

Phase Transitions in a Model of Cooperative Behavior in the Brain

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Abstract

A series of neural network systems have recently been proposed that computationally perform most efficiently, e.g., in relation with the task of associative memory. Their definition and most relevant features, as obtained from several theoretical and numerical studies, are reviewed here. The emphasis is on showing that complex functions of the brain are intimately related to the occurrence of (nonequilibrium) phase transitions.

1 Nonequilibrium Phase Transitions

Complex systems, both living and inert, are generally out of thermodynamic equilibrium. The simplest nonequilibrium condition involves a constant flux of matter, energy, or some other quantity. Such nonequilibrium steady states exhibit instabilities known as **nonequilibrium phase transitions**, a concept closely related with that of bifurcations, synergetics, pattern formation, morphogenesis, and self-organization. As part of a constant interest on nonequilibrium phenomena, detailed investigation of nonequilibrium phase transitions in lattice models has blossomed over the last years. A main reason is that one can deal with many interacting units in a lattice system, and this is most amenable to precise analysis in the computer.[1] Qualitatively new and complex kinds of organization thus emerge that promise to illuminate some fundamental questions in physics, biology and sociology.

As a matter of fact, lattice models of nonequilibrium processes have lately begun to multiply at a dizzying pace. Sand piles, driven fluids, traffic models, contact processes, surface catalytic reactions, branching annihilating random walks, and sequential adsorption are just a few examples. They are oversimplified representations or caricatures of nature, but often capture some of the essential features responsible for nonequilibrium ordering in real systems.[1]

2 Hopfield–like Neural Nets

Some intriguing examples of nonequilibrium phase transitions occur in a class of neural net models. Consider a set of N binary neurons, $s = \{s_x = \pm 1; x =$

$1, \dots, N\}$, that evolve in time by the stochastic equation:[2]

$$\begin{aligned} \frac{\partial P}{\partial t} = & p \sum_{\mathbf{x}} [-\varpi_{\mathbf{J}}(s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}})P + \varpi_{\mathbf{J}}(s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}})P_t(s^{\mathbf{x}}, \mathbf{J})] \\ & + (1-p) \sum_{\mathbf{x}, \mathbf{y}} \sum_{J_{xy}} [-\varpi(J_{xy} \rightarrow J'_{xy})P + \varpi(J'_{xy} \rightarrow J_{xy})P_t(\mathbf{s}, J^{xy})]. \end{aligned} \quad (1)$$

Here $P \equiv P_t(\mathbf{s}, \mathbf{J})$ where $\mathbf{J} = \{J_{xy} \in \mathfrak{R}; \mathbf{x}, \mathbf{y} = 1, \dots, N\}$ stands for the set of synaptic weights, $s^{\mathbf{x}}$ represents \mathbf{s} after the change $s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}}$, and J^{xy} is \mathbf{J} after the change $J_{xy} \rightarrow J'_{xy}$. For simplicity, the rates are

$$\begin{aligned} \varpi(J_{xy} \rightarrow J'_{xy}) & \text{ independent of the current } \mathbf{s}, \\ \varpi_{\mathbf{J}}(s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}}) & = \varphi \left[2T^{-1} s_{\mathbf{x}} h_{\mathbf{x}} \right]. \end{aligned} \quad (2)$$

Here, T is a temperature parameter that controls the stochasticity of the neurons evolution, φ is an arbitrary function, and the local field is $h_{\mathbf{x}} = h_{\mathbf{x}}(\mathbf{s}, \mathbf{J}) = \sum_{\mathbf{y}} J_{xy} s_{\mathbf{y}}$. One further assumes that the system contains information from a set of M previously stored patterns,

$$\xi^{\mu} = \{\xi_{\mathbf{x}} = \pm 1; \mathbf{x} = 1, \dots, N\}, \mu = 1, \dots, M, \quad (3)$$

and that this is stored in the synapses according to Hebb's rule for learning:[3]

$$J_{xy} \propto \sum_{\mu=1}^M \xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu}. \quad (4)$$

For $p = 1$ under these conditions, eq.(1) reduces to the Hopfield model.[4, 3] In this case, neurons evolve in the presence of static synapses and asymptotically go to the equilibrium for temperature T and energy $H = -\sum_{\mathbf{x}} h_{\mathbf{x}} s_{\mathbf{x}}$. The resulting \mathbf{s} sometimes resembles one of the M stored patterns, so that the system is said to exhibit associative memory. This is generally poorly performed, however, e.g., the system eventually goes to a mixture of several of the stored patterns.

