Critical and scaling properties of cluster distributions in nonequilibrium Ising-like systems

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We report on analytical and Monte Carlo studies of d-dimensional nonequilibrium stochastic lattice systems whose dynamical rule incorporates various symmetries. We find that critical behavior is of the Ising variety, and that the cluster distribution has scaling properties proposed earlier for the equilibrium system, and give estimates for the exponents that characterize clusters for $d = 2$. The scaling region is notably larger than the corresponding one for the equilibrium case; ramified *percolating* clusters do not occur below the critical point.

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I. INTRODUCTION

The understanding of nonequilibrium ordering phenomena [1] often involves an extension of the phase transition concept to systems that are open to the environment. The theory supporting this extension is scarce, however, so that investigating simple stochastic lattice systems is sensible. The models of interest have been termed from various perspectives as nonequilibrium Ising models, stochastic spin systems, time Markov processes, probabilistic cellular automata, and interacting particle systems. They are characterized by a dynamical rule that is designed *ad hoc* to produce certain emergent properties, or to simulate the action of external agents, etc. A main question concerns universality, and the dependence of nonequilibrium critical behavior on the details of the dynamical rule [2,3]. Therefore, determining the range of validity of the finding that the ordinary Ising fixed point is locally stable under *small amounts* of irreversibility under certain conditions [4] is intriguing. A few nonequilibrium systems have been reported to agree with this expectation [5–8]. Other cases [2,3,9,10] are more difficult to classify from this point of view, however. This may reveal the existence of other stable fixed points having their own domain of attraction, which is not excluded by the perturbative argument in Ref. [4]; such a possibility has been worked out for $d > 1$ for a certain dynamical rule [11].

We report some exact and numerical results on the nonequilibrium ordering in a class of systems whose dynamical rule includes several different symmetries. Besides critical properties, we have analyzed the cluster distribution in relation to some expectations [12–14]. No evidence is found within the class of systems investigated of any departure from the ordinary Ising critical behavior. But we find what, in a sense, is a surprise: as compared to the equilibrium case, the asymptotic region for scaling extends to a wider range of parameter values for nonequilibrium systems. Furthermore, we address a controversy about the properties of clusters of aligned spins (a description of previous studies of the equilibrium cluster distribution is beyond the scope here: for details we refer to Sec. I of Ref. [15], and to Refs. [16–19], for instance).

II. DEFINITION OF THE MODEL

Consider a regular lattice, e.g., the d-dimensional simple cubic lattice $\mathbf{Z}^d$, with two-state (*spin*) variables, $s_{\mathbf{r}} = \pm 1$, $\mathbf{r} \in \mathbf{Z}^d$. The dynamical rule that induces time changes of the configuration $\mathbf{s} = \{s_{\mathbf{r}}\}$ consists of single *spin-flip* processes, i.e., $s_{\mathbf{r}} \to -s_{\mathbf{r}}$. They characterize a time Markov process that is defined by the master equation

$$\frac{\partial P(\mathbf{s};t)}{\partial t} = \sum_{\mathbf{s}^{\mathbf{r}}} \left[c(\mathbf{s} \leftarrow \mathbf{s}^{\mathbf{r}}) P(\mathbf{s}^{\mathbf{r}};t) - c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s}) P(\mathbf{s};t) \right] \quad (1)$$

for the probability $P(\mathbf{s};t)$ of $\mathbf{s}$ at time t. Here $\mathbf{s}^{\mathbf{r}}$ denotes $\mathbf{s}$ after the indicated flip at $\mathbf{r}$, and the probability (per unit time) for this transition is

$$c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s}) = 1 + a \sum_k q_k \operatorname{sgn}[s_{\mathbf{r}}(\eta_{\mathbf{r}} + \lambda_k)] + b \sum_k q_k \operatorname{sgn}^2[s_{\mathbf{r}}(\eta_{\mathbf{r}} + \lambda_k)] . \quad (2)$$

Here

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$$\eta_{\mathbf{r}} \equiv \frac{1}{2} \sum_{\{\mathbf{r}';|\mathbf{r}-\mathbf{r}'|=1\}} s_{\mathbf{r}'} , \tag{3}$$

where the sum is over the nearest-neighbor (NN) sites of $\mathbf{r}$,

$$\operatorname{sgn}(X) = \begin{cases} 0 & \text{for } X = 0, \\ +1 & \text{for } X > 0, \\ -1 & \text{for } X < 0, \end{cases} \tag{4}$$

and a, b, J, λ_k and q_k, with $k = 1, 2, \ldots$, are system parameters.

