Long-range correlations for conservative dynamics

Pedro L. Garrido
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Joel L. Lebowitz
Department of Mathematics and Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

Christian Maes* and Herbert Spohn†
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903
(Received 26 March 1990)

We investigate the origin of long-range spatial correlations in certain anisotropic translation-invariant stationary nonequilibrium states of systems with conservative dynamics. We consider both lattice-gas models with anisotropic but reflection-invariant stochastic dynamics and driven diffusive systems described by a Ginzburg-Landau equation with an electric field E. Carrying out perturbation expansions about an equilibrium state with short-range correlations we find that, in general, the spatial correlations in the stationary state decay only via a power law; the spatial decay thus reflects the well-known diffusive decay in time for systems with conservative dynamics. The typical spatial decay of the pair correlation behaves like the electrostatic potential produced by a quadrupole charge density at the origin. Exponential decay of spatial correlations, so familiar from equilibrium, appears here as the exception; it occurs generically only when there are special constraints on the dynamics, such as detailed balance and possibly spatial symmetry. The paradigm for this generic long-range behavior or self-organized criticality is found in the solutions of the linear Langevin equation describing the behavior of fluctuations of a conserved macroscopic variable. The fluctuating hydrodynamics underlying such a description is justified here rigorously for the macroscopic scaling limit of certain lattice-gas models.

I. INTRODUCTION

It is well known that in equilibrium systems described by Gibbs measures, with an interaction energy H that is exponentially (or more rapidly) decaying with distance, the spatial correlations at high temperatures and/or low densities also decay exponentially. This is, in fact, expected to be the typical equilibrium behavior for pure phases at all temperatures and densities. Important exceptions are systems at critical points in their parameter space and special models, such as the massless harmonic crystal, where long-range correlations are caused by the presence of a continuous symmetry. These cases are, however, clearly non-"generic" in the class of equilibrium systems.

The situation appears to be quite different for non-equilibrium systems where long-range correlations seem to be the rule,[1,2] giving rise to what Bak *et al.* call *self-organized criticality*[3] (SOC). In this paper we study the origin of such long-range correlations in translation-invariant stationary non-equilibrium systems. To do that we need to say something about the dynamics with respect to which the measure is stationary.

For closed physical systems there is a natural dynamics associated with an interaction H, and a chief characteristic of the Gibbs measure ν_H (when extended also to the momentum variables) is that it is stationary with respect to the time evolution induced by this dynamics. It is, in fact, expected to be the only such stationary measure for isolated macroscopic systems with realistic interactions in the infinite volume (thermodynamic) limit.[4] To obtain such systems in a stationary nonequilibrium, i.e., non-Gibbsian, state, they must be coupled to "external" sources and sinks that maintain steady-state fluxes through them. While such "open systems" have been extensively studied from various points of view, their microscopic structure, even when they are far away from any (obvious) critical points, is still an open problem;[5] hence our interest in the study of model lattice systems with simpler dynamics.

Historically, lattice systems, such as the Ising model, were introduced to model equilibrium behavior of real systems. They were therefore described by Gibbs measures ν_H, with suitable energy H, and their states were referred to as equilibrium states even though there is no *natural* dynamics associated with this H. There are, on the other hand, many *artificial* stochastic dynamics for these lattice systems for which the Gibbs measure ν_H is stationary, Glauber spin-flip and Kawasaki exchange dynamics satisfying detailed balance with respect to ν_H being the best known examples. Now while these dynamics can be quite artificial (they were invented just to get ν_H as the stationary measure), there are good reasons to believe that with *appropriately* chosen dynamics, these lattice systems can also correctly model the nonequilibrium behavior of corresponding physical systems.[6,7] In particular, we expect these models to correctly represent the long-distance behavior of correlations in real systems with appropriate external reservoirs designed to maintain such

nonequilibrium stationary states. We wish to exploit this and use the richness of stochastic models to study the behavior of correlations in lattice systems described by measures ν which are stationary with respect to appropriate local-conservative dynamics, e.g., Kawasaki-type exchanges, whose rates depend only on neighboring sites, but which apparently are not Gibbsian for any interaction H. In addition to lattice gases, we shall also consider continuum models with a conserved scalar-order parameter evolving according to the Ginzburg-Landau equations (model B of Ref. 6). Adding an external electric field E to such a system produces a stationary state not described by any Gibbs measure.[8]

The present investigation originated in the observation made in Ref. 1, using computer simulations on a driven diffusive lattice gas on a square and simple cubic lattice. It was found there that the pair correlation $G(\mathbf{r})$ in the stationary state was very anisotropic, and decayed with distance like a power law $|G(\mathbf{r})| \sim r^{-d}$, for $d=2$ and 3. This occurred even at very high "temperatures," far away from any observed or calculated critical point. In fact, the simulations fitted rather well to the first term in an approximate high-temperature expansion of the pair correlation, which predicted the behavior of $G(\mathbf{r})$ to be like that of a quadrupole field on the lattice. Similar long-range correlations are also implicitly present in the field-theoretic treatment of these driven diffusive systems (DDS) described by the Ginzburg-Landau equations with an external field.[9,10]

The existence of long-range correlations in nontranslation-invariant stationary nonequilibrium systems, in which macroscopic gradients are maintained by boundary reservoirs, is predicted by fluctuating hydrodynamics (cf. Refs. 11–14 for a review). A rigorous derivation of them, for a lattice-gas system evolving according to a symmetric Kawasaki exchange dynamics at infinite temperature, was given by Spohn.[15] In this system a stationary density gradient is induced by a chemical potential difference in particle reservoirs attached to sides perpendicular to the x axis a distance L apart. The system was infinite in the other $d-1$ lattice directions. Spohn found rigorously that for distances $r<L$, the correlations behave as the inverse Laplacian, i.e., they have a power-law behavior $r^{-(d-2)}$, $d>2$, growing linearly in $d=1$ and logarithmically in $d=2$, with a strength proportional to $(\delta/L)^2$, δ is the difference in chemical potential of the two reservoirs.

Fluctuating hydrodynamics is a phenomenological theory which assumes that the system is locally in thermal equilibrium, with a local temperature, density, and velocity field obtained by solving the macroscopic hydrodynamical equations subject to appropriate boundary conditions. Deviations from these local fields are then assumed to be governed by a linear stochastic partial differential equation obtained from a linearization of the hydrodynamical equations to which δ-correlated Gaussian noise sources are added. The strength of these sources is determined by treating the system locally as if it were in global equilibrium (cf. Ref. 16). This prediction of power-law behavior of the pair correlation was recently verified experimentally via light scattering from a fluid whose walls were kept at different temperatures.[17,18]

The long-range correlations in fluctuating hydrodynamics arise from the spatial nonuniformity of the macroscopic solution of the hydrodynamical equations about which the linearization has been made. This appears to be a different mechanism from that which operates in the spatially uniform driven diffuse systems, studied in Ref. 8, where there is no local equilibrium state. This suggests that the existence of such long-range spatial correlations may be typical for systems whose dynamics satisfy a conservation law. Such systems are known to have power-law decay of *temporal* correlations even in equilibrium.[6] This slow decay in time might then be expected to induce similar slow decay in the spatial correlations of the stationary states. In fact, since particles at sites a distance r apart are influenced at time t by the same local fluctuations which happened at time $t \approx r^2$ in the past, and these fluctuations have long time tails typically decaying like $t^{-d/2}$, we would expect to have a $(r^2)^{-d/2}=r^{-d}$ decay of the spatial correlations. The equilibrium state would then be nontypical—there would be some special relation, e.g., detailed balance, which makes these long-range correlations vanish in such systems except at special (critical) values of the parameters. This picture fits in well with the general point of view of SOC advanced by Bak *et al.*[3]

We investigate this problem here by first considering several microscopic model systems. The essential distinction between the behavior of correlations in systems with conservative and nonconservative dynamics is discussed in Sec. II. Conservative lattice-gas models with reflection invariance and Ginzburg-Landau models with an external field are investigated in Secs. III and IV, respectively. For the first, the presence of anisotropy is found to give power-law decay of correlations in a perturbation expansion. In the latter case we also consider the three-point correlations which show power-law decay also in one dimension. This may, however, be an artifact of the expansion.[19] In Sec. V we investigate fluctuating hydrodynamics as the paradigm for the large-scale behavior of correlations in macroscopic systems. We are not able to derive this theory for the microscopic models mentioned above except for an anisotropic model on the lattice when the ratio of vertical rates to horizontal rates becomes large. This is done in Sec. VI. Similar results hold for a very nice simple lattice system in which there can be an arbitrary number of particles at each site (*zero-range process*[16]), studied by van Beijeren.[20]

In Sec. VII we consider the role of anisotropy in producing the power-law decay of correlations. It has been argued recently by Grinstein *et al.*[21] that some anisotropy is essential for slow decay in systems in which both the "deterministic" and "fluctuating" parts of the dynamics are conservative. Their arguments are based essentially on extending the behavior obtained in Sec. V from fluctuating hydrodynamics to nonlinear systems using renormalization-group ideas, and are consistent, in dimension greater than 1, with the results obtained here. We formulate a precise conjecture about the "genericity" of such behavior in Sec. VII A.

We put all technical details in the Appendixes.

II. MICROSCOPIC MODELS

To understand the role of conservation laws in producing long-range correlations, let us consider first an Ising system with a nonconservative (model-A-type) Glauber spin-flip dynamics. It is then easy to show that if the flips are "close to independent," both temporal and spatial correlations decay exponentially fast.[22] For the "close to independent" condition to hold, it is sufficient but not necessary that the flips satisfy detailed balance with respect to an equilibrium state ν_H at high temperatures. The exponential decay is a consequence of there being a gap, when the dynamics is close to independent spin flips, in the spectrum of the generator in the master equation. This property will remain true also when the flip rates are slightly perturbed, i.e., it is true in an open set in the "space" of stochastic dynamics.[23]

As an example, we may consider a system in contact with two thermal reservoirs at different temperatures distributed throughout the bulk. This corresponds mathematically to making the generator of the dynamics a linear combination of two generators, each satisfying detailed balance for the same H, but at different temperatures, β_1^{-1} and β_2^{-1}. If both β_1 and β_2 are small, then there is still a gap in the spectrum of the time generator of the evolution which implies the existence of a unique stationary translation-invariant measure in which the space-time correlations decay exponentially.[24] Thus, on a qualitative level, the violation of detailed balance in a nonconservative dynamics has no effect on the general behavior of the space-time correlations, at least when the "interactions" are weak.

