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Interacting Particle Lattice Systems: Some Recent Results on Nonequilibrium Steady States and Phase Transitions

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Abstract - This is a very short description of results in a series of studies of phase transitions in Ising-like or interacting particle lattice systems which are very convenient for computational purposes. The condition of detailed balance does not hold in general due to the presence of either conflicting kinetics or driving dissipative fields. The systems exhibit interesting behaviour that may sometimes correspond to natural situations, e.g., they model ideal steady nonequilibrium states in complex systems.

INTRODUCTION

Macroscopic phenomena in complex systems often exhibit net stationary currents or influence of constant external agents that induce steady nonequilibrium conditions. No general model that generalizes equilibrium thermodynamics or Gibbs ensemble theory exists for these situations. One may set up relatively simple kinetic Ising-like lattice systems, however, that model (quite ideally, of course) some of the steady states that are observed in nature.[1] The models of interest belong to the class of *probabilistic cellular automata* or *interacting particle/spin systems* in which kinetics consist of local probabilistic rules (That is, it is not convenient to distinguish here between discrete and continuous time stochastic processes;[2] furthermore, finite and infinite lattices are assumed to be equivalent for practical purposes.) These systems may be implemented in the computer as efficiently as the standard kinetic Ising model that evolves towards equilibrium. Therefore, computational physics reveals itself as a most convenient tool to study steady nonequilibrium states. I find interesting the following approach from the point of view of this objective. One first considers any of the (now standard) kinetic lattice models, e.g., the lattice gas, the binary alloy, or the magnetic Ising model. They have in common the fact that the corresponding (particle, spin, etc.) configuration evolves in time by a stochastic process that satisfies

detailed balance which implies an asymptotic tendency towards the canonical equilibrium state. Then, however, detailed balance is impeded by modifying kinetics, e.g., by involving competing tendencies or driving dissipative fields. In any of these cases, the effect is similar to the action of an external, non-Hamiltonian agent. The resulting models might appear somewhat arbitrary at first glance but may model natural situations in practice such as fast ionic conductors, chemically reacting systems, microscopically disordered materials, and other interesting features of complex systems.[1] Furthermore, they allow a systematic theoretical study of nonequilibrium steady states and phase transitions. My lecture tries to illustrate some of the prolific behaviour that has been revealed recently by both analytical and numerical, Monte Carlo study of nonequilibrium lattice systems of that sort. I shall refer very briefly to a reaction diffusion model, to a nonequilibrium model of spin glasses, to a model of neural network, to a nonequilibrium ANNNI model, and to a driven diffusive system.

SYSTEMS WITH COMPETING INTERACTIONS

The models of interest have in common that the configuration, say $\mathbf{s} = \{s_x = \pm 1; \mathbf{x} \in \lambda\}$, where $\lambda = \mathbf{Z}^d$, evolves in time due to a stochastic process which is characterized by the following *master equation* for the probability of $\mathbf{s}$ at time t:

$$\partial P(\mathbf{s};t)/\partial t = \mathbf{L}\, P(\mathbf{s};t) \equiv \sum_{\mathbf{s}'} [c(\mathbf{s}' \to \mathbf{s})\, P(\mathbf{s}';t) - c(\mathbf{s} \to \mathbf{s}')\, P(\mathbf{s};t)]$$

