CHAOTIC HOPPING BETWEEN ATTRACTORS IN NEURAL AUTOMATA

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Abstract

We present a neurobiologically–inspired stochastic cellular automaton whose state jumps with time between the attractors corresponding to a series of stored patterns. The jumping varies from regular to chaotic as the model parameters are modified. The resulting irregular behavior, which mimics the *state of attention* in which a systems shows a great adaptability to changing stimulus, is a consequence in the model of short–time presynaptic noise which induces *synaptic depression*. We discuss results from both a mean–field analysis and Monte Carlo simulations.

I.   INTRODUCTION

Analysis of electroencephalogram time series, though perhaps not conclusive yet, suggest chaotic neural activity in the brain [1–3]. This may reflect a state of collective synchronization of the network during expectation and attention [4, 5], and a condition for efficient search between many patterns, as in olfactory processes [6, 7], for instance. In fact, there is a well–founded recent hypothesis that the human brain might exhibit chaotic itinerancy [8, 9]. Consequently, different types of constructive chaos have been incorporated in neural network modeling [10–13]. Nevertheless, concluding on the significance of chaos in neurobiological systems is still an open issue [14–16].

As a new effort towards better understanding this problem, in the present paper we present, and study both analytically and numerically, a *neural automaton* with chaotic behavior. That is, the model is a neurobiologically inspired cellular automaton [17, 18], in which dynamics concerns the whole, which is simultaneously updated —instead of sequentially updating a small neighborhood at each time step. This automaton (or *Little dynamics* [19]) strategy has already shown to be efficient in modeling several aspects of associative memory [20, 21]. In fact, concerning this property, neural automata often exhibit more interesting behavior than their Hopfield–like, sequentially–updated counterparts, in spite of the fact that any two successive states are stronger correlated in the sequential case. Therefore, we extend here to neural automata our recent study of the effects of synaptic "noise" on the stability of attractors in Hopfield–like networks [22]. We demonstrate below that synaptic fluctuations may determine retrieval in neural automata when coupled to the presynaptic activity in such a way that *synaptic depression* occurs, a phenomenon which has been observed in actual systems [23, 24]. We then find that the attractor may not only turn unstable but even *strange,* so that a chaotic dynamics between the stored patterns sets in. We also show that this kind of behavior implies a high adaptability of the system to a changing environment, which seems to be one of the nature strategies for efficient computation [25–27].

II. THE MODEL

Let a set of N binary neurons with configurations $\mathbf{S} = \{s_i = \pm 1; i = 1, \ldots, N\}$ connected by synapses of intensity [22]:

$$w_{ij} = w_{ij}^{\mathrm{L}} x_j, \quad \forall i, j. \tag{1}$$

Here, x_j stands for a random variable, and w_{ij}^{L} is an average weight. The specific choice for the latter is not essential here but, for simplicity and reference purposes, we shall consider a Hebbian *learning rule* [28]. As an example, let us assume in the following that synapses *store* a set of M binary patterns, $\xi^\mu = \{\xi_i^\mu = \pm 1; i = 1, \ldots, N\}$, $\mu = 1, \ldots, M$, according to the prescription, $w_{ij}^{\mathrm{L}} = M^{-1} \sum_\mu \xi_i^\mu \xi_j^\mu$. The set $\mathbf{X} = \{x_j\}$ of random variables is intended to model microscopic fluctuations that have been reported to constantly perturb the synaptic weights [29]. The origin for such fluctuations is varied in practice, ranging from short–length stochasticities, e.g., associated to the opening and closing of the vesicles and to the local concentration of neurotransmitters, to differences between the incoming long–ranged signals [30, 31]. These short–time fluctuations affect the local fields, namely, $h_i(\mathbf{S}, \mathbf{X}) \equiv \sum_{j \neq i} w_{ij} x_j s_j$ which measure the total presynaptic current arriving to the postsynaptic neuron s_i. In addition to, and competing with the *synaptic noise* in (1), there is stochasticity of the neural activity. As it is customary in many fields, including physics and biology, we shall assume this stochasticity is controlled by a "temperature" parameter, T, which characterizes some underlying "thermal bath" [32–34].