The situation can be quite different when there is no spectral gap, e.g., when the example mentioned above is modified so that both generators are of the Kawasaki type, in which nearest neighbor sites on a lattice exchange their spins at certain rates. Here, even for infinite temperature, $\beta=0$, corresponding to exchange rates which are independent of the configuration, there is no gap in the spectrum, as exhibited by the fact that the time correlations only decay with a power law. Thus there is no *a priori* reason for expecting exponential decay of spatial correlations in the stationary state, even when β_1 and β_2 are both small. In fact, we shall argue later that, in this case, there will be only power-law decay, at least when the rates are anisotropic.

To describe the Kawasaki dynamics more explicitly, let $\sigma=\{\sigma(i)\}_{i\in\mathbb{Z}^d}$ be a spin configuration on the d-dimensional lattice $\mathbb{Z}^d$, and

$$c(i,j;\sigma)=c(j,i;\sigma)>0 \tag{2.1}$$

be the rate of exchange of the spins at neighboring sites i and j, $|i-j|=1$. We assume that the $c(i,j;\sigma)$ depend only on the configuration of spins within a finite range and are translation invariant. The dynamics clearly conserves the magnetization. It satisfies *detailed balance* for the Gibbs state ν_H if and only if

$$c(i,j;\sigma)=c(i,j;\sigma^{ij})\exp\{-\beta[H(\sigma^{ij})-H(\sigma)]\}\ , \tag{2.2}$$

where σ^{ij} is the configuration σ with the spins at sites i and j interchanged. This condition can in principle be checked directly from the $c(i,j;\sigma)$ by noting that (2.2) implies that the rates for a sequence of cyclic transitions, $\sigma\to\sigma'\to\sigma''\to\ldots\sigma$, must be independent of the direction around the cycle. We are interested in the behavior of the correlations in the stationary state of such conservative dynamics when the detailed balance condition (2.2) is not satisfied. Also of interest will be the two-time correlations in this steady state.

Specifically, we will investigate two physically fairly natural models. We believe our conclusions to be valid for most systems with a conservation law.

(i) *Models without net current.* Here the exchange rates are assumed to be reflection invariant. That is, letting $\{e_\alpha\}_{\alpha=1}^d$ be unit vectors pointing in the αth direction,

$$c(0,e_\alpha;\sigma)=c(0,e_\alpha;\theta_\alpha\sigma)\ , \tag{2.3}$$

where $\theta_\alpha\sigma$ is the reflected configuration through a plane perpendicular to the bond $\langle 0,e_\alpha\rangle$:

$$\theta_\alpha\sigma(i)\equiv\sigma(i_1,\ldots,-i_\alpha+1,\ldots,i_d) \quad \text{for } i=(i_1,\ldots,i_\alpha,\ldots,i_d)\in\mathbb{Z}^d\ . \tag{2.4}$$

A particular class are the multiple-temperature models. The jump rates in the αth direction, $\alpha=1,\ldots,d$, satisfy the detailed balance condition (2.2) with inverse temperature β_α. The steady state will be anisotropic, although, since the rates are reflection symmetric; no net current is flowing.

(ii) *Driven diffusive systems.* This is the model initially studied in Ref. 8. The jumps are biased due to an external driving field. If E denotes the driving force, then the exchange rates satisfy a "local detailed balance" of the form[8,16]

$$c(i,j;\sigma)=c(i,j;\sigma^{ij})\exp\{-\beta[U(\sigma^{ij})-U(\sigma)] -E(i-j)[\sigma(i)-\sigma(j)]\}\ . \tag{2.5}$$

For periodic boundary conditions, the driving force maintains a spatially uniform anisotropic steady state with nonzero average current. The same is expected to be true for the infinite system at high temperatures, i.e., we assume that there is a unique translation-invariant stationary measure for fixed density and small β (see Ref. 16).

For both models we expand the steady-state correlations around an exactly soluble case in which the spatial correlations are short range. For model (i) the expansion is around infinite temperature, where there are no spatial correlations. Due to the particle-conservation law, however, the time correlations decay diffusively. To see this more precisely, let $\{\sigma_t\}$ be the spin configuration at time t: we define the translation-invariant time-displaced truncated spin-spin correlation function in the steady state by

$$G(i-j,t-s)=\langle\sigma_t(i)\sigma_s(j)\rangle-m^2\ , \tag{2.6}$$

where $m=\langle\sigma_t(i)\rangle$ is the magnetization, which is time independent in the steady state. The spatial Fourier transform or *structure function* is defined by

$$S(k,t)\equiv\sum_j e^{ik\cdot j}[\langle\sigma_t(j)\sigma_0(0)\rangle-m^2]\ , \tag{2.7}$$

where k is in the first Brillouin zone, $k\in[-\pi,\pi]^d$. Then, at infinite temperature [$\beta=0$ in (2.2)], the structure function (2.7) is, for small k, of the form

$$S(k,t)\approx(1-m^2)\exp(-Dk^2t)\ , \tag{2.8}$$

with D the bulk diffusion coefficient.[16] This implies, for fixed j and large t, the decay

$$G(j,t)\approx t^{-d/2}\ . \tag{2.9}$$

For the driven system, model (ii), at $\beta=0$ (and the standard choice of rates), the steady state is independent of E.[1] However, the time-dependent structure function is complicated and not known explicitly even to lowest order in β. Thus we would need to expand in both β and E. We avoid such an extra complication by considering a continuum Ginzburg-Landau model with a field E. There we can expand around a finite-temperature Gaussian field in the strength of E (Sec. IV).

III. REFLECTION-INVARIANT MODELS

The dynamics is described by giving the evolution of the correlation functions. Let A be a finite set of sites, $\sigma_A\equiv\prod_{i\in A}\sigma(i)$. Then, starting from some initial state μ_0, the correlations at time t satisfy the coupled set of equations

$$\frac{d}{dt}\langle\sigma_A\rangle_t=\sum{}'\langle\sigma_A[\sigma(i)\sigma(j)-1]c(i,j;\sigma)\rangle_t\ , \tag{3.1}$$

where the sum $\sum'$ is over nearest-neighbor sites i and j such that $i\in A$, $j\notin A$.

Defining difference operators

$$\nabla_\alpha f(i)=f(i+e_\alpha)-f(i)\ ,\qquad \nabla^*_\alpha f(i)=f(i-e_\alpha)-f(i)\ , \tag{3.2a}$$

for a function $f(i)$, $i\in\mathbb{Z}^d$, and

$$\Delta_\alpha=-\nabla^*_\alpha\nabla_\alpha,\quad \Delta=\sum_{\alpha=1}^d\Delta_\alpha\ , \tag{3.2b}$$

the equation for the pair correlations takes the form

$$\frac{1}{2}\frac{d}{dt}\langle\sigma(0)\sigma(i)\rangle_t=-\sum_{\alpha=1}^d\{\nabla^*_\alpha\langle c(i,i+e_\alpha,\sigma)\nabla_\alpha\sigma(0)\sigma(i)\rangle_t-\tfrac{1}{2}\Delta_\alpha\delta_{i,0}[\nabla^*_\alpha\langle c(j,j+e_\alpha,\sigma)\nabla_\alpha\sigma(0)\sigma(j)\rangle_t]_{j=0}\}\ . \tag{3.3}$$

Since we are interested in weakly interacting spins, it is convenient to see the influence of these interactions by writing the rates $c(i,i+e_\alpha;\sigma)$ as the sum of a constant which can be fixed to be unity and a "small" configuration-dependent term,

$$c(i,i+e_\alpha;\sigma)=1+c_\alpha(i;\sigma)\ . \tag{3.4}$$

Equation (3.3) can then be rewritten in the suggestive form

$$\frac{1}{2}\frac{d}{dt}\langle\sigma(0)\sigma(i)\rangle_t=\Delta\langle\sigma(0)\sigma(i)\rangle_t-\rho_t(i)\ , \tag{3.5}$$

where

$$\rho_t(i)=\sum_{\alpha=1}^d\nabla^*_\alpha\langle c_\alpha(i;\sigma)\nabla_\alpha\sigma(0)\sigma(i)\rangle_t+\tfrac{1}{2}\sum_{\alpha=1}^d\Delta_\alpha\delta_{i,0}[\nabla^*_\alpha\langle c_\alpha(j;\sigma)\nabla_\alpha\sigma(0)\sigma(j)\rangle_t]_{j=0}+\Delta\delta_{i,0}-\sum_{\alpha=1}^d\Delta_\alpha\delta_{i,0}\langle\sigma(0)\sigma(e_\alpha)\rangle_t \tag{3.6}$$

depends both on the pair and higher-order correlation functions.

Equation (3.5) can also be written as an integral equation using the transition probability $p_t(i,j)=e^{\Delta t}(i,j)$ for a simple random walk on $\mathbb{Z}^d$; i.e., the probability that a random walker starting at site i will be at j at time t:

$$p_{t-s}(i,j)\approx\frac{e^{-(i-j)^2/(t-s)}}{(t-s)^{d/2}}\ .$$

We then have

$$\tfrac{1}{2}\langle\sigma(0)\sigma(i)\rangle_t=e^{\Delta t}\langle\sigma(0)\sigma(i)\rangle_0+\int_0^t ds\,e^{\Delta(t-s)}\rho_s(i)\ . \tag{3.7}$$

This illustrates the remark made in the Introduction that the spins at sites 0 at i are "connected" at time t via space-time walks $(i,t)\to(j,s)\to(0,t)$.

