. \quad (128)$$

- \* Case (b): in this case  $\chi(v_0) = k_0 + k_1 v_0$  that makes that the stationary state have the nice property that  $D(\phi^*(x))\phi^{*'}(x) = Ek_1\phi^*(x) + Ek_0 - J$ , where  $J$  is the stationary current. Therefore, the unknown function  $f_{11}(\beta)$  is given by:

$$f_{11}(\beta) = \frac{2\bar{D}_0}{\beta(2J - (\beta + 2E)(k_0 - k_1\bar{\Lambda}_0))}, \quad (129)$$

where  $\bar{D}_0 = D_0/4$  and  $\bar{\Lambda}_0 = \Lambda_0/2$  so  $D(u) = \bar{D}_0/(\bar{\Lambda}_0 - u)^2$ . Finally, we get:

$$f(v_0, v_1, v_2) = -\bar{\Lambda}_0 + \bar{D}_0 \frac{(D(v_0)\chi'(v_0) - D'(v_0)\chi(v_0))v_1^2 - D(v_0)\chi(v_0)v_2}{v_1 D(v_0)((k_0 E - J - k_1 E \bar{\Lambda}_0)\chi(v_0) - (k_0 - k_1 \bar{\Lambda}_0)D(v_0)v_1)}. \quad (130)$$

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\* Case (c): this order only is accomplished for situations already studied in (a) or (b).

- The remaining condition (EM) is fulfilled once we use the full expressions for  $a$  and  $f$  for each nontrivial case.

We see that in this case we have determined the precise form of  $a$  and  $f$  functions such that conditions (C1T), (C3T) and (EM) are fulfilled for some concrete forms of  $D(u)$  and  $\chi(u)$ . We can now to compute the quasipotential for each case.

(a)  $D(\mathbf{u}) = D_0$ ,  $\chi(\mathbf{u}) = k_0 + k_1 \mathbf{u} + k_2 \mathbf{u}^2$ .

We have found:

$$A[\phi, \tilde{\phi}; x] = 2D_0 \int_{\tilde{\phi}(x)}^{\phi(x)} \frac{du}{\chi(u)} \equiv a(\phi(x), \tilde{\phi}(x)) \quad (131)$$

$$\phi[\tilde{\phi}; x] = \tilde{\phi}(x) - \frac{\chi(\tilde{\phi}(x)) E\chi'(\tilde{\phi}(x))\tilde{\phi}'(x) - D_0\tilde{\phi}''(x)}{k_2\tilde{\phi}'(x) E\chi(\tilde{\phi}(x)) - D_0\tilde{\phi}'(x)}.$$

We use equation (48) to compute the quasipotential that it can be written as:

$$V_0[\eta] = V_0[\phi^*] + I_1 - I_2$$

$$I_1 = \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) \frac{\delta}{\delta \tilde{\phi}(x; \lambda)} \left[ \int_{\Lambda} dy \int du_0 a(u_0, v_0) \Big|_{\substack{u_0=\phi[\tilde{\phi}(\lambda); y] \\ v_0=\tilde{\phi}(y; \lambda)}} \right] \quad (132)$$

$$I_2 = \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) \int du_0 \frac{\partial a(u_0, v_0)}{\partial v_0} \Big|_{\substack{u_0=\phi[\tilde{\phi}(\lambda); x] \\ v_0=\tilde{\phi}(x; \lambda)}}$$

where  $\tilde{\phi}(x; \lambda) = \phi^*(x) + \lambda(\tilde{\eta}(x) - \phi^*(x))$ , and from equation (131)  $\tilde{\eta}(x)$  is the solution of the differential equation:

$$\eta(x) = \tilde{\eta}(x) - \frac{\chi(\tilde{\eta}(x)) E\chi'(\tilde{\eta}(x))\tilde{\eta}'(x) - D_0\tilde{\eta}''(x)}{k_2\tilde{\eta}'(x) E\chi(\tilde{\eta}(x)) - D_0\tilde{\eta}'(x)} \quad (133)$$

for a given  $\eta(x)$  field with given boundary conditions  $\tilde{\phi}(x) = \phi(x) \forall x \in \partial\Lambda$ .

We have used the relation:

$$\int_{\Lambda} dy a(\phi[\tilde{\phi}; y], \tilde{\phi}(y)) \frac{\delta \phi[\tilde{\phi}; y]}{\delta \tilde{\phi}(x)} = \frac{\delta}{\delta \tilde{\phi}(x)} \int_{\Lambda} dy \int du_0 a(u_0, v_0) \Big|_{\substack{u_0=\phi[\tilde{\phi}; y] \\ v_0=\tilde{\phi}(y)}} - \int du_0 \frac{\partial a(u_0, v_0)}{\partial v_0} \Big|_{\substack{u_0=\phi[\tilde{\phi}; y] \\ v_0=\tilde{\phi}(y)}}. \quad (134)$$

The first integral in (132) is just:

$$\begin{aligned}
 I_1 &= \int_0^1 d\lambda \frac{d}{d\lambda} \left[ \int_{\Lambda} dy \int du_0 a(u_0, v_0) \right]_{\substack{u_0 = \phi[\tilde{\phi}(\lambda); y] \\ v_0 = \phi(y; \lambda)}} \\
 &= \int_{\Lambda} dy [v(\eta(y)) - \eta(y)v'(\eta(y)) - v(\phi^*(y)) + \phi^*(y)v'(\phi^*(y))], \tag{135}
 \end{aligned}$$

where  $v(u_0) = 2D_0 \int du_0 \int du_0 / \chi(u_0)$ .