The most relevant case is perhaps that of *very fast* fluctuations of the microscopic variables $\mathbf{X}$. As a matter of fact, it seems that a system relaxation typically occurs in a time which is much larger than that characterizing the synaptic variability [35]. Consequently, we shall assume that the neurons evolve as in presence of a steady distribution for $\mathbf{X}$, namely, $P^{\mathrm{st}}(\mathbf{X}|\mathbf{S})$. The technicalities of this situation have been discussed with detail in Refs.[32, 33], for instance. It then follows that the postsynaptic neuron, s_i, receives an *effective* presynaptic current given by

$$\overline{h_i}(\mathbf{S}) \equiv \int_{\mathbf{X}} h_i(\mathbf{S}, \mathbf{X}) P^{\mathrm{st}}(\mathbf{X}|\mathbf{S}) \mathrm{d}\mathbf{X}. \tag{2}$$

That is, each neuron endures an effective field, one which, in fact, is the average contribution of all possible different realizations of the actual field [34, 36].

Next, we shall model the noise steady distribution in (2) according to some recent neurobiological findings. In particular, we would like it to mimic short–term *synaptic depression*

[23, 24]. This refers to the observation that the synaptic intensity decreases under repeated presynaptic activation. The question is how such mechanism may affect the neural automata dynamics. For simplicity, we shall assume factorization of the noise distribution, so that the resulting simplest hypothesis is that $P^{\mathrm{st}}(\mathbf{X}|\mathbf{S}) = \prod_j P(x_j|\mathbf{S})$ with

$$P(x_j|\mathbf{S}) = \zeta(\vec{m}) \; \delta(x_j + \Phi) + [1 - \zeta(\vec{m})] \; \delta(x_j - 1). \tag{3}$$

Here, $\vec{m} = \vec{m}(\mathbf{S})$ stands for the overlap vector of components $m^\mu(\mathbf{S}) = N^{-1} \sum_i \xi_i^\mu s_i$, and $\zeta(\vec{m})$ is a probability to be determined. The choice (3) therefore implies a depression effect which increases with $\vec{m}$ and, consequently, with the net current arriving to postsynaptic neurons. Consequently, (3) introduces some non–trivial correlations between synaptic noise and neural activity. Looking for an increasing function of the total presynaptic current with proper normalization, a simple choice is $\zeta(\vec{m}) = (1 + \alpha)^{-1} \sum_\mu [m^\mu(\mathbf{S})]^2$, where $\alpha = M/N$ is the load parameter [37]. It then follows after some simple algebra that the effective field is

$$\overline{h_i}(\mathbf{S}) = \left\{ 1 - \gamma \sum_\mu [m^\mu(\mathbf{S})]^2 \right\} \sum_\nu \xi_i^\nu m^\nu(\mathbf{S}), \tag{4}$$

where $\gamma \equiv (1 + \Phi)(1 + \alpha)^{-1}$. We remark that the resulting system lacks any synaptic noise for $\Phi \to -1$. Therefore, the local field reduces in this limit to the one in the Hopfield model [37–39].

We next determine the neuron dynamics under the action of these fields. This is governed by the master equation $\Pi_{t+1}(\mathbf{S}) = \sum_{\mathbf{S}'} \Pi_t(\mathbf{S}') \Omega(\mathbf{S}' \to \mathbf{S})$, where the probability per unit time of a transition from configuration $\mathbf{S}'$ to $\mathbf{S}$ is

$$\Omega(\mathbf{S}' \to \mathbf{S}) = \prod_{i=1}^N \omega(s_i' \to s_i). \tag{5}$$