A main motivation for (2) is as follows. Consider the simpler rule

$$c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s}) = 1 + a \operatorname{sgn}(s_{\mathbf{r}}\eta_{\mathbf{r}}) + b \operatorname{sgn}^2(s_{\mathbf{r}}\eta_{\mathbf{r}}) . \tag{5}$$

This generalizes several cases in the literature. That is, $a = b = -\frac{1}{2}$ corresponds to the zero-temperature limit of the algorithm by Metropolis *et al.* [20], and $a = -1$, $b = 0$ reduces (5) to the zero-temperature limit of the case introduced by Kawasaki [21]; cf. Table I. Both satisfy detailed balance, namely, $c(\mathbf{s} \leftarrow \mathbf{s}^{\mathbf{r}}) \exp[-H(\mathbf{s}^{\mathbf{r}}) / k_B T] = c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s}) \exp[-H(\mathbf{s}) / k_B T]$ with $H(\mathbf{s}) = -J \sum_{\{\mathbf{r},\mathbf{r}';|\mathbf{r}-\mathbf{r}'|=1\}} s_{\mathbf{r}} s_{\mathbf{r}'}$ (where the sum is over all pairs of NN sites in $\mathbf{Z}^d$). Consequently, the two cases imply a tendency of $P(\mathbf{s};t)$ as $t \to \infty$ toward the canonical equilibrium state for $T \to 0$ and energy $H(\mathbf{s})$. The detailed balance property does not hold in general, so that no canonical steady state is reached asymptotically; e.g., for *majority vote processes* that are characterized by either $a = b = (2p-1)/2(1-p)$ with $p \neq 0,1$ [22], $a = p - \frac{1}{2}$ and $b = -\frac{1}{2}$ [7], or $a = 2p - 1$ ($p \neq 0, 1$) and $b = 0$ [23,24]; cf. Table I. One may interpret (2) as a natural generalization of (5) in the sense that the former ensues from the latter family after considering the action of *fields* h_k ($\equiv 2J\lambda_k$), and averaging with distribution

$$g(\lambda) = \sum_k q_k \delta(\lambda - \lambda_k) , \tag{6}$$

where δ is Dirac's delta function. Therefore, (2) is a superposition of different (competing) dynamical rules, which is characterized by the parameters h_k and q_k. Thus any case of (2) will, in general, drive the system to nonequilibrium steady states (i.e., steady states that cannot be described by any simple Hamiltonian) even if the corresponding (5) is canonical. In fact, the state at $\mathbf{r}$ is then not only influenced by its neighbors but also by the external agent, which is represented by the random variable $\lambda = h/2J$ of distribution $g(\lambda)$ that cannot be included in the Hamiltonian (except for a few cases that we consider in Sec. III).

One may think of (2) as the dynamical rule corresponding to a magnet that is under an external field; the value of the latter varies with time sufficiently fast and completely at random within the set $\{h_k\}$ of possible values. We deal below with the zero-temperature limit of this for $d \geq 1$; the behavior with temperature and q_k for $d = 1$ has been reported elsewhere [25]. The present study may thus be of some relevance to the understanding of cluster properties in magnetic systems with random fields [26], and one may imagine other situations, e.g., in the social sciences that are represented by (2). Nevertheless, our interest is rather associated to the fact that (2) is general enough to exhibit several symmetries for different values of the parameters λ_k, q_k, a, and b.

TABLE I. Rates $c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s})$ of the form (5) that characterize several stochastic lattice systems, namely, the zero-temperature limit of two familiar cases, and three majority vote processes, as defined in the references indicated. The present work concerns systems whose dynamical rule is a combination of this sort of rates, as defined in (2).