Here, $c(\mathbf{s} \to \mathbf{s}')$ is the probability per unit time for a transition from $\mathbf{s}$ to a new configuration $\mathbf{s}'$. Different models correspond to different realizations of $\mathbf{L}$. A familiar case is the *Glauber generator*,[3] to be denoted $\mathbf{L}_G$. This is implemented by choosing an arbitrary initial configuration and performing the following step reiteratively: First, a site $\mathbf{x}$ is selected at random; then, one attempts to *flip spin* s_x, i.e., one attempts the change $s_x \to -s_x$, according to some given criterion. The resulting configuration $\mathbf{s}'$ is specifically written as $\mathbf{s}^x$. Alternatively, one may change the initial configuration reiteratively by the following step: First, two nearest-neighbour (NN) sites, $\mathbf{x}$ and $\mathbf{y}$, are selected at random; then, one attempts the interchange, $s_x \rightleftarrows s_y$, of the corresponding occupation variables (only) if $s_x s_y = -1$ (i.e., if any one of the sites is occupied by a particle and the other is empty) according to some given criterion. The corresponding operator may be called the *Kawasaki generator*,[4] to be denoted $\mathbf{L}_K$; $\mathbf{s}'$ is specifically written as $\mathbf{s}^{xy}$ in this case. The simplest criterion one may use in any of the above cases to decide whether or not the attempted move should be performed corresponds to the familiar *Metropolis algorithm*: A configurational energy is defined, e.g., the one that corresponds to the NN Ising Hamiltonian $H_J(\mathbf{s}) = -J \sum_{NN} s_x s_y$. Then, one computes the energy cost associated to the move, i.e., $\Delta H_J = H_J(\mathbf{s}') - H_J(\mathbf{s})$. The move is performed with probability $\phi(\beta \Delta H_J) = \min\{1, \exp(-\beta \Delta H_J)\}$,

where $\beta=(k_BT)^{-1}$ is the inverse temperature that characterizes the bath involved.

The function $\phi(X)$ is critical in determining the asymptotic steady state to which the system evolves as $t\to\infty$. The Metropolis choice satisfies detailed balance, i.e., $\phi(X) = \exp(-X)\,\phi(-X)$. The latter is sufficient (but not necessary) to guarantee that the canonical equilibrium state is reached asymptotically. That is, the stationary solution of the master equation is $P_{st}(s) \propto \exp[-\beta H_J(s)]$ for any function $\phi(X)$ that satisfies detailed balance. Otherwise, the master equation may imply tendency towards a steady nonequilibrium state that may differ essentially (from a conceptual point of view, at least) from the equilibrium state. It occurs, for instance, when the stochastic process defined by $\mathbf{L}$ corresponds to one of the cases defined below. In fact, none of them may be described in general by a proper Hamiltonian, in which they also differ from the ordinary, equilibrium lattice models. A configurational function is still involved in some cases, however; for simplicity, such a function is taken here equal to the NN Ising Hamiltonian, $H_J(s)$.

Consider first the ordinary lattice gas with attractive interactions, i.e., $J>0$. The generator, however, is defined as $\mathbf{L} = p\,\mathbf{L}_K + (1-p)\,\mathbf{L}_G$, where $\mathbf{L}_G$ and $\mathbf{L}_K$ are implemented in the computer by using probabilities $\phi[\Delta H_J(s^x)/k_BT]$ and $\psi[\Delta H_J(s^{xy})/k_BT']$, respectively. Both ϕ and ψ satisfy detailed balance (they may correspond to the Metropolis algorithm but this is not necessary; ϕ and ψ may even differ from each other). The combination of ϕ and ψ which is implied by $\mathbf{L}$ does not satisfy detailed balance in general, however; in addition, one has $T\neq T'$. Consequently, the lattice gas follows only for $p\equiv 1$, and $p\equiv 0$ is the only case that corresponds to the ordinary magnetic Ising system, while steady nonequilibrium states occur otherwise in general. They are induced by the competition between, say *reaction processes* due to $\mathbf{L}_G$, that are activated by a thermal bath at temperature T, and *diffusion processes* due to $\mathbf{L}_K$ consisting of NN exchanges driven by a thermal bath at temperature T'. The two processes are independent in continuous time, with p the *a priori* probability of exchanges per bond and $1-p$ that of reactions per site. I refer elsewhere for details of the resulting behaviour as well as for some related theory and further interesting properties of the model.[1,5-7] Some main global conclusions that ensue from both exact and Monte Carlo studies are as follows.