In the limit $p \rightarrow 0$, (1) transforms into:[1]

$$\frac{\partial P_t(\mathbf{s})}{\partial t} = \sum_{\mathbf{x}} [\varpi(s^{\mathbf{x}}; \mathbf{x})P_t(s^{\mathbf{x}}) - \varpi(\mathbf{s}; \mathbf{x})P_t(\mathbf{s})], \quad (5)$$

where

$$\varpi(\mathbf{s}; \mathbf{x}) = \int d\mathbf{J} f(\mathbf{J}) \varphi \left[2T^{-1} s_{\mathbf{x}} h_{\mathbf{x}}(\mathbf{s}, \mathbf{J}) \right]. \quad (6)$$

The fact that this superposition does not satisfy the condition of detailed balance impedes the system from reaching equilibrium. Instead, it may go asymptotically to a nonequilibrium steady state.[1] In this case, dramatic changes of behavior may be observed for varying f and φ . For certain φ and $f(\mathbf{J})$ such that simulates each synaptic weight taking at each time the value $J_{xy} = \xi_{\mathbf{x}}^{\mu(t)} \xi_{\mathbf{y}}^{\mu(t)}$, independently of the others, so that the Hebb's rule holds as a time average, one obtains the phase diagram of Fig.1(Left).[2] Apart from some metastable behavior, this shows (using the magnetic language) a paramagnetic phase (P) of disordered states in which the system lacks associative memory, a spin-glass phase (SG), in which stable states are undesirable mixtures of several of the stored patterns, and a ferromagnetic phase (F) with the sought behavior. The

latter, however, as for the Hopfield system, is restricted to a small region near the origin, so that the capacity is very limited. In fact the only noticeable difference with the Hopfield case is the curve that separates SG from P, which strongly curves downwards in the present case.[5]

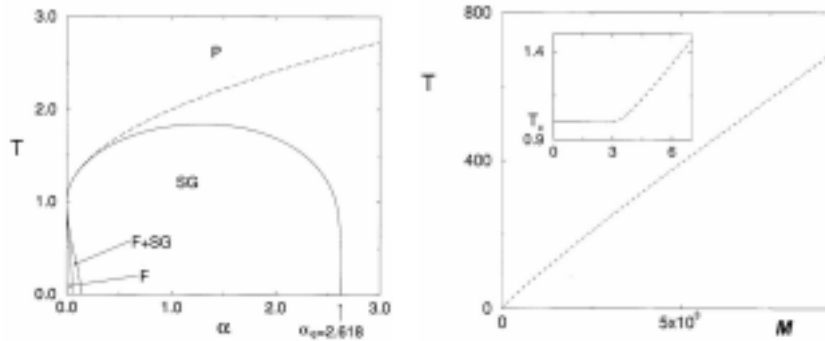


Figure 1: Phase diagram (temperature parameter versus capacity $\alpha \equiv N/M$) for two versions of (1) in the limit $p \rightarrow 0$, i.e., for synapses that fluctuate very rapidly and take at random at each time the Hebb value associated to one of the M stored patterns. Left: Each synaptic weight takes a value at each time which is independent of the others, as if each pair (x, y) would visit a different pattern. The dashed line is the Hopfield result; this transforms in the present case into the upper solid curve going to $\alpha = 2.6$. Right: All synapses visit coherently the same pattern at each time. This was obtained for a macroscopic net, namely, for $N \rightarrow \infty$.

The resulting phase diagram is, however, as in Fig.1(Right) for other choices of φ and f . [6] This is obtained for $f(J)$ simulating that all synapses take at each t the value $J_{xy} = \xi_x^{\mu(t)} \xi_y^{\mu(t)}$ corresponding to one of the patterns. That is, while $\mu(t)$ differs locally from one pair (x, y) to the other in the previous case, it takes the same value throughout the system in Fig.1(Right), which corresponds to coherent synaptic dynamics. Moreover, unlike for Fig.1(Left), the phase transition here is generally of first order, which results in a much smaller error of recovery. This is important, for instance, when one aims at recognizing a pattern out of a set including damaged image of it. [7] Also remarkable is that mixtures do not occur, which is computationally convenient.