This corresponds to parallel updating or *Little dynamics* [19]. For simplicity and concreteness, we shall assume here that $\omega(s_i' \to s_i) \propto \Psi[\beta_i(s_i' - s_i)]$, where $\beta_i \equiv T^{-1}\overline{h_i}(\mathbf{S}')$, and that $\overline{h_i}(\mathbf{S}')$ is independent of s_i', which is a good approximation for a sufficiently large network (technically, this is an exact property in the *thermodynamic limit* $N \to \infty$). The function Ψ is arbitrary except that (in order that the model has well defined limits) we require that $\Psi(u) = \Psi(-u)\exp(u)$, $\Psi(0) = 1$ and $\Psi(\infty) = 0$,[33] as for a normalized exponential function. Then, consistent with the condition $\sum_{\mathbf{S}} \Omega(\mathbf{S}' \to \mathbf{S}) = 1$, we take

$$\omega(s_i' \to s_i) = \Psi[\beta_i(s_i' - s_i)] \left[1 + \Psi(2\beta_i s_i')\right]^{-1}. \tag{6}$$

We remark that the model may easily be extended to cover other interesting cases [40].

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III. SOME RESULTS

Monte Carlo simulations of the resulting system reveal a rich behavior, namely, a more varied and interesting behavior that in the case of sequential updating [22]. Figure 1 illustrates the occurrence of fixed points, cycles, regular and irregular hopping between the attractors, namely, between the pattern $m \equiv m^1 = 1$ and the *anti–pattern* $m = -1$, in the case $M = 1$. On the other hand, assuming the mean–field approximation $\langle s_i \rangle = s_i$, one may obtain the discrete map [37] which describes the time evolution of the overlap. The result for $M = 1$ is

$$m_{t+1} = \tanh\{T^{-1}m_t[1 - m_t^2(1 + \Phi)]\}. \tag{7}$$

As one varies here the "temperature" T and the (fast) noise parameter Φ, one obtains a varying situation which is in full agreement with Monte Carlo simulations, as one should have expected for a fully connected network. In particular, figure 2 confirms the occurrence of chaos in a case in which thermal fluctuations are small compared to the synaptic noise. That is, the Lyapunov exponent, λ, corresponding to the dynamic mean–field map shows different chaotic windows ($\lambda > 0$) as one varies Φ for a fixed T. As illustrated also by figure 2 (upper graph), dynamics is stable for $\Phi = -1$, i.e., lacking any synaptic noise, and the only solutions then correspond to the ones for the familiar Hopfield case with parallel updating [19, 38]. As Φ is increased, the system tends to become unstable, and transitions between $m = 1$ and $m = -1$ finally occur that can be fully chaotic.

This is not the first time in which chaos is reported to occur during the retrieval process in an attractor neural network. In particular, Ref.[11] illustrates many different dynamic behaviors ranging from point attractors to chaotic dynamics. In this case, the shape of the gain functions, namely, $m_{t+1} = G(m_t)$, is used as a parameter. Our model, however, does not involve any tuning of the gain function but just of the noise parameter Φ. For example, our gain function for $T = 0$ is

$$G\left(m_t, \Phi\right) = \text{sign}\left\{m_t \left[1 - m_t^2\left(1 + \Phi\right)\right]\right\}, \tag{8}$$

which may have a sigmoid shape or an oscillating one, as illustrated in figure 3. Only the latter case allows for jumping between the attractors and, eventually, for chaotic behavior, in accordance with the situation in figure 2.

There is also chaotic hopping between the attractors when the system *stores* several

patterns, $M > 1$. As an *order parameter* to detect chaos, it turns out convenient to define

$$\zeta = \frac{1}{1+\alpha} \sum_\mu \left(m^\mu\right)^2.$$ (9)

This is shown in figure 4 as a function of Φ. The graph clearly illustrates a region of irregular behavior which has a width $\Delta\Phi_c$ defined as the distance, in terms of Φ, from the first to the last bifurcation. Interesting enough, we find that the width of this region is practically independent of the number of patterns, namely, $\Delta\Phi_c = 0.575 \pm 0.005$ for $M \in [1, 50]$.