The study of two-dimensional systems reveals that the diffusion rate is irrelevant for $T'=\infty$ (completely random diffusion), but not the reaction rate. The exact solution for microscopically fast diffusion, i.e., within the limit $p\to 1$, indicates that a phase transition exists which is of first order when the reaction rate is implemented by the Metropolis algorithm, and of second order with classical exponents for other realizations. The limit $p\to 1$ is singular, however. Monte Carlo data for $p<1$ reveal that the phase transition is of first order for $p>0.83$ only, i.e., a *tricritical point* exists at $p_t\simeq 0.83$. Moreover, the critical exponents for $p\leq p_t$ have the equilibrium ($p = 0$) values in agreement with renormalization group computations. On the other hand, the behaviour within the limit

$p \to 1$ is mean field for $T' = \infty$ and Metropolis (reaction) rates. Some limited data suggest that no first-order phase transition occurs for Glauber (reaction) rates when $p < 1$, in agreement with the exact result for $p \to 1$. The system with finite T' has been studied for $\phi(X) = \psi(X) = \min\{1, e^{-X}\}$. Three distinct types of qualitative behaviour occur as one varies p, T and T': (a) equilibrium-like, Onsager second-order phase transitions (with some nonequilibrium features, however); (b) nonequilibrium first-order phase transitions whose discontinuities are very pronounced; and (c) some very weak, *rare* discontinuous phase transitions. As for equilibrium, the one-dimensional system with $T' = \infty$ and $p \to 1$ has a zero-T critical point for familiar reaction rates. An exception is $1 - s_x[s_{x-1} + s_{x+1}]\tanh(J/k_BT) + s_{x-1}s_{x+1}\tanh^2(J/k_BT)$ that induces a mean-field second-order phase transition at finite temperature. On the other hand, the Monte Carlo data for $p < 1$, $T' = \infty$, and certain reaction rate might suggest the existence of a phase transition perhaps for $p > 0.5$ at least; the corresponding transition temperature would increase with p.

Next, consider the ordinary magnetic Ising model. The generator is, however, a combination of two different, independent Glauber generators, $\mathbf{L} = p\,\mathbf{L}_G^{+} + (1-p)\,\mathbf{L}_G^{-}$, where the subscript refers to the sign of the exchange interaction J in the corresponding configurational function $H_J(\mathbf{s})$. Therefore, after a site $\mathbf{x}$ is selected at random, one attempts in this case to flip spin s_x according to the following criterion: all the exchange interactions between s_x and its four NN are assumed to have the value $J_o > 0$ with probability p, and $-J_o$ with probability $1-p$. The energy cost of the attempted flip is computed then by using H_J with J equal to the selected value. Thus, one gets either $\Delta H_{J_o} = 2J_o s_x \sum_{\text{NN of } x} s_y$ or $\Delta H_{-J_o} = -2J_o s_x \sum_{\text{NN of } x} s_y$, where the sums extend to all NN of site $\mathbf{x}$. If the Metropolis algorithm is to be used, for example, the flip is performed with probability either $\min\{1, \exp(-\beta\Delta H_{J_o})\}$ or $\min\{1, \exp(-\beta\Delta H_{-J_o})\}$, respectively. I refer elsewhere for the detailed description of this case.[1,7-11] Some of its main features are as follows.

The resulting system is a sort of nonequilibrium impure Ising-like model in which NN interactions J change sign randomly with time. That is, kinetics involve the simultaneous action of two independent spin-flip mechanisms, each satisfying individually detailed balance and occurring *as if* the exchange interaction between the involved spins has a different value. It induces fast and random, spin-configuration-independent diffusion of microscopic disorder. This may be viewed as a first step in modelling spin glasses when magnetic ions diffuse. Under these conditions, the system is driven asymptotically towards pure ferromagnetic and antiferromagnetic steady states for $p = 1$ and 0, respectively, while there will be (non-unique) steady nonequilibrium states otherwise in general. The consideration of such *nonequilibrium* impure model is appealing because some *unusual* observations in disordered materials are consistent with the existence of *nonequilibrium* effects, e.g., a dependence of the steady state on history has been reported. On the other hand, the model, which one may expect *a priori* to behave differently from the (equilibrium) quenched spin-glass model, essentially differs also

from the annealed version of the latter which involves an unrealistic representation (e.g.) of impurity diffusion that leads to impurities that are strongly correlated, which is not observed in general.