Rate for indicated value of $s_{\mathbf{r}}\eta_r$			
> 0	$=0$	< 0	Ref.
0	1	1	[20]
0	1	2	[21]
$p(1-p)^{-1}$	1	1	[22]
p	1	$1-p$	[7]
$2p$	1	$2(1-p)$	[23,24]

III. SOME ANALYTICAL RESULTS

Let us consider explicitly the distribution

$$g(\lambda) = \frac{q}{2}\delta(\lambda - \lambda_0) + \frac{q}{2}\delta(\lambda + \lambda_0) + (1-q)\delta(\lambda) . \tag{7}$$

That is (using the magnetic language), the applied field does not induce net magnetization but takes one of the values $\pm\lambda_0$ or 0 at random with the indicated probabilities. In order to obtain explicit analytical results, we consider first a few extraordinary cases [23] for which the asymptotic solution of (1) is

$$P_{\mathrm{st}}(\mathbf{s}) \propto e^{-E(\mathbf{s})} \equiv \exp\left[K_e \sum_{\{\mathbf{r},\mathbf{r}';|\mathbf{r}-\mathbf{r}'|=1\}} s_{\mathbf{r}} s_{\mathbf{r}'} + \mu_e \sum_{\mathbf{r}} s_{\mathbf{r}}\right] \tag{8}$$

and the symmetry $c(\mathbf{s} \leftarrow \mathbf{s}^{\mathbf{r}}) \exp[-E(\mathbf{s}^{\mathbf{r}})] = c(\mathbf{s}^{\mathbf{r}} \leftarrow \mathbf{s}) \exp[-E(\mathbf{s})]$ holds. This is far from trivial, e.g., an effective temperature cannot exist in general for $K_e \neq 0$ *and* $\mu_e \neq 0$. However, one would expect full nonequilibrium behavior only for cases lacking this sort of *quasicanonical* solution. In any case, the consideration of steady states that have this symmetry is also interesting [2].

The rate (2) reduces for (7) and $d = 1$ to

$$c\left(\mathbf{s}^{\mathbf{r}}\leftarrow \mathbf{s}\right)=\begin{cases}1+as_{\mathbf{r}}\eta_{\mathbf{r}}+b\left[q+(1-q)\,\eta_{\mathbf{r}}^{2}\right] & \text{for } 0<\lambda_0<1,\\ 1+as_{\mathbf{r}}\eta_{\mathbf{r}}\left(1-\frac{q}{2}\right)+b\left[q-\frac{q}{2}\eta_{\mathbf{r}}+(1-q)\,\eta_{\mathbf{r}}^{2}\right] & \text{for } \lambda_0=1,\\ 1+as_{\mathbf{r}}\eta_{\mathbf{r}}\left(1-q\right)+b\left[q+(1-q)\,\eta_{\mathbf{r}}^{2}\right] & \text{for } \lambda_0>1.\end{cases} \tag{9}$$

This has the structure

$$c\left(\mathbf{s}^{\mathbf{r}}\leftarrow \mathbf{s}\right)=\alpha_0+\alpha_1 s_{\mathbf{r}}\eta_{\mathbf{r}}+\alpha_2\eta_{\mathbf{r}}+\alpha_3\eta_{\mathbf{r}}^{2}. \tag{10}$$

One finds then after some algebra that (1) has the simple solution $E\left(\mathbf{s}\right)=K_e\sum_{\{\mathbf{r},\mathbf{r}';|\mathbf{r}-\mathbf{r}'|=1\}}s_{\mathbf{r}}s_{\mathbf{r}'}$, i.e., $\mu_e=0$ in (8) if and only if $\alpha_2=0$. The case $\alpha_2=0$ in (10) happens to correspond in (9) to either $b=0$ (the situation considered in [21,23,24]) for which we obtain

$$\tanh\left(2K_e\right)=\begin{cases}-a & \text{for } 0<\lambda_0<1,\\ -a\left(1-\frac{q}{2}\right) & \text{for } \lambda_0=1,\\ -a\left(1-q\right) & \text{for } \lambda_0>1,\end{cases} \tag{11}$$

or else $b\neq 0$ with $\lambda_0\neq 1$ (the case in [7,20,22]) for which we obtain

$$\tanh\left(2K_e\right)=\begin{cases}-\dfrac{a}{1+b} & \text{for } 0<\lambda_0<1,\\ -\dfrac{a\left(1-q\right)}{1+b} & \text{for } \lambda_0>1\,.\end{cases} \tag{12}$$

This result suggests that one investigate further (for $d=1$) the case $b\neq 0$ with $\lambda_0=1$, which might correspond to the more complex behavior.