The second integral needs a little more work. First, it can be written:

$$I_2 = -2D_0 \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) \frac{\phi[\tilde{\phi}(\lambda); x]}{\chi(\tilde{\phi}(x; \lambda))}. \tag{136}$$

This integral can be separated into two pieces: one without derivatives of  $\tilde{\phi}(x; \lambda)$  and the other with its derivatives:

$$\begin{aligned}
 I_2 &= I_{21} + I_{22} \\
 I_{21} &= \int_0^1 d\lambda \int_{\Lambda} dx \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x; \lambda))} \left[ \tilde{\phi}(x; \lambda) - \frac{1}{k_2} \chi'(\tilde{\phi}(x; \lambda)) \right] \\
 I_{22} &= \frac{D_0}{k_2} \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) \frac{1}{u(x; \lambda) \chi(\tilde{\phi}(x; \lambda))} \frac{u'(x; \lambda)}{E - D_0 u(x; \lambda)}, \tag{137}
 \end{aligned}$$

where  $u(x; \lambda) = \tilde{\phi}'(x; \lambda) / \chi(\tilde{\phi}(x; \lambda))$ .  $I_{21}$  can be evaluated by making the change of variables  $\lambda \rightarrow \tilde{\phi}(x; \lambda)$  at each  $x$  and we get:

$$\begin{aligned}
 I_{21} &= \frac{1}{2D_0} \int_{\Lambda} dx \left[ \tilde{\eta}(x)v'(\tilde{\eta}(x)) - v(\tilde{\eta}(x)) - \frac{2D_0}{k_2} \log \chi(\tilde{\eta}(x)) \right. \\
 &\quad \left. - \phi^*(x)v'(\phi^*(x)) + \frac{2D_0}{k_2} \log \chi(\phi^*(x)) \right]. \tag{138}
 \end{aligned}$$

Integral  $I_{22}$  can be written as:

$$I_{22} = \frac{D_0}{k_2 E} \int_0^1 d\lambda \int_{\Lambda} dx \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x; \lambda))} \frac{d}{dx} \log \left[ \frac{|D_0 u(x; \lambda)|}{|E - D_0 u(x; \lambda)|} \right] \tag{139}$$

and after integrating by parts on  $x$ :

$$I_{22} = -\frac{D_0}{k_2 E} \int_0^1 d\lambda \int_{\Lambda} dx \frac{d}{dx} \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x; \lambda))} \log \left[ \frac{|D_0 u(x; \lambda)|}{|E - D_0 u(x; \lambda)|} \right] \tag{140}$$

we now use the relation:



$$\frac{du(x; \lambda)}{d\lambda} = \frac{d}{dx} \frac{\tilde{\eta}(x) - \phi^*(x)}{\chi(\tilde{\phi}(x; \lambda))} \tag{141}$$

to get

$$\begin{aligned} I_{22} &= -\frac{D_0}{k_2 E} \int_{\Lambda} dx \int_0^1 d\lambda \frac{du(x; \lambda)}{d\lambda} \log \left[ \frac{|D_0 u(x; \lambda)|}{|E - D_0 u(x; \lambda)|} \right] \\ &= -\frac{D_0}{k_2 E} \int_{\Lambda} dx \int_{u(x; 0)}^{u(x; 1)} du \log \left[ \frac{|D_0 u|}{|E - D_0 u|} \right], \end{aligned} \tag{142}$$

where  $u(x, 0) = \phi^{*'}(x)/\chi(\phi^*(x))$  and  $u(x, 1) = \eta'(x)/\chi(\eta(x))$ . After doing the integral of the logarithm and putting together all the pieces  $I_1$ ,  $I_{21}$  and  $I_{22}$ , we get the final expression for  $V_0$ :

$$\begin{aligned} V_0[\eta] &= V_0[\phi^*] + \int_{\Lambda} dx [v(\eta(x)) - v(\tilde{\eta}(x))] \\ &\quad - (\eta(x) - \tilde{\eta}(x))v'(\tilde{\eta}(x)) - \frac{2D_0}{k_2} \log \frac{\chi(\tilde{\eta}(x))}{\chi(\phi^*(x))} \\ &\quad + \frac{2D_0}{k_2 E} \left( \left( \frac{D_0 \tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E \right) \log \left| \frac{D_0 \tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} - E \right| - \frac{D_0 \tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} \log \left| \frac{D_0 \tilde{\eta}'(x)}{\chi(\tilde{\eta}(x))} \right| \right. \\ &\quad \left. - \left( \frac{D_0 \phi^{*'}(x)}{\chi(\phi^*(x))} - E \right) \log \left| \frac{D_0 \phi^{*'}(x)}{\chi(\phi^*(x))} - E \right| \left[ + \frac{D_0 \phi^{*'}(x)}{\chi(\phi^*(x))} \log \left| \frac{D_0 \phi^{*'}(x)}{\chi(\phi^*(x))} \right| \right] \right) \end{aligned} \tag{143}$$

with  $\tilde{\eta}(x)$  solution of the differential equation (133) with boundary conditions  $\tilde{\eta}(x) = \eta(x) \forall x \in \partial\Lambda$ .

We observe that the case  $k_2 \rightarrow 0$  seems to be singular; however, it is not. In order to do the limit let us think that  $k_2$  is a perturbative parameter and then we assume that a well-defined expansion exists:  $\tilde{\eta}(x) = \tilde{\eta}_0(x) + k_2 \tilde{\eta}_1(x) + \dots$ . We apply this expansion to equation (133), and it appears the order  $k_2^{-1}$ . Its coefficient should be zero and therefore:

$$-D_0 \tilde{\eta}_0''(x) + E k_1 \tilde{\eta}_0'(x) = 0 \Rightarrow \tilde{\eta}_0(x) = \phi^*(x), \tag{144}$$

where  $\phi^*(x)$  is now the stationary state when  $k_2 = 0$ . For instance,  $\phi^*(x)$  for the fixed boundary condition case ( $\phi^*(0) = \phi_0$  and  $\phi^*(1) = \phi_1$ ) is

$$\phi^*(x) = \frac{J}{k_1 E} + \frac{\phi_1 - \phi_0}{e^{\tilde{E}} - 1} e^{\tilde{E}x}, \quad J = E k_1 \frac{\phi_0 e^{\tilde{E}} - \phi_1}{e^{\tilde{E}} - 1}, \tag{145}$$

where  $\tilde{E} = E k_1 / D_0$  and  $J$  is the current. The order  $k_2^0$  has the form:

$$\begin{aligned} \eta(x) &= -\phi^*(x) \left( 1 + \frac{2D_0 \phi^{*'}(x)}{k_0 E + J} \right) \\ &\quad - \frac{\chi_0(\phi^*(x))}{(k_0 E + J) \phi^{*'}(x)} (k_1 E \tilde{\eta}_1'(x) - D_0 \tilde{\eta}_1''(x)) \end{aligned} \tag{146}$$