IV. DISCUSSION

Motivated by the fact that analysis of brain waves suggest that the chaos–theory concept of *strange attractor* may be relevant to describe some of the neural activity, we presented here a neurobiologically–inspired model which exhibits chaotic behavior. The model is a cellular automaton with only two parameters, T and Φ, which control the thermal stochasticity of the neural activity and the depressing effect of the (coupled) synaptic fluctuations, respectively. This system reduces to the Hopfield case with Little dynamics (parallel updating) only for $\Phi = -1$ and, otherwise, it exhibits efficient retrieval. In particular, even though mixture (*spin–glass*) steady states may occur at low *temperature* T, the system may escape from them under a very small stimulus. This is illustrated in figure 5 which shows a vivid response of the system to a weak external stimuli when the system is in the chaotic regime as a consequence of synaptic depression, which does not occur for fixed point or regular behavior. It is likely that such a mechanism plays a functional role in attention and in sequential processing of parallel sensory information.

The complex phenomenology of our neural automaton, which is rather robust and independent of the number of stored patterns (within the conditions here investigated), may allow for encoding of large amounts of information in terms of spatiotemporal patterns. In fact, experiments reveal that strange attractors are more efficient for the purpose [9, 41]. This is a consequence in our model of the inherent instability of chaotic motion, which enhances the system ability to adapt to a changing environment by irregular transitions between attractors. That is, chaos allows for an efficient exploration of different opportunities, which might account for the adaptive, reliable, and robust behavior of natural systems. This is consistent with the hypothesis of chaotic itinerancy [8, 9].

V. ACKNOWLEDGMENTS

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[1] A. Babloyantz and A. Destexhe, Proc. Natl. Acad. Sci. USA **83**, 3513 (1986).

[2] T. Elbert, W. J. Ray, Z. J. Kowalik, J. E. Skinner, K. E. Graf, and N. Birbaumer, Physiol. Rev. **74**, 1 (1994).

[3] J. M. Barrie, W. J. Freeman, and M. Lenhart, J. Neurophysiol. **76**, 520 (1996).

[4] D. Hansel and H. Sompolinsky, Phys. Rev. Lett. **68**, 718 (1992).

[5] D. Hansel and H. Sompolinsky, J. Comput. Neurosci. **3**, 7 (1996).

[6] W. J. Freeman, Biol. Cybern. **56**, 139 (1987).

[7] P. Ashwin and M. Timme, Nature **56**, 36 (2005).

[8] I. Tsuda, Behav. Brain Sci. **24**, 793 (2001).

[9] W. J. Freeman, Chaos **13**, 1067 (2003).

[10] L. Wang, E. E. Pichler, and J. Ross, Proc. Natl. Acad. Sci. USA **87**, 9467 (1990).

[11] D. Bolle and B. Vink, Physica A **223**, 293 (1996).

[12] D. R. C. Dominguez, Phys. Rev. E **54**, 4066 (1996).

[13] C. S. Poon and M. Barahona, Proc. Natl. Acad. Sci. USA **98**, 7107 (2001).

[14] M. I. Rabinovich and H. D. I. Abarbanel, Neuroscience **87**, 5 (1998).

[15] P. Faure and H. Korn, C. R. Acad. Sci. III **324**, 773 (2001).

[16] H. Korn and P. Faure, C. R. Biol. **326**, 787 (2003).

[17] U. Frisch, B. Hasslacher, and Y. Pomeau, Phys. Rev. Lett. **56**, 1505 (1986).

[18] B. Chopard and M. Droz, *Cellular Automata Modeling of Physical Systems* (Cambridge University Press, 1998).

[19] P. Peretto, *An Introduction to the modeling of neural networks* (Cambridge University Press, 1992).

[20] N. Ganguly, A. Das, P. Maji, B. K. Sikdar, and P. P. Chaudhuri, in *High Performance Computing - HiPC 2001 : 8th International Conference*, edited by B. Monien, V. Prasanna,

and S. Vajapeyam (Springer-Verlag, 2003), vol. 2228, pp. 115–124.

[21] J. J. Torres, M. A. Munoz, J. Marro, and P. L. Garrido, Neurocomputing **58-60**, 229 (2004).

[22] J. M. Cortes, J. J. Torres, J. Marro, P. L. Garrido, and H. J. Kappen (2005), Neural Comp. In press.

[23] M. V. Tsodyks, K. Pawelzik, and H. Markram, Neural Comp. **10**, 821 (1998).