A similar study for $d>1$ indicates that no solution with the symmetry (8) exists in general but for a few very special cases whose physical interest is quite limited within the present context. That is, (8) holds for $d=2$ for any of the canonical versions of (5) with (7) such that either (i) $q=0$ so that no dynamical competition occurs, and one obtains $K_e=\infty$ (this is the trivial case that corresponds to the rates by Kawasaki or Metropolis, for instance, for $T\to 0$), or (ii) $\lambda_0>2$ and $q=\frac{1}{2}$, which describes an even competition of $\pm 2J\lambda_0$ fields, and one obtains that $K_e=0$ (this corresponds for $d=1$ to the limit $T\to 0$ of a situation studied in detail earlier [25]). Furthermore, one also finds for $d=2$ and (iii) $a=-1$ and $b=0$ with $\lambda_0=2$ and $q=2-\sqrt{2}$ that an effective temperature of $K_e\simeq 0.22$ exists.

Some additional information of interest may be obtained from a necessary property of the rates [22]. That is, the condition

$$\Delta\equiv\frac{1}{2^N}\left[\sum_{\mathbf{r}}c\left(\mathbf{s}^{\mathbf{r}}\leftarrow \mathbf{s}\right)-\sum_{\Omega}|p_\Omega|\right]>0, \tag{13}$$

where $p_\Omega=-\sum_{\mathbf{r}}\left[\prod_{\mathbf{r}\in\Omega}s_{\mathbf{r}}\right]c\left(\mathbf{s}^{\mathbf{r}}\leftarrow \mathbf{s}\right)$ with Ω any set of spins, guarantee stability of the system. A comparison of this with mean-field solutions in a different problem has revealed that (13) sometimes provides a relatively accurate method to bound the relevant region in which the system may exhibit a phase transition [27].

Some consequences of (13) are illustrated in Table II. In agreement with the results above, the one-dimensional system is predicted to evolve with time toward states with only one phase for any $q\,(>0)$ and λ_0. This applies also to the case $\lambda_0=1$, which we have shown above lacks the symmetry (8). For $d>1$, (13) reveals that regions of the phase diagram exist for $\lambda_0\geq d$ in which $\Delta\leq 0$ so that the system might undergo a phase transition.

We remark also that the rate (9) reduces for $\lambda_0>d$ to

$$\begin{aligned}c\left(\mathbf{s}^{\mathbf{r}}\leftarrow \mathbf{s}\right)&=1+a\left(1-q\right)\operatorname{sgn}\left(s_{\mathbf{r}}\eta_{\mathbf{r}}\right)\\&\quad+b\left[q+(1-q)\operatorname{sgn}^2\eta_{\mathbf{r}}\right].\end{aligned} \tag{14}$$

This is similar to a rate used before [28] to represent a local competition of temperature $T=\infty$ with probability q and temperature $T=0$ with probability $1-q$. As this case is relatively well understood, our analysis in this section creates special interest in studying the case $\lambda_0=d>1$, for which no information is available yet. That is, one might expect the latter to exhibit interesting, full nonequilibrium behavior.

TABLE II. The values for q that produce a change of sign of Δ for different cases, as indicated (see the main text in Sec. IV for a definition of the two dynamical rules mentioned here). A phase transition may only occur for q smaller than the value shown for each case. If no value is reported here stability is guaranteed for any q if $d=1$, while one has $\Delta<0$ if $d>1$.