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and  $\chi_0(u) = k_0 + k_1u$ . This differential equation for  $\tilde{\eta}_1(x)$  should be solved with boundary conditions  $\tilde{\eta}_1(x) = 0 \forall x \in \partial\Lambda$  because in the expansion  $\tilde{\eta}_0$  carries the original boundary conditions. Finally, with all of this information we can expand the quasipotential around  $k_2 = 0$  and we get for  $D(u) = D_0$  and  $\chi(u) = k_0 + k_1u$ :

$$\begin{aligned}
 V_0[\eta] = & V_0[\phi^*] + \int_{\Lambda} dx [v(\eta(x)) - v(\phi^*(x)) - (\eta(x) - \phi^*(x))v'(\phi^*(x))] \\
 & + \frac{2D_0^2 E}{\chi_0(\phi^*(x))^2} (\chi_0(\phi^*(x))\tilde{\eta}_1'(x) - k_1\phi^*(x)\tilde{\eta}_1(x)) \\
 & \times \log \left[ \left| 1 - \frac{E\chi_0(\phi^*(x))}{D_0\phi^*(x)} \right| \right] - \frac{2D_0k_1\tilde{\eta}_1(x)}{\chi_0(\phi^*(x))} \quad (147)
 \end{aligned}$$

with the  $\tilde{\eta}_1(x)$  solution of equation (146). Observe that  $\tilde{\phi}_1^*(x) \neq \phi^*(x)$ . We can compute  $\tilde{\phi}_1^*(x)$  from equation (146) for the fixed boundary condition case:

$$\begin{aligned}
 \tilde{\phi}_1^*(x) = & \frac{\phi_1 - \phi_0}{k_1} \frac{1}{(e^{\tilde{E}} - 1)^2} \left[ (\phi_1 - \phi_0) (e^{\tilde{E}x} - 1) (e^{\tilde{E}} - e^{\tilde{E}x}) \right. \\
 & \left. + 2\tilde{E}(\phi_0 e^{\tilde{E}} - \phi_1) \left[ e^{\tilde{E}x}(1 - x) + \frac{e^{\tilde{E}x} - e^{\tilde{E}}}{e^{\tilde{E}} - 1} \right] \right] \quad (148)
 \end{aligned}$$

that in the limit  $\tilde{E} \rightarrow 0$  is reduced to

$$\tilde{\phi}_1^*(x) = \frac{2}{k_1}(\phi_1 - \phi_0)^2 x(1 - x). \quad (149)$$

(b)  $\mathbf{D}(\mathbf{u}) = \bar{\mathbf{D}}_0/(\bar{\Lambda}_0 + \mathbf{u})^2$ ,  $\chi(\mathbf{u}) = \mathbf{k}_0 + \mathbf{k}_1\mathbf{u}$ .

We found in this case:

$$\begin{aligned}
 A[\phi, \tilde{\phi}; x] = & 2 \int_{\tilde{\phi}(x)}^{\phi(x)} \frac{du D(u)}{\chi(u)} \equiv a(\phi(x), \tilde{\phi}(x)) \\
 \phi[\tilde{\phi}; x] = & -\bar{\Lambda}_0 + \frac{\bar{D}_0 \left[ (k_1 D(\tilde{\phi}(x)) - D'(\tilde{\phi}(x))\chi(\tilde{\phi}(x)))\tilde{\phi}'(x)^2 - D(\tilde{\phi}(x))\chi(\tilde{\phi}(x))\tilde{\phi}''(x) \right]}{\tilde{\phi}'(x)D(\tilde{\phi}(x)) \left[ a_0\chi(\tilde{\phi}(x)) - a_1D(\tilde{\phi}(x))\tilde{\phi}'(x) \right]} \quad (150)
 \end{aligned}$$

where  $a_0 = k_0E - J - k_1E\bar{\Lambda}_0$  and  $a_1 = k_0 - k_1\bar{\Lambda}_0$ . We use these expressions in equation (48), and initially, we follow similar steps as in the above case to get:

$$\begin{aligned}
 V_0[\eta] = & V_0[\phi^*] + \int_{\Lambda} dx [v(\eta(x)) - v(\phi^*) - \eta(x)v'(\tilde{\eta}(x)) + \phi^*(x)v'(\phi^*(x))] \\
 & + \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x))v''(\tilde{\phi}(x; \lambda))\phi[\tilde{\phi}; x], \quad (151)
 \end{aligned}$$

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where we remind that  $v(u) = 2 \int du \int du D(u)/\chi(u)$ . After substituting  $\phi[\tilde{\phi}; y]$  from its expression we can decompose the last integral into two pieces:

$$\int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) v''(\tilde{\phi}(x; \lambda)) \phi[\tilde{\phi}; x] = -\bar{\Lambda}_0 I_{11} - \bar{D}_0 I_{12}, \quad (152)$$

where

$$\begin{aligned} I_{11} &= \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) v''(\tilde{\phi}(x; \lambda)) \\ I_{12} &= \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \phi^*(x)) v''(\tilde{\phi}(x; \lambda)) \frac{u'(x; \lambda)}{u(x; \lambda)(a_0 + a_1 u(x; \lambda))}, \end{aligned} \quad (153)$$

where  $u(x, \lambda) = v''(\tilde{\phi}(x; \lambda)) \tilde{\phi}'(x; \lambda)/2$ . The integral  $I_{11}$  is straightforwardly evaluated using the change of variables  $\lambda \rightarrow \tilde{\phi}(x; \lambda)$  at each  $x$ . Then:

$$I_{11} = \int_{\Lambda} dx [v'(\tilde{\eta}(x)) - v'(\phi^*(x))]. \quad (154)$$