In a translational-invariant stationary state μ_{SS}, with $\langle\sigma(i)\rangle_{SS}\equiv m$, the truncated equal-time two-point function of (2.6) satisfies the equation

$$\Delta[\langle\sigma(0)\sigma(i)\rangle_{SS}-m^2]\equiv\Delta G(i,0)=\rho(i)\ , \tag{3.8}$$

while its Fourier transform, the steady-state structure function, becomes

$$S(k,0)=\frac{\hat\rho(k)}{2\sum_{\alpha=1}^d(\cos k_\alpha-1)}\ . \tag{3.9}$$

We can therefore think of $\rho(i)$ as a "charge density" and of $G(i,0)$ as the corresponding "electrostatic potential."

Such an identification does not by itself give us much rigorous information about the behavior of correlations in the steady state. The charge density $\rho(i)$, defined in (3.6), is, after all, also unknown, and (3.8) can be thought of as a member of some infinite hierarchy of such relations. The structures of (3.6) and (3.8) are, however, very suggestive of what one should expect for the "generic" behavior of the decay of the pair correlation in the stationary state. For a finite system in a periodic box Λ, the total charge density $\sum_{i\in\Lambda}\rho(i)=0$ by (3.6), and the dipole moment $\sum_{i_\alpha\in\Lambda} i\rho(i)=0$ by reflection invariance, which follows from translation invariance. To conclude the same for the infinite system, which would assure then that $S(k,0)$ remains bounded as $k\to 0$, requires assumptions about the decay of $\rho(i)$. It suffices that the second moment of $\rho(i)$ is bounded,

$$\sum_i |i|^2|\rho(i)|<\infty\ , \tag{3.10}$$

for the infinite system. This will be verified explicitly later in perturbation theory and will be assumed to be generic (c.f. our discussion in Sec. VII). It then follows that, for small wave vector k,

$$S(k,0)\approx\frac{1}{2d}\sum_i |i|^2\rho(i)+\frac{\sum_{\alpha=1}^d b_\alpha k_\alpha^2}{k^2}\ , \tag{3.11a}$$

with $b_\alpha=(1/2d)\sum_i\rho(i)(di_\alpha^2-|i|^2)$, $|i|^2\equiv\sum_\alpha i_\alpha^2$.

Equation (3.11a) shows that the decay of the pair correlations will have a quadrupole-type power-law behavior unless all the b_α are equal to zero, as would occur if the "charge density" $\rho(i)$ had the same symmetry as the lattice. That is, generically, we have SOC with

$$\langle\sigma(0)\sigma(i)\rangle_{SS}\approx\sum_{\alpha=1}^d b_\alpha\frac{i_\alpha^2}{|i|^{d+2}},\quad |i|\to\infty\ , \tag{3.11b}$$

with appropriate constants $\{b_\alpha\}$.

To evaluate the b_α's is difficult in general; we do it here for a particular model up to first order in a high-temperature expansion. This will also enable us to point out where cancellations occur if detailed balance (2.2) is assumed. Higher-order terms will be discussed in Appendix A, and isotropic dynamics in Sec. VII.

A. Example: Multiple-temperature model

We assume the exchange rates to be given by

$$c(i,i+e_\alpha;\sigma)\equiv\Phi_\alpha(\beta_\alpha[H(\sigma^{i,i+e_\alpha})-H(\sigma)])\ ,\qquad \alpha=1,\dots,d\ , \tag{3.12}$$

where the $\{\Phi_\alpha\}$ are smooth strictly positive functions with $\Phi_\alpha(0)=1$. The energy

$$H(\sigma)\equiv-\sum_A J_A\sigma_A \tag{3.13}$$

is even, and translation and reflection invariant with A, a finite subset of $\mathbb{Z}^d$; $J_A=0$ if the diameter of A is larger than R. If all $\beta_\alpha=\beta$ and $\Phi_\alpha(z)=e^{-z}\Phi_\alpha(-z)$, then (3.12) satisfies the condition of detailed balance, Eq. (2.2), for the Gibbs measure ν_H with the Hamiltonian (3.13). We study the stationary state of the dynamics governed by (3.12) as a perturbation of the product state with zero magnetization. The latter is, of course, invariant for the dynamics with all $\beta_\alpha=0$.

For the moment, we are only interested in the *first*-order correction in the β_α's to the stationary two-point function. We write this, dropping the subscript "SS," as

$$\langle\sigma(0)\sigma(i)\rangle=\delta_{0i}+\sum_{\alpha=1}^d\beta_\alpha G_\alpha^{(1)}(i)+N\ , \tag{3.14}$$

where N represents higher-order terms in the β_α. Using the stationarity condition (3.8), and substituting (3.12) and (3.13), we obtain

$$\sum_{\alpha=1}^d\beta_\alpha\Delta G_\alpha^{(1)}(i)\equiv\Delta G^{(1)}(i)=-2\sum_{\alpha=1}^d\beta_\alpha\Phi_\alpha'(0)[\Delta_\alpha J(i)+J(e_\alpha)\Delta_\alpha\delta_{0i}]-\sum_{\alpha=1}^d\Delta_\alpha\delta_{0i}G^{(1)}(e_\alpha)\equiv\rho(i)\ , \tag{3.15}$$

where $J(i)=J_{\{0,i\}}$ if $i\neq 0$ and is zero if $i=0$. Multispin interactions containing more than two sites do not contribute to this order. As before, due to the reflection invariance of the conservative dynamics, the dipole moment (as well as the total charge) is zero; i.e.,

$$\sum_i\rho(i)=0\ , \tag{3.16}$$

$$\sum_i i_\gamma\rho(i)=0,\quad \gamma=1,\dots,d\ . \tag{3.17}$$

The decay properties of $G^{(1)}(i)$ as $i\to\infty$ will therefore be governed by the quadrupole moment.

For a charge density $\rho(i)$, $i\in\mathbb{Z}^d$ the quadrupole moment $Q_{\alpha\alpha'}(\rho)$, $\alpha,\alpha'=1,\dots,d$ is defined by

$$Q_{\alpha\alpha'}(\rho)\equiv\sum_i\rho(i)(|i|^2\delta_{\alpha\alpha'}-di_\alpha i'_\alpha)\ , \tag{3.18}$$

where $|i|^2=\sum_\alpha|i_\alpha|^2$. For our case, (3.15) can be solved using Fourier transforms. We get

$$\sum_\alpha(\cos k_\alpha-1)S^{(1)}(k)=-2\sum_\alpha\beta_\alpha\Phi_\alpha'(0)[(\cos k_\alpha-1)\hat{J}(k)+J(e_\alpha)(\cos k_\alpha-1)]-\sum_{\alpha,\gamma}(\cos k_\alpha-1)A^{-1}_{\alpha\gamma}M_\gamma\ , \tag{3.19}$$

where A^{-1} is the inverse of matrix A given by

$$A_{\alpha,\alpha'}=\delta_{\alpha,\alpha'}+2\int dk\frac{1}{(2\pi)^d}\frac{\cos(k_\alpha)[\cos(k'_\alpha)-1]}{\sum_\gamma(\cos(k_\gamma)-1)}, \tag{3.20}$$

and

$$M_\gamma=-4\sum_\alpha\beta_\alpha\Phi'_\alpha(0)\int dk\frac{1}{(2\pi)^d}\cos(k_\gamma)\frac{[\hat{J}(k)+J(e_\alpha)][\cos(k_\alpha)-1]}{\sum_{\alpha'}[\cos(k_{\alpha'})-1]}. \tag{3.21}$$

For small k, Eq. (3.19) reads

$$S^{(1)}(k)\approx\text{const}+\sum_\alpha Q_{\alpha\alpha}\frac{k_{\alpha'}^2}{k^2}, \tag{3.22}$$

where

$$Q_{\alpha\alpha}=-4\sum_\gamma\beta_\gamma\Phi'_\alpha(0)\left[(1-d\delta_{\alpha\gamma})[\hat{J}(0)+J(e_\alpha)]\right.$$
$$\left.-2\sum_{\alpha',\gamma'}(1-d\delta_{\alpha\alpha'})A^{-1}_{\alpha'\gamma'}\int dk\frac{1}{(2\pi)^d}\cos(k_{\gamma'})\frac{[\hat{J}(k)+J(e_\gamma)][\cos(k_\gamma)-1]}{\sum_{\alpha''}[\cos(k_{\alpha''})-1]}\right]. \tag{3.23}$$

Taking the inverse Fourier transform, we obtain, for $|i|\to\infty$,

$$G^{(1)}(i)\approx\sum_\alpha Q_{\alpha\alpha}\frac{i_\alpha^2}{|i|^{d+2}}. \tag{3.24}$$

The equal-time *structure function* $S(k,0)\equiv S(k)$, defined in (2.7), will thus have the form

$$S(k)=1-m^2+\sum_{\alpha=1}^{d}Q_{\alpha\alpha}\frac{k_\alpha^2}{k^2}+N. \tag{3.25}$$

The decay properties of the stationary two-point function are therefore extremely different, according to whether the $Q_{\alpha\alpha}$ are all equal or not. Since $\sum_\alpha Q_{\alpha\alpha}=0$, there are two possibilities.

(i) Not all $Q_{\alpha\alpha}$ are zero; then the two-point function has the form of a quadrupole field and decays for large $|i|$ like a power law $\sim|i|^{-d}$, at least to first order in β_α.

(ii) All $Q_{\alpha\alpha}=0$; then the structure function is analytic at the origin and the two-point function decays exponentially.

The cancellations required in the second case occur if all the coefficients $\Phi'_\alpha(0)\beta_\alpha=-\frac{1}{2}\beta$ are equal. We then find the explicit solution

$$G^{(1)}(i)=\beta J(i). \tag{3.26}$$

This happens, of course, in one-dimensional systems, but also—in any dimension—if the detailed balance condition is satisfied for the rates (3.1). Then (3.26) gives the first term in a high-temperature expansion for the equilibrium two-point correlations in the Gibbs measure ν_H with Hamiltonian (3.13).

In the Appendixes we investigate this cancellation mechanism also for *higher* orders in a perturbation expansion. Every order satisfies a relation of the type (3.19) where the charge is a function of lower-order multiple spin correlation functions. For the general asymmetric case, the two-point function has a power-law decay order by order except in the case of detailed balance, where each order is strictly local. We are, however, not able to prove the convergence of the perturbation series.