	K rule			*M rule*		
	$\lambda_0<d$	$\lambda_0=d$	$\lambda_0>d$	$\lambda_0<d$	$\lambda_0=d$	$\lambda_0>d$
$d=1$						
$d=2$		0.50	0.50		0.56	0.64±0.02
$d=3$		0.69±0.02	0.71±0.02		0.79±0.01	0.81±0.02

IV. NUMERICAL RESULTS

Consequently with the above, we have analyzed in detail $\lambda_0 = d = 2$ by the Monte Carlo (MC) method. In particular, we have considered the cases $a = b = -\frac{1}{2}$ (denoted hereafter as *M rule*) and $a = -1$ and $b = 0$ (denoted hereafter as *K rule*). Results refer to $L \times L$ lattices with $L = 128$ for the *K rule*, and $L = 16, 32, 64, 128$ for the *M rule*. No systematic finite-size scaling analysis was performed for the former rule given the similarity of results obtained for the two cases for $L = 128$; that is, we believe that the qualitative results reported here apply to both *M* and *K* rules. The stationary regime involves between 2×10^5 and 10^6 MC steps (per site) with data collected every 200 MC steps. We have monitored the long-ranged order parameter or *magnetization*, $m = N^{-1} \langle \sum_{\mathbf{r}} s_{\mathbf{r}} \rangle$, where $\langle \cdots \rangle$ represents the average over configurations, the short-ranged order parameter or *energy*, that we define $e = \frac{1}{2} N^{-1} \left\langle \sum_{\{\mathbf{r},\mathbf{r}';|\mathbf{r}-\mathbf{r}'|=1\}} s_{\mathbf{r}} s_{\mathbf{r}'} \right\rangle$, its fluctuations, and $C = \partial e / \partial q$.

A main conclusion is that, independently of the dynamical rule, the system exhibits a second-order phase transition with a critical point of the $(2-d)$ Ising variety at q_C. This is illustrated in Figs. 1 and 2. The values for q_C that have been obtained by extrapolating a linear fit m^8 vs q are 0.146 ± 0.002 (which is confirmed in Fig. 2 by finite-size scaling analysis) for the *M rule*, and 0.215 ± 0.005 for the *K rule*. An interesting observation that is made evident in the figures is that the asymptotic critical region is wider than for the Ising model. That is, $m \sim \epsilon^{1/8}$, with $\epsilon \equiv (q - q_C) q_C^{-1}$, holds here for finite lattices more generally, namely, from zero up to a value of $|\epsilon|$, which is around 0.8 for the *M rule* and 0.5 for the *K rule* while this is around 0.1 for the Onsager case [with $\epsilon \equiv (T - T_C) T_C^{-1}$ instead] for the system sizes considered; this is illustrated in Fig. 1(b). It may be an indication of the respective range of validity near the critical point of homogeneity of the correlation function.

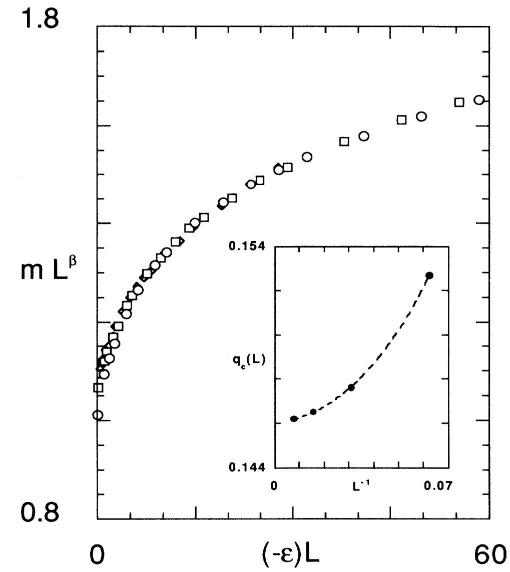

FIG. 2. Finite-size scaling of data for the *M rule* with $\beta = 1/8$, $\nu = 1$, and $q_C = 0.146$ for $L \times L$ lattices with $L = 32$ (◇), 64 (□), and 128 (○). The inset is a plot of $q_C(L)$ defined as the location of the maximum in the temperature derivative of the *specific heat* $C = \partial e / \partial q$. The dashed curve corresponds to the best fit given in Eq. (17).