The integral  $I_{12}$  is done by first rewriting  $u'/(u(a_0 + a_1 u)) = (\log|u/(a_0 + a_1 u)|)'/a_0$ . Second, we integrate by parts where the surface term is zero due to the boundary conditions and third, we use the relation:

$$\frac{du(x; \lambda)}{d\lambda} = \frac{1}{2} \frac{d}{dx} \left[ v''(\tilde{\phi}(x; \lambda)) (\tilde{\eta}(x) - \phi^*(x)) \right]. \quad (155)$$

Finally, we put all of the terms together and we find:

$$\begin{aligned} V_0[\eta] &= V_0[\phi^*] + \int_{\Lambda} dx \left[ v(\eta(x)) - v(\phi^*) - (\eta(x) + \bar{\Lambda}_0) v'(\tilde{\eta}(x)) + (\phi^*(x) \right. \\ &\quad \left. + \bar{\Lambda}_0) v'(\phi^*(x)) + \frac{\bar{D}_0}{a_0} v''(\phi^*(x)) \phi^{*'}(x) \log \left[ a_1 + \frac{2a_0}{v''(\phi^*(x)) \phi^{*'}(x)} \right] \right. \\ &\quad \left. - \frac{\bar{D}_0}{a_0} v''(\tilde{\eta}(x)) \tilde{\eta}'(x) \log \left[ a_1 + \frac{2a_0}{v''(\tilde{\eta}(x)) \tilde{\eta}'(x)} \right] \right. \\ &\quad \left. - \frac{2\bar{D}_0}{a_1} \log \left[ \frac{2a_0 + a_1 v''(\tilde{\eta}(x)) \tilde{\eta}'(x)}{2a_0 + a_1 v''(\phi^*(x)) \phi^{*'}(x)} \right] \right], \end{aligned} \quad (156)$$

where  $\tilde{\eta}(x)$  is the solution of the differential equation:

$$\begin{aligned} \eta(x) &= -\bar{\Lambda}_0 \\ &\quad + \frac{\bar{D}_0 [(k_1 D(\tilde{\eta}(x)) - D'(\tilde{\eta}(x)) \chi(\tilde{\eta}(x))) \tilde{\eta}'(x)^2 - D(\tilde{\eta}(x)) \chi(\tilde{\eta}(x)) \tilde{\eta}''(x)]}{\tilde{\eta}'(x) D(\tilde{\eta}(x)) [a_0 \chi(\tilde{\eta}(x)) - a_1 D(\tilde{\eta}(x)) \tilde{\eta}'(x)]} \end{aligned} \quad (157)$$

for any given  $\eta(x)$  field.

(iii)  $(n, m, l, s) = (0, 0, 1, 1)$ .

In this case, we choose  $a = a(\phi(x), \tilde{\phi}(x))$  and  $\phi^{(1)}(x) = f(\phi(x), \tilde{\phi}(x), \tilde{\phi}'(x))$ . The corresponding operators (equations (87)–(90)) have the form:

$$\begin{aligned} P_0(x) &= a^{(1,0)}(u_0, v_0) \\ Q_0(x) &= a^{(0,1)}(u_0, v_0) \\ L_1(x) &= f^{(1,0,0)}(u_0, v_0, v_1) - \frac{d}{dx} \\ S_1(x) &= f^{(0,1,0)}(u_0, v_0, v_1) + f^{(0,0,1)}(u_0, v_0, v_1) \frac{d}{dx}. \end{aligned} \tag{158}$$

The conditions for the existence of  $V_0$  are given by equation (101):

- (C1T): this condition is fulfilled because  $P_0(x)$  is self-adjoint by construction.
- (C3T) and (EM): they are polynomials on  $v_k$ .

The strategy we have followed to unveil the form of the functions  $a$  and  $f$  is:

- (1) Coefficient of  $v_2$  from (C3T) equal to zero:

$$f^{(0,0,2)}(u_0, v_0, v_1) = 0 \Rightarrow f(u_0, v_0, v_1) = f_0(u_0, v_0) + v_1 f_1(u_0, v_0). \tag{159}$$

- (2) Rest of (C3T) after substituting equation (159):

$$f_0(u_0, v_0) f_1^{(1,0)}(u_0, v_0) = f_0^{(0,1)}(u_0, v_0) + f_1(u_0, v_0) f_0^{(1,0)}(u_0, v_0). \tag{160}$$

- (3) Coefficient  $v_3$  from (EM):

There are two possibilities,

$$\begin{aligned} f_1(u_0, v_0) &= -\frac{a^{(0,1)}(u_0, v_0)}{a^{(1,0)}(u_0, v_0)} \\ f_1(u_0, v_0) &= \frac{\chi(u_0) a^{(0,1)}(u_0, v_0)}{2D(u_0) - \chi(u_0) a^{(1,0)}(u_0, v_0)}. \end{aligned}$$

One can show that the first case implies the case  $(n, m, l, s) = (0, 0, 0, 0)$ .

- (4) Coefficient  $v_2$  from (EM):

We get a partial differential equation on  $f_0(u_0, v_0)$  that, together with equation (160), implies:

$$\begin{aligned} a(u_0, v_0) &= \tilde{a}(u_0) - \tilde{a}(v_0) \\ f_0(u_0, v_0) &= c_0 \frac{\chi(u_0)}{2D(u_0) - \chi(u_0) \tilde{a}'(u_0)} \\ f_1(u_0, v_0) &= -\frac{\tilde{a}'(v_0) \chi(u_0)}{2D(u_0) - \chi(u_0) \tilde{a}'(u_0)}. \end{aligned} \tag{161}$$

These relations imply that (C3T) is fulfilled.