IV. DRIVEN DIFFUSIVE SYSTEM

As mentioned in Sec. II, there are serious difficulties with obtaining explicitly even the lowest-order expansion terms in β for DDS on a lattice in which there is at most one particle per site. In this section we therefore consider the case of a single conserved field ϕ, $\phi\in\mathbb{R}$, in the continuum $\mathbf{r}\in\mathbb{R}^d$. The time evolution of ϕ is described by the stochastic partial differential equations (SPDE) (model *B* Ref. 6)

$$\frac{\partial}{\partial t}\phi_t(\mathbf{r})+\text{div}\mathbf{j}_t(\mathbf{r})=0, \tag{4.1}$$

with the flux $\mathbf{j}$, in the absence of an external driving field, given by

$$\mathbf{j}_t(\mathbf{r})=-\text{grad}\frac{\delta H}{\delta\phi}+\gamma^{1/2}\mathbf{J}_t(\mathbf{r}). \tag{4.2}$$

Here $\mathbf{J}_t(\mathbf{r})$ is a δ-correlated random current with covariance

$$\langle J_{t,\alpha}(\mathbf{r})J_{t',\alpha'}(\mathbf{r}')\rangle=\delta_{\alpha\alpha'}\delta(t-t')\delta(\mathbf{r}-\mathbf{r}'). \tag{4.3}$$

Clearly, $\phi_t(\mathbf{r})$ is locally conserved. Equations (4.1)–(4.3) admits the one-parameter family of stationary equilibrium distributions

$$\frac{1}{Z}\exp\left[-\beta H(\phi)-h\int d\mathbf{r}\,\phi(\mathbf{r})\right], \tag{4.4}$$

with $\beta=2/\gamma$. The chemical potential h regulates the average value of ϕ. In the following we will set $h=0$, which implies $\langle\phi\rangle=0$ when H is chosen to be symmetric under ϕ going to $-\phi$. For the quadratic Hamiltonian

$$H(\phi)=\tfrac{1}{2}\int d\mathbf{r}(\kappa|\mathrm{grad}\phi|^2+g\phi^2), \tag{4.5}$$

$g,\kappa>0$, (4.1)–(4.3) is a Gaussian field theory with a structure function in the stationary state (4.4), $h=0$, of the form

$$S(k,t)=\frac{1}{(\tau+k^2)}\exp[-k^2(\tau+k^2)|t|], \tag{4.6}$$

where we assume for simplicity $\beta\kappa=1$, and we define $\tau=\beta g$ and $\lambda=1/\beta$. We are interested in a situation where the system is driven by an external force $\mathbf{E}$. We have to add then to the current (4.2) the externally driven current (c.f. Refs. 9 and 10)

$$\mathbf{j}_{ex}(\mathbf{r},t)=\sigma(\phi)\mathbf{E}, \tag{4.7}$$

where σ is the field-dependent conductivity. We expand $\sigma(\phi)$ about $\phi=0$. The zeroth- and first-order terms can be absorbed through a Galilean transformation. The relevant contribution is found by putting $\sigma(\phi)=\sigma_0\phi^2$. Combining this with (4.1)–(4.4), the ϕ field is governed by the SPDE

$$\frac{\partial}{\partial t}\phi_t(\mathbf{r})=\lambda(\tau\Delta-\Delta^2)\phi_t(\mathbf{r})-\sigma_0\mathbf{E}\cdot\mathrm{grad}\phi_t^2(\mathbf{r}) -(2\lambda)^{1/2}\nabla\cdot\mathbf{J}_t(\mathbf{r}). \tag{4.8}$$

For the equations of motion to be well defined, one has to regularize at short distances. A physically reasonable choice is to discretize and to put the ϕ field on the lattice $\mathbf{Z}^d$. For our purposes it suffices and is computationally simpler to merely cut off the large wave numbers; i.e., we choose a maximum wavelength Λ and set the Fourier transform of ϕ equal to zero for values of $k>\Lambda$.

To study the behavior of the correlation functions from (4.8), we follow Refs. 9 and 10 and use the Martin-Siggia-Rose (MSR) formalism.[25,26] We rewrite the model in terms of a dynamic functional

$$L(\phi,\psi,\eta)=\int dt\,d\mathbf{r}\lambda\left[\psi\Delta\psi+\psi\left[\lambda^{-1}\frac{\partial}{\partial t}-\tau\Delta+\Delta^2\right]\phi +\frac{1}{2\lambda}\sigma_0(\mathbf{E}\cdot\mathrm{grad}\psi)\phi^2+\frac{1}{\lambda}\eta\phi\right], \tag{4.9}$$

where η is an auxiliary external field and ψ is the MSR response field. Correlation functions can now be expressed as functional averages with weight e^{-L}. In particular, the truncated correlation functions are given by

$$\langle\phi_{t_1}(\mathbf{r}_1)\cdots\phi_{t_n}(\mathbf{r}_n)\rangle_T =\frac{\delta}{\delta\eta_{t_1}(\mathbf{r}_1)}\cdots\frac{\delta}{\delta\eta_{t_n}(\mathbf{r}_n)}\ln Z(\eta)\Big|_{\eta=0}, \tag{4.10}$$

where

$$Z(\eta)=\int\delta\phi\,\delta\psi\exp[-L(\phi,\psi,\eta)]. \tag{4.11}$$

To compute $Z(\eta)$ we follow a perturbative scheme where the expansion parameter is E; see Appendix B. Due to symmetry, the first-order term in the truncated pair-correlation function vanishes. The structure function in the stationary nonequilibrium state $\hat{S}(k)$ can therefore be written in the form

$$S(k)=S^{(0)}(k)+(\sigma_0E)^2S^{(2)}(k)+\mathcal{N},$$

where $\mathcal{N}$ represents higher-order terms, and

$$S^{(0)}(k)=\frac{1}{k^2+\tau}. \tag{4.12a}$$

The second-order term has the form (see Appendix B)

$$S^{(2)}(k)=\frac{k_\parallel^2}{k^2}D(k^2), \tag{4.12b}$$

Here $k_\parallel=kE/|E|$ is the component of k parallel to the field and $D(x)$ is an analytic function around x equal zero. $S^{(2)}(k)$ is always analytic in one dimension. In higher dimensions we have to examine the behavior of $D(x)$ near the origin; i.e., if $D(x)\approx x^n$, $n\geq 1$, when $x\to 0$. In particular, we can compute explicitly $D(x)$ when $1/\tau$ is small enough. For $\tau\to\infty$, we find in Appendix B

$$D(x)=-\frac{1}{(2\tau)^5\lambda^2}\frac{1}{4(4\pi)^{d/2}}\left[\frac{9}{4}I\left[\frac{d}{2}-1\right]x^2+3(2d+1)I\left[\frac{d}{2}\right]x+(4+8d+3d^2)I\left[\frac{d}{2}+1\right]\right]+O\left[\frac{1}{\tau}\right]^6, \tag{4.13}$$

where d is the space dimension, and

$$I(n)=\int_{\Lambda^{-2}}^{\infty}dv\frac{\exp(-\frac{3}{2}vk^2)}{(2v)^n}. \tag{4.14}$$

It is easy to check from (4.13) that $D(0)\neq 0$, implying a nonanalytic behavior of the structure function for dimensions higher than 1. Thus, at least in second-order perturbation, we find a large distance decay of the steady-state correlations identical to the one found for the two-temperature model in the last section, although the physics of the two mod-

els are quite difficult.

We have also computed the three-point truncated correlation function to first order in E. It can be written as

$$\langle\phi(0)\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle_T=\int\int\frac{dk_1dk_2}{(2\pi)^{2d}}e^{ik_1\mathbf{r}_1+ik_2\mathbf{r}_2}T(k_1,k_2)\ , \tag{4.15}$$

where

$$T(k_1,k_2)=-i\frac{\sigma_0E}{\lambda}\frac{k_{1,\parallel}k_2^2+k_{2,\parallel}k_1^2+2(k_{1,\parallel}+k_{2,\parallel})k_1k_2}{\Pi(k_1)\Pi(k_2)\Pi(k_1+k_2)}\frac{1}{k_1\Pi(k_1)+k_2^2\Pi(k_2)+(k_1+k_2)^2\Pi(k_1+k_2)}\ , \tag{4.16}$$

and $\Pi(x)=x^2+\tau$. In particular, we have computed the leading term of (4.15) when $\tau\to\infty$. In this case, whenever $\mathbf{r}_1$ and $\mathbf{r}_2$ are different from zero, the behavior of T when the cutoff Λ goes to infinity is

$$\langle\phi(0)\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle_T=\frac{\sigma_0E}{2\lambda\tau^4}(d+1)!\left[\frac{3}{4\pi^2}\right]^{d/2}\frac{x_1^2(2x_{1,\parallel}-x_{2,\parallel})+x_2^2(2x_{2,\parallel}-x_{1,\parallel})-2x_1x_2(x_{1,\parallel}+x_{2,\parallel})}{(x_1^2+x_2^2-x_1x_2)^{d+2}}\ . \tag{4.17}$$

Thus the truncated three-body correlation function behaves as $r^{-(2d+1)}$ when $r\to\infty$ whenever the points x_1 and x_2 are not symmetric with respect to the origin; in the symmetric case T is identically zero. Notice that this power-law decay behavior occurs even in one dimension where the two-body correlation has an exponential decay. As noted in the Introduction, however, this may be an artifact of the expansion.

V. FLUCTUATING HYDRODYNAMICS

In this section we try to understand the slow decay of correlations from the point of view of a macroscopic fluctuation theory, in the spirit of fluctuating hydrodynamics.[14] We consider a density fluctuation field $\xi_t(\mathbf{r})$ around the uniform average density ρ. $\xi_t(\mathbf{r})$ is small because it is a spatial average over a cell centered at $\mathbf{r}$ containing many microscopic degrees of freedom. Its time evolution will therefore be described by a linear SPDE. Since the system is away from criticality, it is natural to assume that $\xi_t(\mathbf{r})$ is a Gaussian field which is governed by the linear Langevin equation [c.f. (4.1) and (4.2)]

$$\frac{\partial}{\partial t}\xi_t+\mathrm{div}(\mathbf{c}\xi_t+D\cdot\mathrm{grad}\xi_t+R^{1/2}\mathbf{J}_t)=0\ . \tag{5.1}$$

Here $\mathbf{c}$ is the velocity of propagation for a density fluctuation, and D is the diffusion matrix. R is a $d\times d$ matrix, which determines the strength of the fluctuation current [the covariance of $\mathbf{J}_t$ is given in (4.3)]. The quantities $\mathbf{c}$, D, and R are to be thought of as phenomenological coefficients to be determined from the microscopic model under consideration: $\mathbf{c}$ and D can be determined through the response of the system to a small imposed density deviation, while R is given by the static response to a small uniform force.