[24] L. Pantic, J. J. Torres, H. J. Kappen, and S. C. A. M. Gielen, Neural Comp. **14**, 2903 (2002).

[25] K. J. Stam, D. L. Tavy, B. Jelles, H. A. Achtereekte, J. P. Slaets, and R. W. Keunen, Brain Topogr. **7**, 141 (1994).

[26] Q. Lu, G. Shen, and R. Yu, Comput. Biol. Chem. **27**, 363 (2003).

[27] N. Schweighofer, K. Doya, H. Fukai, J. V. Chiron, T. Furukawa, and M. Kawato, Proc. Natl. Acad. Sci. USA **101**, 4655 (2004).

[28] D. O. Hebb, *The Organization of Behavior: A Neuropsychological Theory* (Wiley, 1949).

[29] L. F. Abbott and W. G. Regehr, Nature **431**, 796 (2004).

[30] K. M. Franks, C. F. Stevens, and T. J. Sejnowski, J. Neurosci. **23**, 3186 (2003).

[31] S. Hilfiker (2005), private communication.

[32] J. J. Torres, P. L. Garrido, and J. Marro, J. Phys. A: Math. Gen. **30**, 7801 (1997).

[33] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, 1999).

[34] C. W. Gardiner, *Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, 2004).

[35] A. Zador, J. Neurophysiol. **79**, 1219 (1998).

[36] D. Bibitchkov, J. M. Herrmann, and T. Geisel, Network: Comput. Neural Syst. **13**, 115 (2002).

[37] J. Hertz, A. Krogh, and R. G. Palmer, *Introduction to the theory of neural computation* (Addison-Wesley, 1991).

[38] J. J. Hopfield, Proc. Natl. Acad. Sci. USA **79**, 2554 (1982).

[39] D. J. Amit, *Modeling brain function: the world of attractor neural network* (Cambridge University Press, 1989).

[40] J. M. Cortes et al. (To be published).

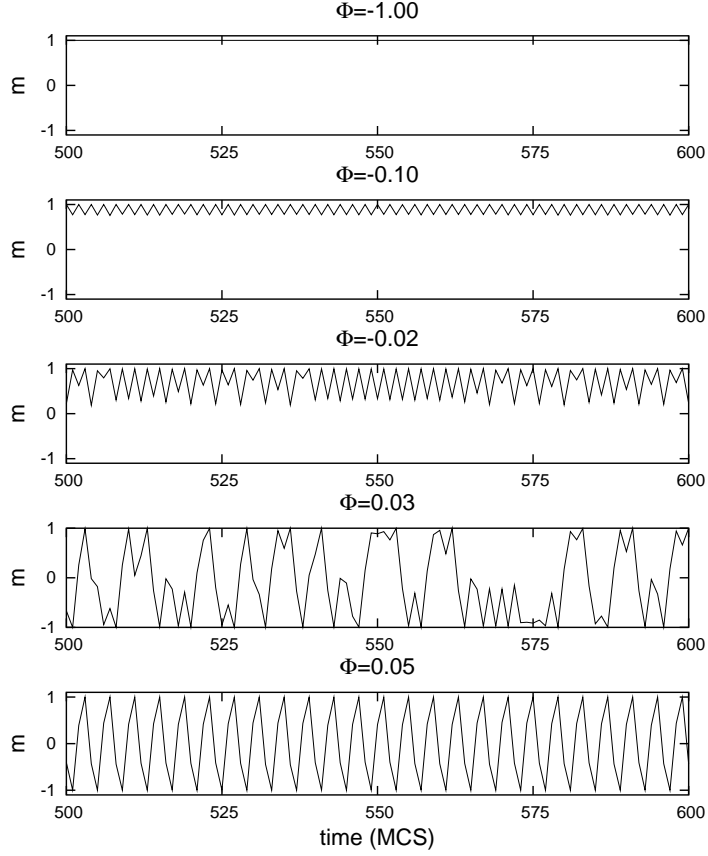[41] W. J. Freeman, J. Theor. Biol. **171**, 13 (1994).