- (5) Rest of condition (EM):

$$c_0(c_0 - 2E)\chi(u_0)^2(c_0 - v_1\tilde{a}'(v_0))\chi''(u_0) = 0. \tag{162}$$

There are three possibilities:  $c_0 = 0$ ,  $c_0 = 2E$  or  $\chi''(u_0) = 0$ . In order to elucidate which one of these is the correct, we apply our equation to the stationary state. First we know that  $\phi^*(x) = \tilde{\phi}^*(x)$  because  $a(\phi^*(x), \phi^*(x)) = 0$ . Then, we apply this to our equation  $u_1 = f(u_0, v_0, v_1)$  and we find that  $c_0 = 2\phi^{*'}(x)D(\phi^*(x))/\chi(\phi^*(x))$ . On the other hand, we know that the stationary state is the solution of  $\phi^{*'}(x)D(\phi^*(x))/\chi(\phi^*(x)) = E - J/\chi(\phi^*(x))$  that implies  $\chi(u_0) = \chi_0$ . This is coherent with one of the conditions in equation (162) and (EM) is fulfilled. Finally, we can write equation (159) as:

$$\phi'(x)[2D(\phi(x)) - \chi_0\tilde{a}'(\phi(x))] = 2(E\chi_0 - J) - \chi_0\tilde{\phi}'(x)\tilde{a}'(\tilde{\phi}(x)) \tag{163}$$

that it can be integrated:

$$\tilde{a}(\phi(x)) - \tilde{a}(\tilde{\phi}(x)) = \frac{2}{\chi_0} \int_{\phi^*(x)}^{\phi(x)} du D(u). \tag{164}$$

The quasipotential can be calculated following similar steps as in the above cases. We know that  $V_0$  can be written as:

$$V_0[\eta] = V_0[\phi^*] + I_1 - I_2, \tag{165}$$

where

$$I_1 = \int_{\Lambda} dx \left[ \int_{\phi^*(x)}^{\eta(x)} du \tilde{a}(u) - \eta(x)\tilde{a}(\tilde{\eta}(x)) + \phi^*(x)\tilde{a}(\phi^*(x)) \right] \tag{166}$$

$$I_2 = \int_0^1 d\lambda \int_{\Lambda} dx (\eta(x) - \phi^*(x)) \phi(x; \lambda) \tilde{a}'(\tilde{\phi}(x; \lambda))$$

$I_2$  can be simplified by using equation (164) with  $\tilde{\phi}(x; \lambda)$  and  $\phi(x; \lambda)$  and after doing a  $\lambda$  derivative in it we get the relation:

$$\tilde{a}'(\tilde{\phi}(x; \lambda)) (\eta(x) - \phi^*(x)) = \frac{d\phi(x; \lambda)}{d\lambda} \left( \tilde{a}'(\phi(x; \lambda)) - \frac{2}{\chi_0} D(\phi(x; \lambda)) \right). \tag{167}$$

Finally, we do the change of variables  $\lambda \rightarrow \phi(x; \lambda)$  at each  $x$  and we find:

$$I_2 = \int_{\Lambda} dx \int_{\phi^*(x)}^{\eta(x)} du u \left[ \tilde{a}'(u) - \frac{2}{\chi_0} D(u) \right]. \tag{168}$$

Putting together  $I_1$  and  $I_2$  we get:

$$V_0[\eta] = V_0[\eta^*] + \frac{2}{\chi_0} \int_{\Lambda} dx \int_{\phi^*(x)}^{\eta(x)} du \int_{\phi^*(x)}^u dv D(v) \tag{169}$$

for any  $D(u)$  and  $\chi(u) = \chi_0$ .  $\phi^*(x)$  is solution of

$$D(\phi^*(x))\phi^{*'}(x) = \chi_0 E - J. \tag{170}$$

Let us finish this section by commenting that we also attempted other values for  $(n, m, l, s)$ . For instance,  $(0, 1, 0, 1)$  reduces to the case  $(0, 0, 1, 1)$  we have studied explicitly. However, we could not find anything for cases  $(0, 0, 2, 2)$ ,  $(0, 0, 0, 4)$ , and  $(0, 2, 0, 2)$ .

### 6. Quasipotentials for some one-dimensional reaction–diffusion models

We first study the reaction–diffusion model whose Langevin equation is given by (8) with

$$F[\phi; x] = g(\phi)\phi''(x) + w(\phi(x)). \tag{171}$$

We choose  $(n, m, l, s) = (0, 0, 0, 0)$ . As happened in the similar case of diffusion dynamics, conditions (C1T) and (C3T) are fulfilled by construction. The condition (EM) is built by using the functionals:

$$\begin{aligned} R_1[\phi, \tilde{\phi}; x] &= g(\phi)\phi''(x) + w(\phi(x)) + a(x)h^2(\phi(x)) \\ R_2[\phi, \tilde{\phi}; x] &= -a(x) [g'(\phi(x))\phi''(x) + w'(\phi(x)) + a(x)h(\phi(x))h'(\phi(x))] \\ &\quad - (a(x)g(\phi(x)))'', \end{aligned} \tag{172}$$

where  $a(x) = a(\phi(x), \tilde{\phi}(x))$ . After substitutions and some trivial algebra, the condition (EM) becomes a polynomial on  $v_k$  where their coefficients are equal to zero:

- Coefficient of  $v_2$  from (EM) equal to zero:

$$a(u_0, v_0; u_s) = \frac{C}{g(u_0)}. \tag{173}$$

This is a singular case because at the stationary state  $a = 0$  and therefore  $g(u_s)^{-1} = 0$ . Let us assume that

$$g(u) = \bar{g}(u)(u - u_s)^{-\alpha} \quad \alpha > 0. \tag{174}$$

This is coherent whenever the stationary state is a constant:  $\phi^*(x) = \phi^*$  (because we are not considering in this paper local functions of  $g$ ,  $w$  or  $h$ ). Observe that with this election the deterministic evolution equation is:

$$\dot{\phi}_D(x, t) = \bar{g}(\phi_D(x, t))(\phi_D(x, t) - \phi^*)^{-\alpha} \phi_D''(x, t) + w(\phi_D(x, t)). \tag{175}$$

Near the equilibrium,  $\phi(x, t) = \phi^* + \epsilon(x, t)$  the dominant terms of this equation for very small values of  $\epsilon$  can be written:

$$\partial_t \epsilon(x, t) = w'(\phi^*)\epsilon(x, t) + \epsilon^{-\alpha} \bar{g}(\phi^*) \partial_{xx}^2 \epsilon(x, t), \tag{176}$$

where  $\phi^*$  is a solution of  $w(\phi^*) = 0$ . The diffusion term is singular but it goes very fast to zero because it behaves as if the system had almost an infinite diffusivity and, therefore, it homogenizes any initial profile very quickly in such a way that the spatial second derivative becomes zero. The reaction term makes  $\epsilon$  evolve exponentially quickly towards zero whenever  $w'(\phi^*) < 0$ . Therefore, this singular system is well behaved near the stationary solution.