The time-displaced structure function $S(k,t)$ is now simply the Fourier transform of the covariance of the Gaussian stationary process of (5.1),

$$S(k,t)=\frac{kRk}{2kDk}\exp(-kD|t|+ikct)\ . \tag{5.2}$$

In particular, the static structure function is

$$S(k,0)=\frac{kRk}{2kDk}\ , \tag{5.3}$$

which is, for small k, of the same form as that found for the two microscopic models studied before. We can therefore identify the coefficients R and D in these models at least perturbatively. From (5.3) we learn that, in general, we have to expect long-range correlations because of the conservation law. There are two notable exceptions.

(i) The microscopic system satisfies detailed balance. Then the macroscopic field satisfies the fluctuation dissipation theorem, which implies that

$$R=2\chi D \tag{5.4}$$

in (5.1) for some constant χ, which is to be identified as the equilibrium compressibility. In that case, $S(k)=\chi$ and $S(r)=\chi\delta(r)$.

(ii) The system is isotropic. Then, even while the microscopic dynamics may not satisfy the condition of detailed balance, the linearized dynamics (5.1) for the macroscopic fluctuations will satisfy (5.4) because D and R become multiples of the unit matrix. This is consistent with the results obtained from the microscopic models, as we discuss in Sec. VII. In addition, since D and R depend in a complicated way on the density and the microscopic exchange rate, there could be isolated points in parameter space where accidentally the proportionality (5.4) holds.

In principle, the fluctuating hydrodynamics (5.1) should be derived from a given microscopic model. In particular, this would yield R and D in terms of the microscopic dynamics. Unfortunately, the rigorous version of such a program is too difficult. More modestly, we analyze in the next section a model where the mechanism to derive fluctuating hydrodynamics is more explicit, because it relies on a scale separation between horizontal and vertical jumps in a two-dimensional lattice gas.

VI. AN EXACT MODEL: FROM MICROSCOPICS TO MACROSCOPICS

In this section we derive the macroscopic equation of fluctuating hydrodynamics, generalizing and making rigorous a microscopic model of the type considered by van Beijeren and Schulman[27] and also investigated by Krug *et al.*[28] We do this by an appropriate space-time rescaling of the type used by De Masi *et al.*[29] We should, however, add that the space rescaling is only partial, keeping in this way the connection between the microscopic models of Sec. III and the fully macroscopic equations of Sec. V. The detailed theorems together with their proofs will be presented elsewhere.[30]

Consider a lattice gas on the square lattice. It is convenient to use the language of occupation variables $\eta(i)=\frac{1}{2}[1+\sigma(i)]$ with value 0 if the lattice site is empty, 1 if it is occupied. The rate at which a particle jumps to a neighboring empty site, i.e., the rates at which the occupations at i and j are exchanged, is denoted as before by $c(i,j;\eta)$. We choose them of the form (3.12)

$$c(i,j,\eta)=\begin{cases}\epsilon^{-2} & \text{if } i-j=\pm e_2\\ \Phi(\beta[H(\eta^{ij})-H(\eta)]) & \text{if } i-j=\pm e_1\\ 0 \text{ otherwise .}\end{cases} \tag{6.1}$$

ϵ is a small number; e_1 and e_2 are the horizontal and vertical unit vectors, respectively, on $\mathbb{Z}^2$, and the energy $H(\eta)$ is parametrized by a vertical (K_2) and horizontal (K_1) coupling parameter:

$$H(\eta)=-4K_1\sum_{i-j=\pm e_1}\eta(i)\eta(j) -4K_2\sum_{i-j=\pm e_2}\eta(i)\eta(j) . \tag{6.2}$$

The function Φ is strictly positive and bounded, but is arbitrary otherwise. We thus have a two-temperature model as in (3.12) with $K_\alpha=J(e_\alpha)$ in the notation of Sec. III, but with *fast* exchanges in the vertical direction. The physical ideas to study the system in the fast-rate limit are well understood.[29]

For $\epsilon\downarrow 0$ the system has two well-defined time scales: (i) a microscopic one in which there are no exchanges of particles between vertical columns, and the system reaches a stationary state within each column corresponding to the infinite-temperature dynamics in the vertical direction, and (ii) a macroscopic scale in which the system has exchanges of particles between columns with rates which are configuration dependent.

In order to get a semimacroscopic equation we have to rescale the vertical space coordinate by ϵ^{-1}. This was not done in Refs. 27 or 28, and in this way spatial variations of the density field were lost, in the limit $\epsilon\downarrow 0$, and the system became effectively one dimensional. We take initial data for the process (6.1) distributed according to a product measure μ^ϵ, with densities

$$\mu^\epsilon(\eta(i_1,i_2))\equiv\langle\eta(i_1,i_2)\rangle^\epsilon=\rho_0(i_1,\epsilon i_2) , \tag{6.3}$$

where we write $i=i_1e_1+i_2e_2=(i_1,i_2)\in\mathbb{Z}^2$ for a general site, and $\rho_0(i_1,y)$, $i_1\in\mathbb{Z}$, $y\in\mathbb{R}$ is a smooth function of y with $0\le\rho_0(i_1,y)\le 1$. If η_t^ϵ, $t>0$ denotes the process, then we can prove that in the limit $\epsilon\downarrow 0$, $\langle\eta_t^\epsilon(i_1,i_2)\rangle^\epsilon$ is *close* to $\rho_t(i_1,\epsilon i_2)$ with $\rho_t(i_1,y)$, $i_1\in\mathbb{Z}$, $y\in\mathbb{R}$, the solution of the following nonlinear mixed diffusion equation:

$$\frac{\partial}{\partial t}\rho_t(i_1,y)=\frac{\partial^2}{\partial y^2}\rho_t(i_1,y)-\nabla_1^*J_1(\rho_t(i_1,y)) , \tag{6.4}$$

where the last term in (6.4) is the discrete divergence [see (3.2a)] of the current $J_1(\rho_t(i_1,y))$ between (i_1,y) and (i_1+1,y) which, itself, is a difference of the particle transport per unit time from (i_1,y) to (i_1+1,y) and back. Explicit formulae for J_1 are given in Appendix C. The initial condition for (6.4) is given by ρ_0, and, under sufficient smoothness in y of ρ_0 (6.4) has a unique solution. It is a deterministic conservation equation for the densities, obtained in much the same way as reaction-diffusion equations.[29] The fast environment-independent exchanges in the vertical direction, combined with the vertical space rescaling, produce the linear diffusion term. The exchanges in the horizontal direction occurs on a much slower time scale, and their influence on the change in the density must be calculated in the so-called "local equilibrium measure;" i.e., the distribution at time t looks, over vertical distances $\sim\epsilon^{-1}$, like a product measure with constant instantaneous densities which vary from column to column in the horizontal direction, which is kept discrete. We refer to Ref. 29 for more details and the derivation of Eq. (6.4), which involves proving a *law of large numbers.*

To go beyond (6.4), we investigate the fluctuations about the deterministic evolution equation. Formally (following the notation of Ref. 28), the limiting process describing the density fluctuation field will be given, in some generalized weak form, by a linear Langevin equation [compare with (5.1)]:

$$\frac{d}{dt}\xi_t(i_1,y)=\frac{\partial^2}{\partial y^2}\xi_t(i_1,y) -\nabla_1^*j_1(\xi_t(i_1,y))+W(i_1,y,t) , \tag{6.5}$$

with linearized horizontal current j_1 and the "white noise" given in (C4)–(C7). A special choice corresponds to the constant solution $\rho_t(i_1,y)\equiv\rho, 0\le\rho\le 1$, of Eq. (4.4). For that choice, the currents $\{J_1\}$ in (6.4) vanish identically. The equal time correlations of the fluctuation field are now most easily described by Fourier transforming the formal stochastic differential equation (6.5) [see (C8) and (C9)], and yield the static *structure function*

$$S(k_1,k_2)=\frac{k_2^2\rho(1-\rho)+2R(\rho)(1-\cos k_1)}{k_2^2+2[R_{23}(\rho)+R_{14}(\rho)(1+2\cos k_1)](1-\cos k_1)} , \tag{6.6}$$

where $R^{*}_{\alpha,\gamma}(\rho)$ and $R(\rho)$ are given in Appendix C. This has to be compared with (4.9): while $S(k)$ remains bounded at $k=0$, it is not analytic whenever

$$F(\rho)\equiv R(\rho)-\rho(1-\rho)[R_{23}(\rho)+3R_{14}(\rho)]\neq 0 . \tag{6.7}$$

In that case, we recover the phenomenon of power-law decay of the static covariance, as predicted; i.e., for large $i_1^2+y^2$,

$$\langle \xi(0,0)\xi(i_1,y)\rangle \approx F(\rho)\frac{i_1^2-y^2}{(i_1^2+y^2)^2} . \tag{6.8}$$

Of course, (6.7) is not true, and $F(\rho)=0$ at $\beta=0$, for which $R(\rho)=\Phi(0)\rho(1-\rho)$, $R_{23}(\rho)=\Phi(0)$, $R_{14}(\rho)=0$.