FIG. 1: Monte Carlo time–evolution of the overlap between the current state and the (single) stored pattern for $M = 1$, $N = 10^4$ neurons, $T = 0.1$, and different values of Φ, as indicated. This illustrates, from top to bottom, the fixed point solution in the absence of any synaptic noise, i.e., $\Phi = -1$, a cyclic behavior, the onset of irregular periodic behavior, and fully irregular and regular jumping between the stored pattern and its anti–pattern, respectively.
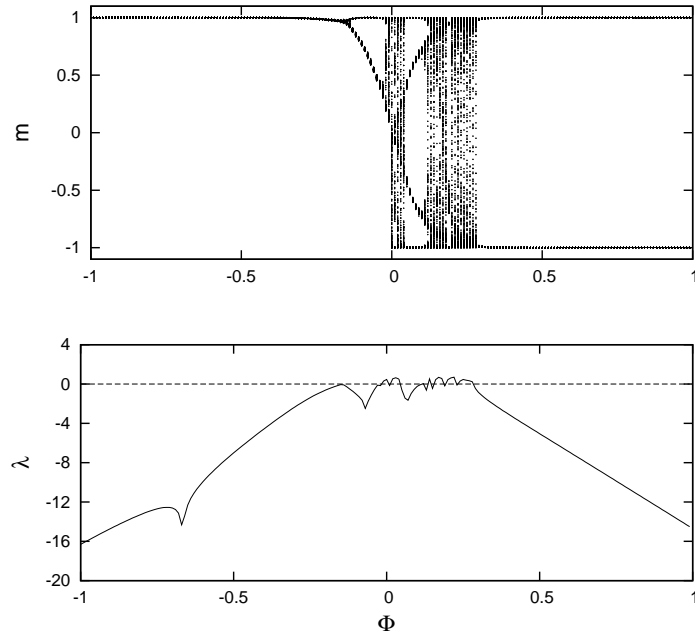
FIG. 2: Bifurcation diagram demonstrating chaotic activity. The upper graph shows the steady overlap between the current state and the stored pattern (for $M = 1$) as a function of the fast noise parameter Φ. This is for Monte Carlo simulations of a network with $N = 10^4$ neurons. The bottom graph depicts the corresponding Lyapunov exponent, λ, as obtained from the mean–field description (7). In particular, this shows some *chaotic windows,* in which $\lambda > 0$. The *temperature* parameter is set $T = 0.1$ here; this is low enough so that the effect of thermal fluctuations is negligible compared to that of presynaptic noise.
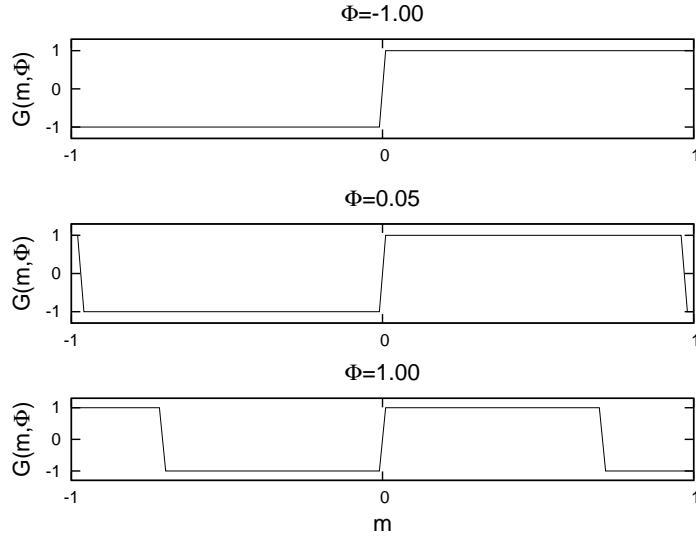
FIG. 3: The gain function (8) versus m_t for different values of Φ, as indicated. Notice that the gain function becomes non-sigmoidal (i.e., oscillatory, which allows for jumping between the attractors) for $\Phi > 0$, while it is monotonic in the Hopfield case $\Phi = -1$.