- The rest of the (EM) condition is fulfilled when:

$$h^2(u) = -2g(u)w(u). \tag{177}$$

One can check that with the form of  $a$  and  $h$ , the Hamiltonian  $H[\phi, \pi] = 0$  along the trajectory.

The quasipotential can be written as

$$V_0[\eta] = V_0[\phi^*] + C \int_0^1 d\lambda \int_{\Lambda} dx \frac{(\tilde{\eta}(x) - \phi^*(x)) \partial f(v; \phi^*)}{g(f(\tilde{\phi}(x; \lambda)))} \Big|_{v=\tilde{\phi}(x; \lambda)}, \tag{178}$$

where  $\phi[\tilde{\phi}; x] = f(\tilde{\phi}(x); \phi^*(x))$  and  $\tilde{\phi}(x; \lambda) = \phi^*(x) + \lambda(\tilde{\eta}(x) - \phi^*(x))$ . We do the change of variable  $\lambda \rightarrow \tilde{\phi}(x; \lambda)$  at each  $x$  point and we get

$$V_0[\eta] = V_0[\phi^*] + \int_{\Lambda} dx \int_{\phi^*}^{\eta(x)} \frac{du}{g(u)}, \tag{179}$$

where  $g(\phi) = \bar{g}(\phi)(\phi - \phi^*)^{-\alpha}$ ,  $\alpha > 0$ .

The other interesting reaction–diffusion model we have studied is the Poissonian reaction–diffusion dynamics. We already defined this model in section 2. We choose  $(n, m, l, s) = (0, 0, 0, 0)$ . Again, conditions (CT1) and (CT3) are fulfilled by construction, and the (EM) condition is built with

$$\begin{aligned} R_1[\phi, \tilde{\phi}; x] &= \phi''(x) - 2(a'(x)\phi(x)(1 - \phi(x)))' + b(\phi(x))e^{a(x)} - d(\phi(x))e^{-a(x)} \\ R_2[\phi, \tilde{\phi}; x] &= -a''(x) - a'(x)^2(1 - 2\phi(x)) + b'(\phi(x))(1 - e^{a(x)}) \\ &\quad + d'(\phi(x))(1 - e^{-a(x)}). \end{aligned} \tag{180}$$

The coefficient for  $v_2$  equalled to zero gives us the first condition:

$$f'(v_0) - f(v_0)(1 - f(v_0)) \frac{da(f(v_0), v_0)}{dv_0} = 0 \Rightarrow a(u_0, v_0) = \log \frac{u_0}{1 - u_0} + C, \tag{181}$$



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where  $C$  is a constant that is fixed by the condition:  $a(\phi^*, \phi^*) = 0$ :  $C = \log[(1 - \phi^*)/\phi^*]$ . Let us repeat that we are using periodic boundary conditions and that the stationary state is homogeneous:  $\phi^*(x) = \phi^*$ , where  $\phi^*$  is the solution of  $b(\phi^*) = d(\phi^*)$ .

We use the result on  $a$  to get the rest of the (EM) condition:

$$\begin{aligned} u_0^2(1 - \phi^*)^2 b(u_0) - \phi^{*2}(1 - u_0)^2 d(u_0) \\ = u_0(1 - u_0)(u_0 - \phi^*) [\phi^*(1 - u_0)d'(u_0) - u_0(1 - \phi^*)b'(u_0)]. \end{aligned} \quad (182)$$

The solution of this equation is:

$$b(u) = \phi^*(1 - u)h(u), \quad d(u) = (1 - \phi^*)uh(u), \quad (183)$$

where  $h(u)$  is a positive function.

Once we know  $a(u_0, v_0)$  and the functions  $b(u)$  and  $d(u)$  that fulfills the conditions we can compute the quasipotential. We almost follow the same steps as we did for the diffusion case for  $(n, m, l, s) = (0, 0, 0, 0)$  and we get

$$V_0[\eta] = V_0[\phi^*] + \int_{\Lambda} dx \left[ \eta(x) \log \left[ \frac{\eta(x)}{\phi^*} \right] + (1 - \eta(x)) \log \left[ \frac{1 - \eta(x)}{1 - \phi^*} \right] \right]. \quad (184)$$

This case was already studied by Gabrielli *et al* [20, 25].

We also studied the case  $(n, m, l, s) = (0, 0, 0, 2)$ , but no situation fulfilled the conditions.

## 7. Summary of results and conclusions

The mesoscopic description of non-equilibrium systems given by the MFT [20] is a solid background to study their generic properties. Even though MFT is mathematically simpler than its microscopic original description, it is still difficult to extract precise information from our actual analytical tools. This paper intends to build a method to get the stationary measure represented by the quasipotential at the small noise limit. Formally, the quasipotential is obtained by a time integral of some variables along a path defined by a Hamiltonian that depends on the studied system,  $(\phi(t), \pi(t))$ :

$$V_0[\eta] \sim \int_{-\infty}^0 dt \int_{x \in \Lambda} \pi(x, t) \partial_t \phi(x, t). \quad (185)$$