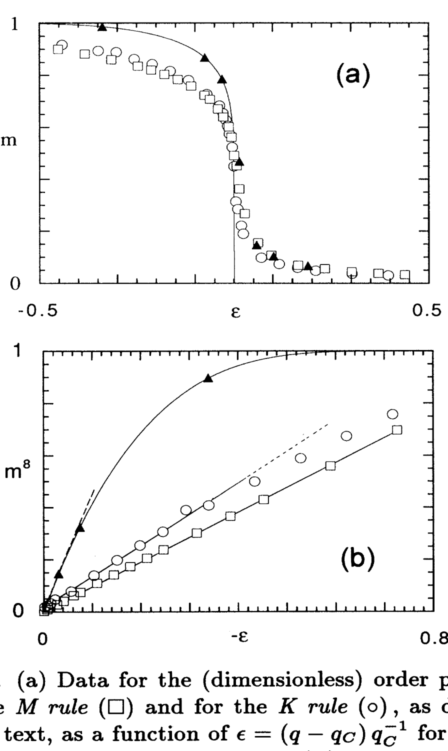

FIG. 1. (a) Data for the (dimensionless) order parameter m for the *M rule* (□) and for the *K rule* (○), as defined in the main text, as a function of $\epsilon = (q - q_C) q_C^{-1}$ for 128×128 lattices, and for the equilibrium case (▲) for 64 × 64 lattices with $\epsilon = (T - T_C) T_C^{-1}$; the solid line is the Onsager exact solution. (b) The same data and exact solution plotted $m^{1/\beta}$ vs ϵ for $\beta = 1/8$, the Onsager value. Linear fits to the data near $\epsilon = 0$ are also indicated.

It would be interesting to have a detailed explanation of the fact that one needs to approach the critical point much more closely in equilibrium than in nonequilibrium in order to enter the Ising regime. Therefore we have investigated also some of the cluster properties below criticality trying to obtain information about crossover phenomena. In addition, it may help to determine more precisely the relation between the above systems, e.g., one would like to figure out in this way any possible topological differences between equilibrium and nonequilibrium configurations. Furthermore, our data are good enough to analyze previous proposals with confidence [15–19].

Consider the cluster size distribution, $p(\ell)$, and mean surface *energy*, $s(\ell)$. Here ℓ represents the number of spins aligned in a given direction that have at least another NN aligned spin, and s is the number of NN broken bonds associated with the boundary of the cluster averaged for each value of ℓ. (In addition, our data for p and s involve the steady-state average indicated above.) With proper normalization (one needs to assume also that no *anomalous*, e.g., percolating, clusters occur), one has that

$m(\epsilon) = 1 - 2\sum_{\ell=1}^{\infty} \ell p(\ell)$, and $e(\epsilon) = 2\sum_{\ell=1}^{\infty} s(\ell)\, p(\ell)$, which one expects to behave as $m(\epsilon) \approx \epsilon^{\beta}\tilde{m}$, and $e(\epsilon) \approx e_0(\epsilon) + \epsilon^{1-\alpha}\tilde{e}$, where e_0 is the regular part, sufficiently near q_C. Then one may interpret the latter as a consequence of the following asymptotic behavior:

$$p(\ell) \approx \ell^{-\tau}\tilde{p}(\epsilon\ell^{y}), \qquad s(\ell) \approx \ell^{\sigma}\tilde{s}(\epsilon\ell^{y}) \tag{15}$$

for sufficiently large ℓ and small ϵ. Here, τ, y, and σ are the exponents that characterize the cluster distribution, and the argument requires that $\sigma = 1 - y(\phi - 1)$, $\alpha + \beta + \phi = 2$, and $\beta = (\tau - 2)\, y^{-1}$. Cambier and Nauenberg [15] have studied these scaling forms for the Ising model in two (and three) dimensions by MC simulations below the critical temperature. They report a narrow asymptotic region in which these properties seem to hold and, consequently, observe deviations from scaling and temperature dependence of exponents. Figures 3–5 describe a situation in which (15) is confirmed more definitely and generally *for the same definition of clusters.* This seems related to the fact that our data statistics are good, and also to a peculiarity of the systems (1) and (2), e.g., no percolating clusters have hampered our analysis slightly below q_C (unlike in the Ising case); of course, an *infinite* cluster develops *at* q_C.