It remains to investigate condition (6.17) to find more explicit situations under which this nonequilibrium effect occurs. $F(\rho)$ is also a function of Φ, $A\equiv -4\beta K_1$, and $B\equiv -4\beta K_2$, and can be calculated starting from (6.6) [see (C10)]. The following special cases can be considered: If $B=0$ [no vertical coupling in the Hamiltonian (6.2)], then

$$F(\rho)=3\rho^2(1-\rho)^2[\Phi(A)-\Phi(-A)] , \tag{6.9}$$

which is never zero unless $\rho=0$ or $\rho=1$, or $\Phi(A)=\Phi(-A)$, corresponding to an infinite-temperature dynamics; if $K_1\geq 2K_2>0$ (ferromagnetic case with large enough horizontal coupling), then using the extra (detailed balance) condition $\Phi(z)=e^{-z}\Phi(-z)$, it can be checked that all terms in (6.19) are strictly positive whenever $\rho\neq 0,1$, implying again power-law decay of the static pair correlations. Of course, the same is true for $K_1\leq 2K_2<0$.

VII. SELF-ORGANIZED CRITICALITY AND THE ROLE OF ANISOTROPY

We consider two types of models which are frequently encountered in the discussion on self-organized criticality.[3,2,31]

(i) Strictly conservative stochastic dynamics. These have been the subject of this paper and we now summarize the role played by the *anisotropy* in having long-range correlations. If the dynamics is isotropic (invariant under lattice rotations) and if the structure function inherits this symmetry, then the coefficients b_α in Eq. (3.11a) have to vanish. Therefore, anisotropy is a *necessary* condition to have long-range correlations of the form discussed in this paper. It is, however, far from being a *sufficient* condition, since we can always take a detailed balance dynamics corresponding to an equilibrium model with anisotropic interactions. We also note, as pointed out in Sec. V, that in one dimension, *higher-order* correlations can have a power-law decay. We are thus led to the following schematic picture: For given short-range exchange rates $c(i,j;\sigma)$, the static two-point function could decay exponentially. However, if we perturb to $c(i,j;\sigma)+\delta c(i,j;\sigma)$, then, excluding special directions in the parameter space, we expect to have long-range correlations of quadrupole type. In this sense, the slow decay of correlations is generic for conservative dynamics.

(ii) Stochastic dynamics, where the density is conserved "only on average:" for example, discrete "sandpile" models,[3] deterministic dynamics with nonconserving noise. To model spin-flip processes with this property we choose flip rates $c(i;\sigma)\geq 0$, $i\in\mathbb{Z}^d$, of the form

$$c(i;\sigma)=-\tfrac{1}{2}\sigma(i)\sum_{\alpha=1}^{d}\beta_\alpha\Delta_\alpha f_i(\sigma) . \tag{7.1}$$

Here $f_i(\sigma)$ is the local function $f(\sigma)$ translated to lattice site i, and β_α's are numbers. f and β_α's have to be chosen such that the flip rates are positive. The form (7.1) of the spin-flip rates assures that the total magnetization is conserved "on the average," since the average magnetization at time t satisfies

$$\frac{\partial}{\partial t}\langle\sigma(i)\rangle_t=-2\langle c(i;\sigma)\sigma(i)\rangle_t$$
$$=\sum_\alpha\beta_\alpha\Delta_\alpha\langle f_i(\sigma)\rangle_t . \tag{7.2}$$

To be specific, let us only discuss the example of the voter model,[22] where $\beta_\alpha=1$ and $f_i(\sigma)=\sigma(i)$. In dimension $d=1,2$ for the steady state of the voter model, all spins are either up or down. However, for $d\geq 3$, there is a continuum of stationary states just as for strictly conservative dynamics. For a translational-invariant initial state, the two-point function $G(i,t)=\langle\sigma(0)\sigma(i)\rangle_t$ satisfies

$$\frac{\partial}{\partial t}G(i,t)=(1-\sigma_{i,0})2\Delta G(i,t) . \tag{7.3}$$

In particular, for the steady-state covariance

$$\Delta G(i)=0,\quad i\neq 0,\quad G(0)=1 . \tag{7.4}$$

For dimension $d\geq 3$, this implies the slow decay

$$\langle\sigma(0)\sigma(i)\rangle-\langle\sigma(0)\rangle^2\simeq c|i|^{-(d-2)} , \tag{7.5}$$

just as in a critical Gaussian theory.

ACKNOWLEDGMENTS

We thank H. van Beijeren, G. Grinstein, and J. S. Wang for useful discussions. Hospitality at the Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, is gratefully acknowledged. Partial support was provided by National Science Foundation Grant No. DMR 90-12369. P. L. G. was partially supported by Grant No. DGICYT, PB85-OC0062, Spain.

APPENDIX A

The main goal of this appendix is to set up a formal perturbation expansion around an infinite-temperature dynamics for reflection-invariant microscopic models without net current, and to establish power-law decay of the two-point function for each order in this expansion. In particular, we want to understand the cancellation mechanism leading to an exponential decay in the exceptional case of a detailed balance dynamics.

We slightly modify the strategy. The exchange rates are of the form

$$c(i,j;\sigma)=1+\sum_{n=1}^{\infty}\gamma^n\sum_A p_A^{(n)}(i,j)\sigma_A\ , \tag{A1}$$

where γ is a small parameter. If $\gamma=0$ (*simple symmetric exclusion process*), then the correlations at time t are expressed in terms of the initial correlations via the *duality relation*[22]

$$\langle\sigma_A\rangle_t^{\gamma=0}=\mathcal{E}^{A(0)=A}(\langle\sigma_{A(t)}\rangle_0)\ . \tag{A2}$$

$\mathcal{E}^{A(0)=A}$ is the expectation with respect to the dual process $A(t)$ ($\subset\mathbb{Z}^d$) started from $A(0)=A$, in which the labels of nearest-neighbor sites i and j are exchanged after random times distributed according to independent Poisson processes associated with each bond $\langle i,j\rangle$ of the lattice; i.e., for a local function $f(B)$ on finite sets $B\subset\mathbb{Z}^d$:

$$\frac{d}{dt}\mathcal{E}^A(f(A(t)))=\sum_{\langle i,j\rangle}\mathcal{E}^A(f(A(t)^{ij})-f(A(t)))\ , \tag{A3}$$

where B^{ij} is the set obtained from B by exchanging the labels of sites $\langle i,j\rangle$. Allowing γ to be different from zero brings in extra *scattering* at certain points in space time; i.e., at certain points, the *free* evolution $A\to A(t)$ is interrupted, and $A(t)$ is replaced by the set B with probabilities derived from the coefficients $p_{A(t)B}^{(n)}$. The following perturbation expansion is the formal expression of this interpretation. A detailed mathematical analysis will be presented elsewhere.[31]

Assume that we start the process from a Bernoulli (or product) state with zero magnetization. The time-evolved correlations satisfy an equation of the form

$$\frac{d}{dt}\langle\sigma_A\rangle_t=\sum_{\langle i,j\rangle}\langle\sigma_A^{ij}-\sigma_A\rangle_t - \sum_{\langle i,j\rangle}\langle(\sigma_A^{ij}-\sigma_A)[1-c(i,j,\sigma)]\rangle_t\ , \tag{A4}$$

where the sums are over all nearest-neighbor bonds $\langle i,j\rangle$. The first term on the right-hand side of (A4) corresponds to the simple exclusion process ($\gamma=0$ in the discussion above). Let $P(X\to Y:t)$ be the probability that $A(t)=Y$ if $A(0)=X$ for the dual process $A(t)$. Then, (A4) can be rewritten as

$$\langle\sigma_A\rangle_t=\delta_{A0}+\sum_B\int_0^t P(A\to B;t-\tau)v_\tau(B)\ , \tag{A5}$$

with

$$v_\tau(B)=-\sum_{\langle i,j\rangle}\langle(\sigma_B^{ij}-\sigma_B)[1-c(i,j,\sigma)]\rangle_\tau = \sum_n\gamma^n\sum_C p_{BC}^{(n)}\langle\sigma_C\rangle_\tau\ , \tag{A6}$$

where the $p_{BC}^{(n)}$ are defined for each pair of (finite) sets B and C:

$$p_{BC}^{(n)}=\sum_{\langle i,j\rangle}[p_{B^{ij}\Delta C}^{(n)}(i,j)-p_{B\Delta C}^{(n)}(i,j)]\ , \tag{A7}$$

with Δ the *symmetric difference* between two sets. We plug in (A6) into (A5) and interate:

$$\langle\sigma_A\rangle_t=\sum_{n=1}^{\infty}\gamma^n V_A^{(n)}(t)\ , \tag{A8}$$

defined recursively by

$$V_A^{(0)}(t)\equiv\delta_{A0}\ ,$$

$$V_A^{(n)}(t)\equiv\sum_{k=0}^{n-1}\sum_B\int_0^t\mathcal{E}^A(p_{A(t-\tau)B}^{(n-k)})V_B^{(k)}(\tau)d\tau\ , \tag{A9}$$

$$n>0\ .$$

It is rather easy to see that the expansion (A8) converges for small t, but we will assume that the same is true for all t.