After a canonical transformation of type 1 with generator  $L[\phi, \tilde{\phi}]$ , the situation does not change too much, and the quasipotential still is mainly a time integral like equation (185) along a path defined by the same Hamiltonian expressed in the new variables  $(\tilde{\phi}(t), \tilde{\pi}(t))$ . However, we can effectively deform this last path into a straight line by using the properties of the canonical transformation and assuming that there exists a map between the paths followed by the original fields and the transformed ones:

$$\text{Main assumption: } \phi(x, t) = \phi[\tilde{\phi}(t); x] \quad \forall t \in [-\infty, 0]. \quad (186)$$

Therefore, the quasipotential can be expressed in the *convenient form*, equation (48):

$$V_0[\eta] = V_0[\phi^*] + \int_0^1 d\lambda \int_{\Lambda} dx (\tilde{\eta}(x) - \tilde{\phi}^*(x)) \int_{\Lambda} dy A[\phi[\tilde{\phi}(\lambda)], \tilde{\phi}(\lambda); y] K[\tilde{\phi}(\lambda); y, x], \tag{187}$$

where  $\tilde{\phi}(x, \lambda) = \tilde{\phi}^*(x) + \lambda(\tilde{\eta}(x) - \tilde{\phi}^*(x))$ . In this way, we manage to get rid of highly non-trivial path integration. However, we pay the price of introducing two local functionals:  $A[\phi; \tilde{\phi}; x] = \delta L[\phi, \tilde{\phi}] / \delta \phi(x)$  and  $K[\tilde{\phi}; x, y] = \delta \phi[\tilde{\phi}; x] / \delta \tilde{\phi}(y)$ . At this point, our method’s goal is to design a way to directly determine both functionals:  $A[\phi; x]$  and  $K[\tilde{\phi}; x, y]$ . We know that they depend on the canonical transformation and on Hamilton’s equations that define the paths. Therefore, they cannot take any functional form. In fact, we show that they should fulfill three compatibility conditions (C1T), (C3T), and (EM) given by equations (53), (55) and (61), respectively. These conditions are the core of our method: we initially give functional forms for  $A$  and  $\phi[\tilde{\phi}]$ , using the compatibility conditions to determine their fine structure.

In this paper (section 4), we only use the family of functionals given by equations (84) and (85):

$$A[\phi, \tilde{\phi}; x] = a(\phi(x), \phi'(x), \dots, \phi^{(n)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(m)}(x), \phi^*(x)) \tag{188}$$

$$\phi^{(l)}(x) = f(\phi(x), \phi'(x), \dots, \phi^{(l-1)}(x), \tilde{\phi}(x), \tilde{\phi}'(x), \dots, \tilde{\phi}^{(s)}(x), \phi^*(x)). \tag{189}$$

Observe that  $K$ -functional can be derived from equation (189) just by a functional derivative on both sides (see main text). This family permits to write the compatibility conditions compactly by using differential operators (see equations (93), (95) and (96)) where some inverse operators appear. This fact makes their algebraic use very difficult, if not impossible. To go forward, we restrict ourselves to cases where such inverse differential operators are just functions. Then the compatibility conditions simplify and become equations (100) or (101), and they are algebraically manageable.

We show in the paper (sections 3.4, 5 and 6) how to algebraically deal with such conditions once we fix a particular set of  $(n, m, l, s)$  values for the functions  $a$  and  $f$  (equations (188) and (189), respectively). We observe that the compatibility conditions can determine the form of the functions  $a$  and  $f$ . At the same time, they select a subfamily of Langevin equations (for instance, the functional form for the diffusion or the mobility for diffusive systems). We explicitly apply this method to already well-known cases, discovering new solutions that may be of general interest. In particular, for one dimensional diffusive systems that are characterized by the functions  $D(\phi)$  and  $\chi(\phi)$  (see main text) we get the explicit quasipotential for the cases: (i)  $D(u) = c\chi'(u)$  in equation (78), (ii)  $D(u) = D_0$ ,  $\chi(u) = k_0 + k_1u + k_2u^2$  in equation (143), (iii)  $D(u) = D_0/(\Lambda_0 + u)^2$ ,  $\chi(u) = k_0 + k_1u$  in equation (156) and (iv)  $D(u), \chi(u) = \chi_0$  in equation (169).

We observe sets of values  $(n, m, l, s)$  that do not fulfill the compatibility conditions. In general, this method does not guarantee *a priori* that any given functional structure for  $a$  and  $f$  should be associated with a well-defined canonical transformation. Therefore,

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this is a kind of trial-and-error method at this moment. It would be relevant in the future to have an *a priori* deeper knowledge about compatible functional forms for  $A$  and  $\phi[\tilde{\phi}]$ .

We think that this method may be developed and improved further in several ways. For instance, applying it to higher dimensional systems, at least initially, for simple cases as the zero-range model or even the SSE could be interesting. On the other hand, in the paper, we focused on transformations whose compatibility conditions do not contain generic differential operators inverse. For these cases, we handle the structure of the compatibility conditions easily. We think that there is vast work to be done dealing with the more generic cases where such inverse operators appear. Probably they carry stronger non-local properties necessary to describe the behavior of more complex situations. It could also be interesting to define some other functional structures for  $A$  and  $\phi[\tilde{\phi}; x]$  as integrals of functions over local domains or similar.

In this paper, we only dealt with canonical transformations of type I, and the study of other types may imply new quasipotential structures. Finally, it could be interesting to set up a systematic perturbation theory similar to Bouchet *et al* [26] but associated with the canonical transformation we have presented.