The following picture emerges from a Monte Carlo study of square lattices for $p > \frac{1}{2}$ (the system is symmetrical around $p = \frac{1}{2}$): (a) A transition to a ferromagnetic state occurs at $T_C(p)$ for p large enough. (b) The parameter p has three main effects on the phase transition: both the degree of saturation and the transition temperature tend to decrease with p, and there is no low-T ferromagnetic state for any $p \leq 0.91$. (c) The indicated transition is qualitatively similar to the one in the pure system, i.e., to the equilibrium, Onsager one for $p = 1$. In particular, it is of second order, and it seems consistent with the Onsager critical exponents. The magnetization for $p = 0.93$ departs from the Onsager behaviour as p is decreased, however. It may indicate that critical exponents differ from the Onsager values for $p \approx 0.93$. A finite-size study confirms the picture. There is also some evidence (whose significance needs to be evaluated) for the existence of order at low temperature between the ferromagnetic and antiferromagnetic regions. In any case, no *freezing* phenomena have been observed. It is remarkable as well that this system has a probabilistic cellular-automaton representation when $T = 0$ which is very convenient for numerical analysis, e.g., thermal fluctuations are absent, and finite-size effects are relatively small. It consists of the following rule to attempt the flip of spin s_x: Compute $\alpha_x \equiv s_x \Sigma_{NN} s_y$, where the sum extends to the 2d NN of site x; the flip is performed with probability p or $1-p$ according to whether α_x is negative or positive, respectively; the flip is performed anyway if α_x is zero. This criterion, which is close to the kinetic rule that characterizes the so-called *majority vote models*, induces steady states exhibiting a sort of nonequilibrium percolation phenomena whose critical exponents are the ones for the ordinary Ising model.

Three variations of this nonequilibrium spin-glass model have been studied that are worth to be mentioned. A competition between different values for the applied magnetic field, h, occurs during the evolution of time in one of them. In addition to impure systems in which local fields diffuse, this may model a magnetic system under a very rapidly fluctuating field.[12] The case in which several values for both J and h compete during the evolution has been shown analytically to exhibit a varied and interesting, e.g., non-universal behaviour in one dimension. This may perhaps be relevant to the study of neuronal activity noise and the so-called *proton glasses*.[13] Also interesting is a nonequilibrium version of the ANNNI problem.[14] It is remarkable that some analytical results have been obtained exactly for dimension larger than one in some of these systems.[13,14]

QUASI TWO DIMENSIONAL LATTICE GASES

This section concerns a different method for inducing irreversibility. Consider first the ordinary

lattice gas, λ, on (e.g.) a square lattice of volume $|\lambda|$ with periodic boundary conditions. The energy is $H(\mathbf{s}) = -4J\sum_{NN} s_x s_y$, $J>0$, where $\mathbf{s}=\{s_x; \mathbf{x}\in\mathbf{Z}^2, s_x=0,1\}$, and the sum is over all pairs $(\mathbf{x},\mathbf{y})$ of NN sites. $\rho \equiv |\lambda|^{-1}\sum_x s_x$ is a density, and $N \equiv \rho|\lambda|$ is the number of particles. In the infinite-volume limit, λ exhibits a continuous phase transition for $\rho=½$ at the *Onsager temperature* T_C, while there is a discontinuous transition for $\rho\neq½$ at temperature $T_{LG}(\rho)$. Next, consider $\Lambda\equiv\lambda_1\bigcup\lambda_2$ of volume $|\Lambda| = 2|\lambda|$; here both λ_1 and λ_2 are defined as λ, and $\lambda_1\bigcap\lambda_2=\varnothing$.[15,16] That is, Λ consists of two twin square lattices such that any site has five NN with one of them in the other plane. The configurations have energy $H_\Lambda(\mathbf{s})=H(\mathbf{s}^1)+H(\mathbf{s}^2)$, where $\mathbf{s}^i=\{s_r;\mathbf{r}\in\mathbf{Z}^2\}$ is a configuration of λ_i, and $i=1,2$. That is, all bonds between the planes are broken. No restriction exists on the possible configurations $\mathbf{s}$ of Λ, however; in particular, any given particle has access to any of the two sub-lattices. Irreversibility is introduced by assuming that particles are driven by a constant external electric field $\mathbf{E}$ that points along one of the principal lattice directions. For simplicity, $E\to\infty$, so that no particle can jump against the field direction under the influence of thermal fluctuations. Therefore, one implements case $\mathbf{L}_K$ in the computer, and the Metropolis criterion is modified in the sense that one adds the work done by the field to the energy cost. That is, $\Delta H_J \equiv H_\Lambda(\mathbf{s}^{xy}) - H_\Lambda(\mathbf{s}) - \mathbf{E}\bullet(\mathbf{x}-\mathbf{y})(s_x-s_y)$, where $H_\Lambda(\mathbf{s})$ does not involve any interaction between planes. In addition to the fact that detailed balance only holds locally here, due to the last term in the expression for ΔH_J, $\mathbf{E}$ is not the gradient of a potential, so that no obvious Hamiltonian including the electric energy exists, and a net steady *dissipative* current sets in (e.g.) for periodic boundary conditions. Let us denote by λ_E and Λ_E the nonequilibrium systems one obtains by applying such dynamics to λ and Λ, respectively. (Then, Λ_∞ may denote Λ_E within the limit $E\to\infty$.) The case λ_E is the (2d) *driven diffusive system* that may model solid electrolytes or fast ionic conductors.[17] The case Λ_E is a non-trivial variation of the latter.[18]