We say that a function $\rho(A)$ on the finite subsets $A\subset\mathbb{Z}^d$ is a *quadrupole* if for all $n=1,2,\ldots$,

$$\sum_{A:|A|=n}\rho(A)=0\ ,$$

$$\sum_{A:|A|=n}\sum_{i\in A}i_\alpha\rho(A)=0,\quad\alpha=1,\ldots,d\ , \tag{A10}$$

$$b_\alpha\equiv\sum_{A:|A|=n}\sum_{i\in A}|i_\alpha|^2\rho(A)<\infty,\quad\alpha=1,\ldots,d\ .$$

The *potential* $V_A\equiv G\rho(A)$ generated by the quadrupole $\rho(A)$ is the function

$$G\rho(A)\equiv\sum_B G(A,B)\rho(B)\equiv\int_0^\infty dt\,\mathcal{E}^A(\rho(A(t)))\ . \tag{A11}$$

Formally, the operator G is minus the inverse of the generator (A3): if

$$\rho(A)=\sum_{\langle i,j\rangle}(v_{A^{ij}}-v_A)\ ,$$

then

$$G\rho(A)=-v_A\ . \tag{A12}$$

The appropriate change of variables allows us to let $t\uparrow\infty$ in each term of (A8):

$$\langle\sigma_A\rangle_\infty=\sum_{n=0}^{\infty}\gamma^n V_A^{(n)}\ , \tag{A13}$$

where the potentials $V_A^{(n)}$ are generated by quadrupoles $\rho_A^{(n)}$; i.e.,

$$V_A^{(n)}\equiv G\rho_A^{(n)}\ , \tag{A14}$$

defined iteratively by putting

$$\rho_A^{(n)}\equiv\sum_{k=0}^{n-1}\sum_B p_{AB}^{(n-k)}V_B^{(k)},\quad n=1,2,\ldots, \tag{A15}$$

and $V_A^{(0)}\equiv\delta_{A0}$. Since, from the definitions (A7) and (A15), there are $q_A^{(n)}(i,j)$ such that

$$\rho_A^{(n)}=\sum_{\langle i,j\rangle}[q_{A^{ij}}^{(n)}(i,j)-q_A^{(n)}(i,j)]\ , \tag{A16}$$

and since we have assumed that the system is reflection invariant, the $\rho_A^{(n)}$ must satisfy the two first conditions of (A10). Since the third condition of (A10), if violated, already implies certain long-range correlations in the system, we just assume it here. In the case where $A=\{0,i\}$ consists of just two sites, this assumption implies the summability of the two-point function in analogy with the discussion presented in (3.10) and (3.11). It can then

be checked in the same way as was done for the first-order term in the example of Sec. III, that the Fourier transform of the potential $V^{(n)}_{\{0,i\}}$, i.e., the nth-order equal-time *structure function* [see (2.5)] must have the form

$$S'^{(n)}(k,0)=\frac{\sum_{\alpha=1}^{d} b_\alpha^{(n)} k_\alpha^2}{k^2} \qquad \text{(A17)}$$

for small k.

Therefore, in dimensions $d>1$, the *structure function* is in general nonanalytic at the origin, as it will depend on how $k\to 0$. As a consequence, every order $n>0$ in the expansion (A13) corresponds to a power-law decay of the two-point function consistent at long distances with the behavior of a *quadrupole field* whenever the coefficients $b_\alpha^{(n)}$ are not all equal. Obviously, this cannot happen in the case where the dynamics has the symmetry of the lattice, for example, in one dimension, but there is another exception: We *a priori* know from equilibrium theory that if the dynamics satisfies the condition of detailed balance, then all the terms in the expansion (A13) must be strictly local. This can also be understood from the formulation above. Detailed balance requires that given the potentials $V_A^{(k)}$ and the numbers $p_A^{(n-k)}$, $k=0,\ldots,n-1$, there is a function K_A with $K_A\to 0$ as the diameter of A tends to infinity, such that, for all bonds $\langle i,j\rangle$,

$$\sum_{k=0}^{n-1}\sum_{B}[p_{A^{ij}\Delta B}^{(n-k)}(i,j)-p_{A\Delta B}^{(n-k)}(i,j)]V_B^{(k)}=K_A-K_{A^{ij}} . \qquad \text{(A18)}$$

The solution is given by the next-order potential $K_A=V_A^{(n)}$. From (A18) we can see that if $p_A^{(n-k)}(i,j)=0$ whenever $\mathrm{diam}(A\Delta\{i,j\})>R$, and $V_A^{(k)}=0$ whenever $\mathrm{diam}\,A>kR$, $k=0,\ldots,n-1$, then $V_A^{(n)}=0$ whenever $\mathrm{diam}\,A>nR$, hence the strict locality of the expansion. Moreover, (A18), if combined with (A11), is responsible for the disappearance of the massless modes caused by the operator G. A rather long but straightforward calculation shows that if we choose the rates according to (2.2),

$$c(i,j;\sigma)=\Phi(\beta[H(\sigma^{ij})-H(\sigma)]) , \qquad \text{(A19)}$$

with $\Phi(z)=e^{-z}\Phi(-z)$, and $H(\sigma)=-\sum_A J_A\sigma_A$ is a local Hamiltonian, then, from (A15) and (A16),

$$V_A^{(n)}=\frac{1}{n!}\sum_{A_1\Delta,\ldots,\Delta A_n=A}\prod_{r=1}^{n} J_{A_r} , \qquad \text{(A20)}$$

and (A12)—with $\gamma\equiv\beta$—simplifies to the usual equilibrium high-temperature expansion and converges for small enough β.

APPENDIX B

In this appendix we discuss in more detail the computation done in Sec. IV for the driven diffusive system case following the Martin-Siggia-Rose (MSR) field-theory formalism.[25] Our starting point is a general stochastic partial differential equation (SPDE) or Langevin equation of the form

$$\frac{\partial}{\partial t}\phi_t(\mathbf{r})=K(\phi_t;\mathbf{r})+\zeta_t(\mathbf{r}) , \qquad \text{(B1)}$$

where ζ is usually a stochastic white-noise field; i.e.,

$$\langle\zeta_t(\mathbf{r})\rangle=0 ,$$
$$\langle\zeta_t(\mathbf{r})\zeta_{t'}(\mathbf{r}')\rangle=\Gamma(\mathbf{r},\mathbf{r}')\delta(t-t') . \qquad \text{(B2)}$$

The Langevin equation (B1) gives us the space-time behavior of a stochastic fluctuating field around its deterministic path. Then it is natural to translate the problem in terms of a path integral formalism where each possible trajectory of the field is weighted in an adequate way around its deterministic one. To carry out explicitly this point of view, we define a stochastic generating functional Z_ζ,

$$Z_\zeta(\eta)=\exp\left[\int dt\,d\mathbf{r}\,\eta_t(\mathbf{r})\phi_t(\mathbf{r})\right] , \qquad \text{(B3)}$$

where ϕ has to be solution of (B1). From (B3) we can compute any correlation function using the identity

$$\langle\phi_{t_1}(\mathbf{r}_1),\ldots,\phi_{t_n}(\mathbf{r}_n)\rangle=\frac{\delta}{\delta\eta_{t_1}(\mathbf{r}_1)}\cdots\frac{\delta}{\delta\eta_{t_n}(\mathbf{r}_n)}\langle Z_\zeta(\eta)\rangle|_{\eta=0} . \qquad \text{(B4)}$$

and the truncated correlation functions following (4.10). Assuming the noise has a Gaussian distribution of zero average, it is possible to write $Z(\eta)\equiv\langle Z_\zeta(\eta)\rangle$ in terms of a functional integral as (4.11), where now

$$L(\phi,\psi,\eta,\eta^*)=\int dt\,d\mathbf{r}_1\left[\int d\mathbf{r}_2\psi_t(\mathbf{r}_1)\Gamma(\mathbf{r}_1,\mathbf{r}_2)\psi_t(\mathbf{r}_2)+\frac{1}{2}\frac{\delta}{\delta\phi_t(\mathbf{r}_1)}K(\phi_t;\mathbf{r}_1)\right.$$
$$\left.-i\psi_t(\mathbf{r}_1)\left[\frac{\partial}{\partial t}\phi_t(\mathbf{r}_1)-K(\phi_t;\mathbf{r}_1)\right]-\phi_t(\mathbf{r}_1)\eta_t(\mathbf{r}_1)-\psi_t(\mathbf{r}_1)\eta_t^*(\mathbf{r}_1)\right] , \qquad \text{(B5)}$$

where we introduced a new external field η^* coupled with the so-called MSR response field ψ (see Ref. 25 for more details). It is straightforward then to get L for the DDS in Sec. IV only by identification of the equations (4.1) and (4.2) with (B1) and (B2) and putting $\eta^*=0$. To get insight from this field-theory formalism we need to achieve a perturbation expansion for small electric fields, i.e., around the equilibrium state. We can then decompose the Lagrangian as $L=L_0+\sigma_0 E L_1$ and rewrite the partition function Z in the form

$$Z(\eta,\eta^*)=\exp\left[-\sigma_0 E L_1\left[\frac{\delta}{\delta\eta},\frac{\delta}{\delta\eta^*}\right]\right]Z_0(\eta,\eta^*)\ , \tag{B6}$$

where

$$Z_0(\eta,\eta^*)=\int\delta\phi\,\delta\psi\exp[-L_0(\phi,\psi,\eta,\eta^*)]\ . \tag{B7}$$

Notice that the action of the electric field only appears in the operator in front of the unperturbed partition function Z_0. The perturbation scheme is naturally based on the expansion of the exponential operators for small E. For practical purposes, instead of working in real space, i.e., $(t,\mathbf{r})$, it is convenient to do it in Fourier space $(w,\mathbf{k})$, in which the expressions for L_1 and Z_0 are simply

$$L_1(\hat\phi,\hat\psi)=\tfrac{1}{2}\int dw_1dw_2dk_1dk_2\frac{1}{(2\pi)^{2d+2}}ik_{1,\parallel}\hat\psi(w_1,k_1)\hat\phi(w_2,k_2)\hat\phi(-w_1-w_2,-k_1-k_2) \tag{B8}$$

and

$$Z_0(\eta,\eta^*)=N\exp\left[\int dwdk\frac{1}{(2\pi)^{d+1}}[\tfrac{1}{2}\hat\eta(w,k)\hat G_{\phi,\phi}(w,k)\hat\eta(-w,-k)+\hat\eta(w,k)\hat G_{\phi,\psi}(w,k)\hat\eta^*(-w,-k)]\right], \tag{B9}$$

where N is a normalization factor, $\hat\phi$, $\hat\psi$, $\hat\eta$ and $\hat\eta^*$, are the corresponding Fourier transforms of the fields ϕ, ψ, η, and η^*, and

$$G_{\phi,\phi}(w,k)=\frac{2\lambda k^2}{w^2+\lambda^2[\tau k^2+(k^2)^2]^2}\ , \tag{B10}$$

$$G_{\phi,\psi}(w,k)=\frac{1}{-iw+\lambda[\tau k^2+(k^2)^2]} \tag{B11}$$

are the free propagators of the model. It is now straightforward to get the expressions (4.12a) and (4.12b) for the structure function at zeroth and second order in E. In particular, $D(k^2)$ is explicitly given by

$$D(k^2)=-\lambda^2\int_\Lambda dq\frac{1}{(2\pi)^d}\frac{k^2(k-q)^2}{A(k)A(k-q)}\frac{1}{A(k)+A(q)+A(k-q)}\left[\frac{2q_\parallel k^2}{k_\parallel A(k)}-\frac{q^2}{A(q)}\right], \tag{B12}$$

where $A(x)=\lambda[\tau x^2+(x^2)^2]$. Assuming $\tau\to\infty$, it is possible to expand the expression of D in powers of $1/\tau$ and solve explicitly the integrals. The resulting expression for the leading term is given in (4.13). In particular, when $k\to 0$, we get

$$D(x)\approx-\frac{\Lambda}{16\lambda^2\tau^5}\left[\frac{2}{\pi}\right]^{1/2}(15-8\pi^{1/2}u+3u^2)+O\left[\frac{1}{\tau}\right]^6 \tag{B13a}$$

for $d=1$,

$$D(x)\approx-\frac{\Lambda^2}{16\pi\lambda^2\tau^5}[2-(1+3C)u^2-3u^2\log_{10}(u^2)]+O\left[\frac{1}{\tau}\right]^6 \tag{B13b}$$

for $d=2$, and

$$D(x)\approx-\frac{\Lambda^3}{24\lambda^2\tau^5}\left[\frac{2}{\pi^3}\right]^{1/2}\left[2\pi^{1/2}u^3+\frac{55-81u^2}{16}\right]+O\left[\frac{1}{\tau}\right]^6 \tag{B13c}$$

for $d=3$. Here $u^2=3k^2\Lambda^{-2}/2$, and C is Catalan's constant.