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### Appendix A

Let  $R[\phi; x]$  be a given local functional on  $\phi$ . We know that

$$\frac{\delta B[\phi]}{\delta \phi(x)} = R[\phi; x] \tag{190}$$

and let us assume that  $B[\phi]$  exists, that is

$$\frac{\delta R[\phi; x]}{\delta \phi(y)} = \frac{\delta R[\phi; y]}{\delta \phi(x)}. \tag{191}$$

Then we can show that

$$B[\phi] = B[\phi^*] + \int_0^1 d\lambda \int_{\Lambda} dx (\phi(x) - \phi^*(x)) R[\phi(\lambda); x], \tag{192}$$

where

$$\phi(x; \lambda) = \phi^*(x) + \lambda(\phi(x) - \phi^*(x)). \tag{193}$$

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*Demonstration:* let us show that the derivative of (192) is (190).

$$\frac{\delta B[\phi]}{\delta \phi(y)} = \int_0^1 d\lambda \int_{\Lambda} dx \left[ R[\phi(\lambda); x] \delta(x - y) + (\phi(x) - \phi^*(x)) \frac{\delta R[\phi(\lambda); x]}{\delta \phi(y)} \right]. \quad (194)$$

We can use the relations:

$$\begin{aligned} \frac{\delta R[\phi(\lambda); x]}{\delta \phi(y)} &= \lambda \frac{\delta R[\phi; x]}{\delta \phi(y)} \Big|_{\phi=\phi(\lambda)}, \\ \frac{dR[\phi(\lambda); x]}{d\lambda} &= \int_{\Lambda} dy \frac{\delta R[\phi; y]}{\delta \phi(x)} \Big|_{\phi=\phi(\lambda)} (\phi(y) - \phi^*(y)), \end{aligned} \quad (195)$$

where we have made use of (191). Then we get

$$\frac{\delta B[\phi]}{\delta \phi(y)} = \int_0^1 d\lambda \left[ R[\phi(\lambda); y] + \lambda \frac{dR[\phi(\lambda); y]}{d\lambda} \right] = R[\phi(\lambda); y] \quad \text{c.q.d.} \quad (196)$$

## Appendix B. Self adjoint conditions for an $n$ -differential operator in one dimension

Let us  $L$  be a linear differential operator of  $n$ th order:

$$L = \sum_{k=0}^n a_k(x) \frac{d^k}{dx^k}, \quad (197)$$

where  $a_k(x) \in \mathcal{R} \forall x \in \Lambda \subset \mathcal{R}$ . We define the inner product for two real analytic functions  $u(x), v(x)$ :

$$\langle u, v \rangle = \int_{\Lambda} dx u(x)v(x). \quad (198)$$

The adjoint of  $L$ ,  $L^\dagger$ , is then defined by

$$\langle L^\dagger u, v \rangle = \langle u, Lv \rangle. \quad (199)$$

Therefore

$$L^\dagger = \sum_{k=0}^n (-1)^k \frac{d^k}{dx^k} a_k(x), \quad (200)$$

where it is assumed that the set of real functions  $v(x), u(x)$ , where  $L$  and  $L^\dagger$  respectively apply, has boundary conditions such that

$$\sum_{k=1}^n \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dx^l} (a_k(x)u(x)) \frac{d^{k-1-l}}{dx^{k-1-l}} v(x) \Big|_{\partial\Lambda} = 0. \tag{201}$$

$L$  is called *self-adjoint* if  $L = L^\dagger$  and the set of boundary conditions for the  $v(x)$  and  $u(x)$  functions coincide and fulfills equation (201). Therefore, the coefficients of  $L$  such that  $L = L^\dagger$  should be related by

$$a_k(x) = \sum_{l=k}^n (-1)^l \binom{l}{k} \frac{d^{l-k}}{dx^{l-k}} a_l(x) \quad k = 1, \dots, n. \tag{202}$$

Observe that only the operators  $L$  with  $n$  even can be self-adjoint. That can be shown by applying relation (202) to the case  $k = n$ .

One realizes that not all of the  $n$ -relations defined by (202) are independent. In fact, we can show that the independent set of relations that define a self-adjoint operator is given by:

$$a_{2l+1}(x) = \frac{1}{(2l+1)!} \sum_{s=0}^{m-l-1} (2(s+l+1))! c_s \frac{d^{2s+1}}{dx^{2s+1}} a_{2l+2s+2}(x), \tag{203}$$

where  $l = 0, \dots, m-1$  and  $n = 2m$ . The  $c$ 's are a set of numbers generated by the recurrence:

$$c_l = \frac{1}{2(2l+1)!} - \frac{1}{2} \sum_{k=0}^{l-1} \frac{c_k}{(2l-2k)!}, \quad l > 0, \quad c_0 = 1/2. \tag{204}$$

For instance:  $c_1 = -1/24$ ,  $c_2 = 1/240$ ,  $c_3 = -17/40\,320$ ,  $c_4 = 31/725\,760$ , ... Curiously enough, we find that  $c$ 's follow another relation:

$$\frac{1}{(2l+2)!} = \sum_{k=0}^l \frac{c_k}{(2l-2k+1)!}. \tag{205}$$

We have computed the first 100 values of  $c$ 's and found that they alternate signs and their modulus decreases exponentially quickly:  $|c_l| \simeq 1.82 \exp[-2.29l]$ .

As an example, for  $n = 2$  the condition for self-adjointness is:

$$a_1(x) = \frac{da_2(x)}{dx} \tag{206}$$

and for  $n = 4$  we have two conditions:

$$a_1(x) = \frac{da_2(x)}{dx} - \frac{d^3a_4(x)}{dx^3}, \quad a_3(x) = 2 \frac{da_4(x)}{dx}. \tag{207}$$

Let us define the Green functions  $G$  and  $G^\dagger$  solutions of the equations:

$$LG(x, x_0) = \delta(x - x_0), \quad L^\dagger G^\dagger(x, x_0) = \delta(x - x_0). \tag{208}$$

*Proposition:*  $L$  is self-adjoint if and only if  $G(x_1, x_2) = G(x_2, x_1)$ .

To prove the proposition, let us choose  $u(x) = G^\dagger(x, x_2)$  and  $v(x) = G(x, x_1)$  for the inner product in equation (198). Then we get:

$$G^\dagger(x_1, x_2) = G(x_2, x_1). \quad (209)$$

If  $L$  is self-adjoint then  $G^\dagger(x_1, x_2) = G(x_1, x_2)$  and using equation (209) we prove the right implication. If we assume  $G(x_1, x_2) = G(x_2, x_1)$  we see from (209) that  $G^\dagger(x_1, x_2) = G(x_1, x_2)$ . Assuming that there is a unique solution for each operator, we get  $L^\dagger = L$ .

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