The main facts concerning ordering in these systems are as follows. For $\rho=½$ below a temperature T^*, Λ_0 (which denotes Λ_E when *the field is turned off*) exhibits a *liquid* (equilibrium) phase of density $\rho_L(T)=\rho_o(T)$ that fills one of the planes, and a *gas* of density $\rho_G(T)=1-\rho_o(T)$ that fills the other plane; $\rho_o(T)$ is the Onsager solution, e.g., $\rho_o(T) = ½ + ½\{1-[\sinh(2J/k_BT)]^{-4}\}^{1/8}$ at low temperature. For $\rho<½$ below $T^*(\rho)$, one plane holds a *liquid drop* of density $\rho_o(T)$ coexisting with gas of density $1-\rho_o(T)$, and there is only gas in the other plane. Then, $\rho=½\rho_G+½[x\rho_L+(1-x)\rho_G]$, and there follows $x = 2(\rho-1+\rho_o)/(2\rho_o-1)$. The particles tend to be distributed homogeneously for $T>T^*(\rho)$ where $\rho=1-\rho_o(T^*)$; $T^*(\rho)=T_{LG}(\rho)$, where the latter is the transition temperature for λ; in particular, $T^*(\rho=½) = T_C$. Define the difference of density between the two planes, $\Delta\rho(T) \equiv (1/2\rho)|\rho_1(T)-\rho_2(T)|$; $\Delta\rho(T) = \rho^{-1}|\rho-1+\rho_o(T)|$. This behaves continuously for any ρ. More precisely, Λ_0 has the Onsager critical point for $\rho=½$, and the order parameter is continuous for $\rho<½$ also.

The situation for Λ_∞ is qualitatively different. At high enough T , the particles distribute evenly between the two planes. The distribution within each plane is not completely homogeneous but some anisotropic clustering is evident (cf. the graphs (a) in Figs. 1 and 2, for $\rho = 0.5$ and 0.2 , respectively). The latter is related both to the anisotropy of the state and to the slow decay of spatial correlations. As T is lowered starting from a random distribution, the system exhibits phase segregation and, apparently, two kinds of phase transitions occur: First, for (say) $T_\infty^*(\rho) > T > T_\infty'(\rho)$, one observes that the liquid separates into two approximately equal strips, *one on top of the other* in a different plane (graphs (b) in the figures). For very low temperature, say $T < T_\infty'(\rho)$, the particle-rich or *liquid* phase is in one plane (graphs (c)). This is qualitatively similar to the segregation that occurs in λ_∞ except for the existence of the second plane holding gas. The field induces clear anisotropies below $T_\infty'(\rho)$. That is, the liquid configurations are strip-like oriented parallel to the field for $\rho < ½$, as for λ_∞ . Moreover, the gas seems to exhibit the sort of anisotropic clustering mentioned above for high T .