APPENDIX C

We collect here the formulas used in Sec. VI. The particle current is given by

$$J_1(\rho(\cdots,y))(i_1)=R(\rho_t(i_1+2,y),\rho_t(i_1+1,y),\rho_t(i_1,y),\rho_t(i_1-1,y))$$
$$-R(\rho_t(i_1-1,y),\rho_t(i_1,y),\rho_t(i_1+1,y),\rho_t(i_1+2,y))\ . \tag{C1}$$

$R(\ )$ is the expected rate of a particle jumping to a nearest-neighbor site in the horizontal direction,

$$R(\rho^{(1)},\rho^{(2)},\rho^{(3)},\rho^{(4)})\equiv\langle\Phi(-4\beta[K_1\eta(2,2)\eta(4,2)+K_1\eta(3,2)\eta(1,2)$$
$$+K_2\eta(2,2)\eta(3,3)+K_2\eta(3,2)\eta(2,3)+K_2\eta(3,2)\eta(2,1)$$
$$+K_2\eta(2,2)\eta(3,1)-K_1\eta(3,2)\eta(4,2)-K_1\eta(2,2)\eta(1,2)$$
$$-K_2\eta(3,2)\eta(3,3)-K_2\eta(3,2)\eta(3,1)-K_2\eta(2,2)\eta(2,3)$$
$$-K_2\eta(2,2)\eta(2,1)])\eta(2,2)[1-\eta(3,2)]\rangle_{\rho^{(1)},\ldots,\rho^{(4)}}, \tag{C2}$$

with $\langle\ \rangle_{\rho^{(1)},\ldots,\rho^{(4)}}$ the average with respect to the product measure having densities

$$\langle\eta(k,l)\rangle_{\rho^{(1)},\ldots,\rho^{(4)}}\equiv\rho^{(k)},\quad k=1,2,3,4;\quad l=1,2,3\ . \tag{C3}$$

The linearized current and the "white noise" are

$$j_1(\xi_t(\cdots,y))(i_1)=R_{14}(\rho_t(i_1+2,y),\rho_t(i_1+1,y),\rho_t(i_1,y),\rho_t(i_1-1,y))[\xi_t(i_1+2,y)-\xi_t(i_1-1,y)]$$
$$+R_{23}(\rho_t(i_1+2,y),\rho_t(i_1+1,y),\rho_t(i_1,y),\rho_t(i_1-1,y))[\xi_t(i_1+1,y)-\xi_t(i_1,y)]\ , \tag{C4}$$

$$R_{kl}\equiv\left[\frac{\partial}{\partial\rho^{(k)}}-\frac{\partial}{\partial\rho^{(l)}}\right]R,\quad k,l=1,2,\ldots,4\ . \tag{C5}$$

The function R is defined in (6.6), and $W(i_1,y,t)$ in (6.9) is a "white noise" with covariance

$$\langle W(i_1,y,t)W(i_1',y',t')\rangle=\delta(t-t')\left[\delta_{i_1i_1'}\frac{\partial^2}{\partial y\partial y'}[\rho_t(i_1,y)[1-\rho_t(i_1,y)]\delta(y-y')]\right.$$
$$\left.+\delta(y-y')\nabla^*_{i_1}\nabla^*_{i_1'}[a(\rho_t(\cdots,r))(i_1)\delta_{i_1i_1'}]\right] \tag{C6}$$

and

$$a(\rho_t(\cdots,r))(i_1)\equiv R(\rho_t(i_1+2,y),\rho_t(i_1+1,y),\rho_t(i_1,y),\rho_t(i_1-1,y))$$
$$+R(\rho_t(i_1-1,y),\rho_t(i_1,y),\rho_t(i_1+1,y),\rho_t(i_1+2,y))\ . \tag{C7}$$

The Fourier transform of (6.9) is

$$d\hat{\xi}_t(k_1,k_2)=\hat{W}(k_1,k_2,t)-k_2^2\hat{\xi}_t(k_1,k_2)-2\hat{\xi}_t(k_1,k_2)\{R_{23}(\rho)+[R_{14}(\rho)-R_{23}(\rho)]\cos k_1-R_{14}(\rho)\cos 2k_1\}\ , \tag{C8}$$

where

$$\langle\hat{W}(k_1,k_2,t)\hat{W}(k_1',k_2',t')\rangle\equiv\delta(k-k')\delta(t-t')[k_2^2\rho(1-\rho)+2R(\rho)(1-\cos k_1)]\ , \tag{C9}$$

with $k=(k_1,k_2)$ in the strip $-\pi\le k_1<\pi$, $-\infty<k_2<\infty$, and we have used in (C8) and (C9) $R(\rho)\equiv R(\rho,\rho,\rho,\rho)$, etc. as short-hand notations.

The expression for (6.17) is

$$F(\rho)=\rho^2(1-\rho)^2(2[\Phi(B)-\Phi(-B)][\rho^2+(1-\rho)^2]^2+3[\Phi(A)-\Phi(-A)]\{[\rho^2+(1-\rho)^2]^2+2\rho^2(1-\rho)^2\}$$
$$+[8\Phi(A+B)-8\Phi(-A-B)+4\Phi(A-B)-4\Phi(-A+B)+2\Phi(2B)-2\Phi(-2B)]$$
$$\times\rho(1-\rho)[\rho^2+(1-\rho)^2]+[5\Phi(A+2B)-5\Phi(-A-2B)+\Phi(A-2B)-\Phi(-A+2B)]\rho^2(1-\rho)^2)\ . \tag{C10}$$

---

*Permanent address: Aangesteld Navorser National Fonds voor Wetenschappelijk Onderzoek, Instituut voor Theoretische Fysika, K.U. Leuven, Belgium.

†Permanent address: Theoretische Physik, Universität München, Federal Republic of Germany.

[1]M. Q. Zhang, J. S. Wang, J. L. Lebowitz, and J. L. Vallés, J. Stat. Phys. **52**, 1461 (1988).

[2]T. Hwa and M. Kardar, Phys. Rev. Lett. **62**, 1813 (1989).

[3]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[4]B. M. Gurevich and Y. M. Suchov, Commun. Math. Phys. **84**, 333 (1982).

[5]S. Goldstein, J. L. Lebowitz, and E. Presutti, *Colloquia Mathematica Societatis Janos Bolyai Random Fields Esztergom, 1979* (North-Holland, Amsterdam, 1981).

[6]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[7]G. Eyink, J. L. Lebowitz, and H. Spohn, Commun. Math. Phys. (to be published).

[8]S. Katz, J. L. Lebowitz, and H. Spohn, J. Stat. Phys. **34**, 497

(1984).
[9]H. K. Janssen and B. Schmittmann, Z. Phys. B **63**, 517 (1986).
[10]Kwan-tai Leung and J. L. Cardy, J. Stat. Phys. **44**, 567 (1986).
[11]T. R. Kirckpatrick, E. G. D. Cohen, and J. R. Dorfman, Rev. A **26**, 950 (1982).
[12]D. Ronis and I. Procaccia, Phys. Rev. A **25**, 1812 (1982).
[13]G. Nicolis and M. Malek Mansour, Phys. Rev. A **29**, 2845 (1984).
[14]E. G. D. Cohen and R. Schmitz, in *Hydrodynamic Instabilities in Stationary States—An Introduction,* Vol. VI of *Fundamental Problems in Statistical Mechanics,* edited by E. G. D. Cohen (North-Holland, Amsterdam, 1985), pp. 229–252.
[15]H. Spohn, J. Phys. A **16**, 4275 (1983).
[16]H. Spohn (unpublished).
[17]B. M. Law, R. W. Gammon, and J. V. Sengers, Phys. Rev. Lett. **60**, 1554 (1988).
[18]B. M. Law, P. N. Segrè, R. W. Gammon, and J. V. Sengers, Phys. Rev. A **41**, 816 (1990).
[19]G. Grinstein (private communication).
[20]H. van Beijeren, J. Stat. Phys. (to be published).
[21]G. Grinstein, D.-H. Lee, and Subir Sachdev, Phys. Rev. Lett. **64**, 1927 (1990).
[22]T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[23]C. Maes and S. B. Shlosman (unpublished).
[24]P. A. Ferrari, Ann. Prob. (to be published).
[25]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[26]C. De Dominicis and L. Petiti, Phys. Rev. B **18**, 353 (1978).
[27]H. van Beijeren and L. S. Schulman, Phys. Rev. Lett. **53**, 806 (1984).
[28]J. Krug, J. L. Lebowitz, H. Spohn, and M. Q. Zhang, J. Stat. Phys. **44**, 535 (1986).
[29]A. DeMasi, P. A. Ferrari, and J. L. Lebowitz, J. Stat. Phys. **44**, 589 (1986).
[30]C. Maes, J. Stat. Phys. (to be published).
[31]C. Maes and F. Redig (unpublished).