3 Neural Automata

The above suggests one to investigate a cellular automaton version of the latter case. This is theoretically more involved —see, however, Ref.[8, 9] for some cases that can be solved analytically— so that its time evolution was simulated in the computer by the Monte Carlo method. The simulation may proceed by choosing at random any pattern, say μ , and updating all the neurons in the lattice assuming the synapses corresponding it, namely, $J_{xy} = \xi_x^\mu \xi_y^\mu$. This step is repeated again and again. The draw is such that the time average for each local J_{xy} gives Hebb's rule, i.e., $\langle J_{xy} \rangle \propto \frac{1}{M} \sum_{\mu=1}^M \xi_x^\mu \xi_y^\mu$. A main result from these

simulations is that even the good performance of the corresponding sequential version —Fig.1(Right)— is improved.[9]

A further improvement of this neural automaton ensues by allowing for the stochasticity of neurons and synapses to be controlled by different (say) temperature parameters. Consider the system described above whose state at time t is determined by $S^t \equiv \{s_x^t\}$ and $J^t \equiv J_{xy}^t$. One also has the set (3) of M binary patterns. For the case of interest, we can identify J^t by the pattern $\mu(t)$ that determines all the synapses at time t ; therefore, the state of the system is characterized by the set (S, μ) . It turns out convenient to define the overlap at time t of the current state S^t with each of the memorized patterns, $m^t = \frac{1}{N} \sum_{x=1}^N \xi_x^\mu s_x^t$. Time evolution proceeds according to

$$P^{t+1}(S, \mu) = \sum_{S^0} \sum_{\mu^0} T[(S, \mu) \rightarrow (S', \mu')] P^t(S', \mu'), \quad (7)$$

where $T[(S, \mu) \rightarrow (S', \mu')]$ is the probability for the indicated transition. Let us assume that this is separable, namely, that $T[(S, \mu) \rightarrow (S', \mu')] = T_n^{\mu^0}(S \rightarrow S') \cdot T_s^S(\mu \rightarrow \mu')$, and that

$$T_n^{\mu^0}(S \rightarrow S') = \prod_{i=1}^N \varphi \left[-\beta_n \xi_x^{\mu^0} (s_x - s'_x) \right] m^{\mu^0} - s'_x \xi_x^{\mu^0} N^{-1} \quad (8)$$

and

$$T_s^S(\mu \rightarrow \mu') = \psi \left[-\frac{1}{2} \beta_s N ((m^\mu)^2 - (m^{\mu'})^2) \right]. \quad (9)$$

Here, φ and ψ are arbitrary functions, and β_n and β_s are the inverse temperatures that control the neuron and synaptic sub-systems, respectively.

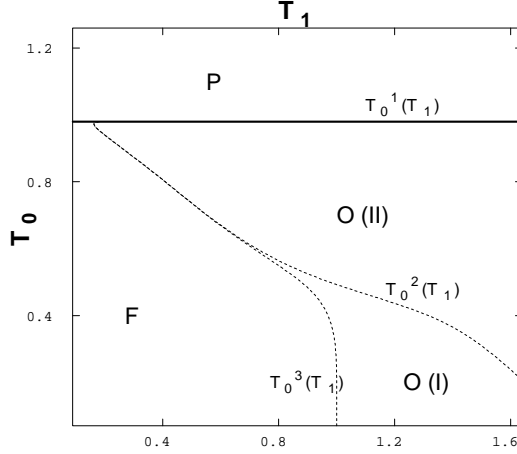


Figure 2: The phase diagram (neuron versus synaptic temperatures in arbitrary units) for the neural automata defined by equations (7)-(9). See the main text for characterization of phases.

Fig.2 depicts the resulting phase diagram for certain choices of functions φ and ψ . [10] (It is remarkable that modifying these functions may essentially change the diagram. That is, there is an —in principle— enormous variety of

phases the model may exhibit and, consequently, a great versatility concerning possible applications.). In addition to paramagnetic (P) and ferromagnetic (F) phases, the system in the regions marked O has dynamic associative memory: After a transient time in which one of the stored patterns is recovered, the system jumps to one of the other possible attractors, and keeps doing so. Furthermore, this jumping is not completely at random but computer simulations uncover some non-trivial structure of time correlations. More specifically, jumping between patterns depends on the activities, and non-trivial time correlations develop as the neuron temperature is lowered, namely, in region O(I). (However, the system in O(II) stays the same amount of time wandering in each attractor.) This behavior suggests one many interesting applications of the algorithm.

Finally, before going to a discussion (which —due to lack of space— is omitted in this written version of my talk) we mention that different network architectures have been investigated. One thus concludes that a power-law or scale-free topology is advantageous compared to the corresponding diluted network. There is also some definite evidence that hubs, i.e. the few most highly connected nodes, play then a fundamental role in making the retrieval of information more robust and efficient. This is most interesting as the scale-free property is known to hold in many natural systems, including cultured neurons.[11]

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