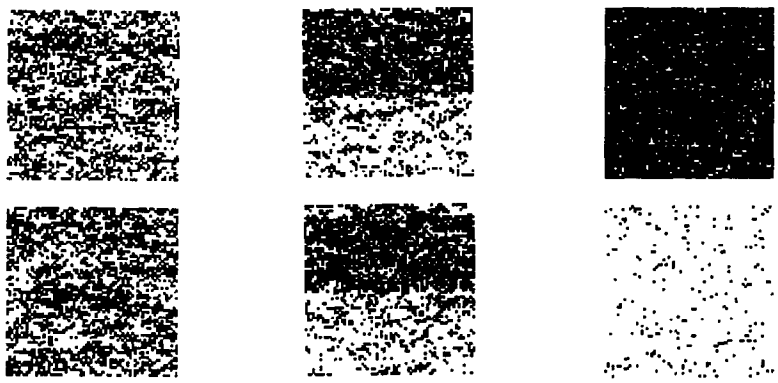

Figure 1: Typical steady state Monte Carlo configurations for $\rho = ½$; cf. the main text. The field acts horizontally.

As a consequence of the above effects, the relation between Λ and Λ_∞ is not straightforward. It seems that a main effect of the field is to modify the interface (in addition to the correlations) so that the density of the liquid which is $\rho_0(T)$ at equilibrium changes to $\rho_\infty(T)$, and the density of the gas is $1-\rho_\infty(T)$. That is, $\rho_\infty(T) \neq \rho_0(T)$ which makes the liquid fraction x and, consequently, the transition temperature to differ from the ones for the equilibrium case. On the contrary, one may expect that Λ_∞ and λ_∞ have the same transition temperature and, probably, the same critical behaviour. In

a sense, the relation between λ_∞ and Λ_∞ may be similar to the one found before at equilibrium between λ and Λ .[15] More specifically, one may assume that $\rho=\frac{1}{2}(1-\rho_\infty)+\frac{1}{2}[x\rho_\infty+(1-x)(1-\rho_\infty)]$ and, therefore, $x=2(\rho-1+\rho_\infty)/(2\rho_\infty-1)$ for Λ_∞ . This is to be compared with $x = 2(\rho-1+\rho_0)/(2\rho_0-1)$ for Λ . One would expect accordingly that $x = (\rho-1+\rho_\infty)/(2\rho_\infty-1)$ for λ_∞. This is basically confirmed by the Monte Carlo data. The difference between $\rho_\infty(T)$ and $\rho_0(T)$ is large for striped configurations due to the presence of the nonequilibrium interface all along the system; if no interface exists, the difference between $\rho_\infty(T)$ and $\rho_0(T)$ is extremely small in general; the difference between the phases that is observed then is to be associated to the different nature of correlations.

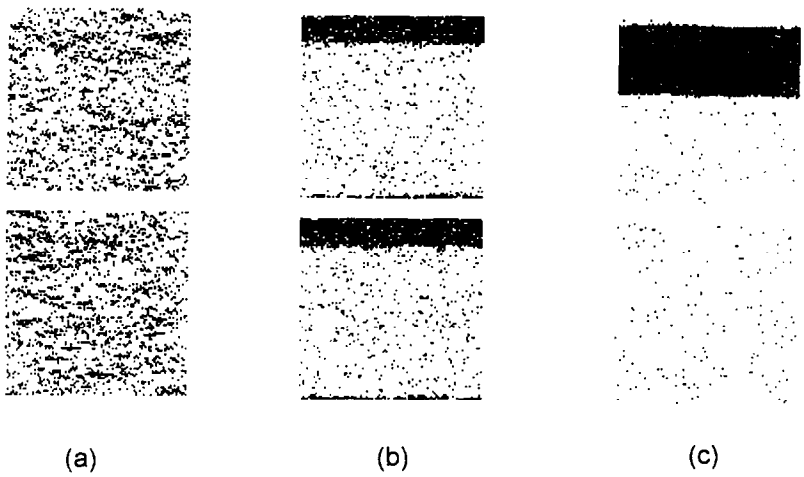

Figure 2: The same as Fig. 1 for $\rho = 0.2$; cf. the main text.