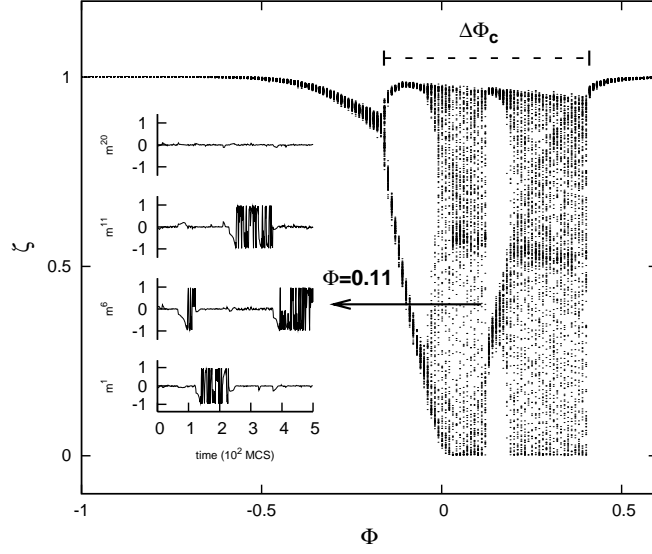
FIG. 4: The function $\zeta(\Phi)$, as defined in the main text, from Monte Carlo simulations at $T = 0.15$ for $N = 10^4$ neurons and $M = 20$ stored random patterns. A region of irregular behavior which extends for $\Delta\Phi_c$, as indicated, is depicted. The insets show the time evolution of four out of the 20 overlaps within the irregular region, namely, for $\Phi = 0.11$.
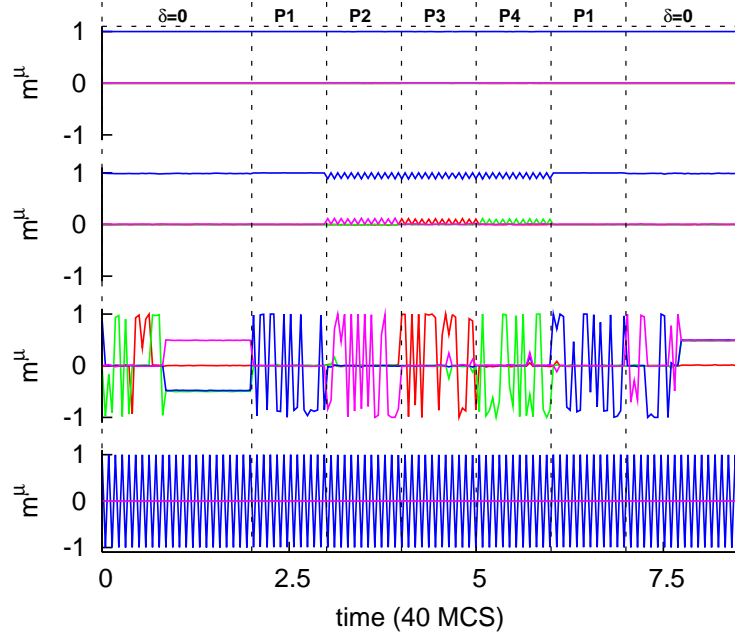
FIG. 5: (Color on-line) Time evolution of the overlap m^μ in a Monte Carlo simulation with $N = 10^4$ neurons, $M = 4$ stored random patterns, $T = 0.05$, and, from top to botom, $\Phi = -0.2, -0.1, 0.12$, and 0.2. This illustrates that, under regular behavior (as for the first two cases and the last one), the system is unable to respond to a weak external stimulus, namely, an extra current $h_i^{\text{ext}} = \delta \xi_i^\mu$, where $\delta = 0.05$ and μ changes ($\mu = 1, 2, 3, 4, 1$) every 40 MCS as indicated in the upper row. The situation is qualitatively different when the regime is chaotic, as for $\Phi = 0.12$ in this figure. After some wandering in the evolution showed here, the system activity is trapped in a mixture state around $t = 80$ MCS. However, the external stimulus induces a continuous jumping to the more correlated attractor. This illustrates how chaos may enhance the system sensitivty.