The Monte Carlo study has monitored the difference of density between the planes, and between the phases, e.g., $\delta\rho(T) \equiv \rho_L(T)-\rho_G(T) = 2\rho_\infty(T)-1$; they are closely related to each other (only) under the above assumption. The data confirm that $\rho_L+\rho_G=1$ for any ρ and T , and indicate that $\rho_\infty(T,\rho)\neq\rho_0(T,\rho)$ in general. That is, the nonequilibrium liquid and gas phases for $\rho=\frac{1}{2}$ and $T<T_\infty'(\rho)$ differ from the equilibrium ones. It is observed that $\delta\rho(T,\rho)$ $(=2\rho_\infty(T,\rho)-1)$ is discontinuous at $T = T_\infty^*(\rho)$ for $\rho<<\frac{1}{2}$, and at $T=T_\infty'(\rho)$ for any ρ . The latter fact probably reflects the discontinuous behaviour of the interface, i.e., $\delta\rho(T)$ apparently increases when one crosses $T_\infty'(\rho)$ for any ρ as T is increased. It is observed also an apparent tendency of $\delta\rho(T)$ to decrease with decreasing ρ for a given $T<T_\infty'(\rho)$. These facts are confirmed by $\Delta\rho(T,\rho)$ when possible. Also

interesting is the behaviour of the order parameter $m \equiv \max\{m_1, m_2\}$, where $m_i = \frac{1}{2}[\rho(1-\rho)]^{-\frac{1}{2}} \mid <M_h^2> - <M_v^2> \mid^{\frac{1}{2}}$ with $M_{h(v)}^2 = \mid \lambda \mid^{-3/2} \sum_{h(v)} [\sum_{v(h)} (1-2s_r)]^2$. Here, h (v) indicates that summation is along the horizontal, i.e., field (vertical) direction on each plane. This confirms that the phase transition at $T_\infty'(\rho)$ is discontinuous for any ρ , while the one at the higher temperature $T_\infty^*(\rho)$ is discontinuous for $\rho << \frac{1}{2}$ but the discontinuities are very small if any as one approaches $\rho = \frac{1}{2}$, and $T_\infty^*(\rho = \frac{1}{2})$ should correspond to a critical point for the infinite system. Moreover, one concludes with confidence that $m \sim \mid T - T_\infty^*(\rho = \frac{1}{2}) \mid^\beta$ as $T \to T_\infty^*(\rho = \frac{1}{2})$ with $T_\infty^*(\rho = \frac{1}{2}) = 1.30 \pm 0.01$ and $\beta = 0.27 \pm 0.02$ for the infinite lattice. On the contrary, the Onsager case $\beta = \frac{1}{8}$ and the Landau case $\beta = \frac{1}{2}$ (even if one allows for a logarithmic correction) are clearly excluded. This suggests again the crucial effect of the field due to the existence of a nonequilibrium interface. ($\Delta\rho(T)$ and $\delta\rho(T)$ exhibit a similar critical behaviour near $T_\infty^*(\rho = \frac{1}{2})$, while $\delta\rho$ is characterized by $\beta \approx \frac{1}{8}$ for $E = 0$.) Previous numerical experiments have suggested that $\beta \approx \frac{1}{4}$ for other 2d nonequilibrium conservative lattice systems that involve anisotropies. (Approximately, the same value has been reported for a 2d field-theoretic driven diffusive system when the applied field is random but not for the present case of constant field. In spite of the latter result, I suggest that $\beta \approx \frac{1}{4}$ characterizes the two-dimensional driven-diffusive lattice system as well as some related anisotropic nonequilibrium systems. It seems desirable and potentially very interesting trying to confirm the existence of a novel universality class in three- and low-dimensional fast ionic conductors under proper nonequilibrium conditions.[16-18])

Finally, I mention that a specific question in nonequilibrium systems is the nature of correlations. One might argue that correlations should perhaps not decay here as slowly as for λ_∞ given that particles can hop to other plane in Λ_∞ (e.g., the structure function is easily stabilized due to this effect). Nevertheless, a log-log plot of both $G_h(r)$ and $G_v(r)$ produces straight lines of slope -2 (e.g.) for $2 \leq r$ and $T \geq 2.5T_C$; this supports here the result $G(x,y) = (ax^2 - by^2)(x^2 + y^2)^{-2}$ found before for the 2d driven diffusive system. On the other hand, one may estimate the correlation length from the phenomenological fit $G_{h,v}(\mathbf{r}) \sim [1 + (\mathbf{r}/\xi_{h,v})^2]^{-1}$ that is confirmed by the data for *large enough* $\mathbf{r}$. The values of ξ_h and ξ_v obtained in this way have critical behaviour at $T_\infty^*(\rho = \frac{1}{2})$ that may be characterized by the exponents ν_h and ν_v , respectively, and it seems that $\nu_h \approx 2\nu_v$ (this is based on the assumption that $\nu_h \neq \nu_v$, and I believe, however, that one should not rule out completely the possibility that $\nu_h \equiv \nu_v$ if the latter are properly defined). Summing up, one may relate to each other the nonequilibrium system Λ_∞ , the ordinary two-dimensional driven diffusive system λ_∞ , the ordinary lattice gas λ_o , and the lattice gas in two planes Λ_o . The nature of both short- and long-range correlations, and the existence for some values of T and ρ of a nonequilibrium interface whose length is proportional to one of the system linear dimensions, makes Λ_∞ essentially different from its equilibrium counterpart Λ_o (and also from λ_o). There is a close relation between Λ_∞ and λ_∞ ,

however, that, in particular, makes the study of an apparently artificial model rather relevant for nonequilibrium theory (Λ_∞ has an intrinsic interest and, perhaps, some practical relevance as well). On the other hand, the numerical study of Λ_∞ turns out to be more rewarding than that of λ_∞ , and even more than the MC study of the familiar λ_0 . Further numerical effort should parallel a better understanding of scaling behaviour for anisotropic situations. I refer elsewhere for further details.[1,7,15-18]

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REFERENCES

1. J. Marro, *Nonequilibrium Statistical Physics*, Lecture Notes, *Troisième Cycle de la Physique en Suisse Romande (Physique Théorique)*, Laussane, 1992.
2. T. M. Liggett, *Interacting Particle Systems*, Springer-Verlag, New York, 1985; R. Durrett, *Particle Systems and Percolation*, Wadsworth, California, 1988.
3. R. J. Glauber, *J. Math. Phys.* **4**, 294 (1963).
4. K. Kawasaki, *Phys. Rev.* **145**, 224 (1966).
5. J. M. González-Miranda, and J. Marro, *J. Stat. Phys.* **61**, 1285 (1990).
6. J. J. Alonso, J. Marro, and J. M. González-Miranda, *Phys. Rev. E* **47**, 885 (1993).
7. J. Marro, A. Achahbar, and J. J. Alonso, in *II Granada Lectures in Computational Physics*, P. L. Garrido and J. Marro eds., World Scientific, Singapore 1993, pp. 173-188.
8. J. M. González-Miranda, A. Labarta, M. Puma, J. F. Fernández, P. L. Garrido, and J. Marro, *Phys. Rev. E* **49**, 2041 (1994).
9. J. Marro, J. F. Fernández, J. M. González-Miranda, and M. Puma, *Phys. Rev. E* **50**, 3237 (1994).
10. P. L. Garrido, and J. Marro, *Europhys. Lett.* **15**, 375 (1991).
11. J. J. Alonso, and J. Marro, *Phys. Rev. B* **45**, 10408 (1992).
12. A. I. López-Lacomba, and J. Marro, *Phys. Rev.* **B46**, 8244 (1992).
13. A. I. López-Lacomba, and J. Marro, *Europhys. Lett.* **25**, 169 (1994).
14. A. I. López-Lacomba, and J. Marro, *J. Phys. A: Math. Gen.* **27**, 1111 (1994).
15. A. Achahbar, J. Marro, and P. L. Garrido, "On the Equilibrium and Time Relaxation of a Lattice Gas in Several Boxes", preprint.
16. A. Achahbar, and J. Marro, *J. Stat. Phys.* **78**, 1493 (1995).
17. J. Marro, P. L. Garrido, and J. L. Vallés, *Phase Transitions* **29**, 129 (1991).
18. P. L. Garrido, J. Marro, and R. Dickman, *Ann. Phys.* (N. Y.), **199**, 366 (1990).