The cluster distribution has been studied in two dimensions for the equilibrium case, and for the nonequilibrium case denoted above as *M rule.* Figure 3 illustrates $s(\ell)$ for the two systems. This gives $\sigma \simeq 0.68$. The graphs in Fig. 3 reveal again that the nonequilibrium data are better described by theory than the equilibrium data (in spite of the fact that both refer to comparable simulations with similar statistics). In particular, no percolating clusters have been observed during the nonequilibrium steady state up to $q = q_C$ (while the behavior observed for the nonequilibrium case is quite similar to the equilibrium data, at and above the corresponding critical value, q_C and T_C, respectively). The same ensues from the study of the cluster size distribution in Fig. 4. The associated fitting parameter follows as $\tau \simeq 2.05$ for the nonequilibrium system. We have obtained roughly the same value for equilibrium but it is remarkable that, as reported before, only data within the temperature range $2.15 < T < 2.25$ have to be considered to avoid the counting then of too large clusters that do not follow the same behavior as the rest.

A stringent test may be obtained from the study of the moment $\sum_{\ell} \ell^{\tau} p(\ell)$. For example, one may try to scale the data according to the function

$$F(\epsilon n^{y}) \equiv \epsilon^{1/y} \sum_{\ell=1}^{n} \ell^{\tau} p(\ell;\epsilon) \tag{16}$$

with proper values for the exponents y and τ. Figure 5 illustrates the result. The best values we have obtained for the parameters in this way are $\tau = 2.054 \pm 0.005$ and $y = 0.44 \pm 0.01$. These values describe both equilibrium and nonequilibrium data.

Finally, we mention that the size dependence of q_C

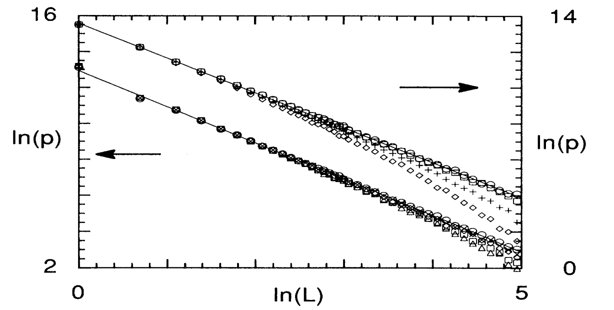

FIG. 4. Log-log plot of $P(\ell)$, i.e., total number of clusters of size ℓ. The lower set corresponds to the *M rule* for $-\epsilon = 0.013\,(\circ)$, $0.027\,(\square)$, $0.041\,(\triangle)$, $0.062\,(*)$, and $0.075\,(\times)$. The upper set corresponds to the equilibrium case for $-\epsilon = 0.004\,(\circ)$, $0.009\,(\square)$, $0.030\,(+)$, and $0.052\,(\diamond)$. The best fit here gives $\tau = 2.05$.

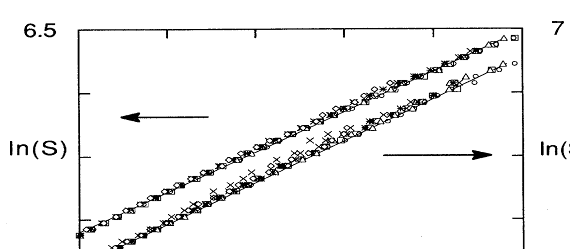

FIG. 3. Log-log plot of $S(\ell)$ defined as the total *surface* of all the clusters of size ℓ for $20 < \ell < 300$. The upper set corresponds to the *M rule* for $-\epsilon = 0.013\,(\circ)$, $0.027\,(\square)$, $0.041\,(\triangle)$, $0.062\,(*)$, $0.075\,(\times)$, and $0.144\,(\diamond)$. The lower set corresponds to the equilibrium case for $-\epsilon = 0.009\,(\circ)$, $0.022\,(\square)$, $0.030\,(\triangle)$, $0.039\,(*)$, and $0.052\,(\times)$. The best fit here gives $\sigma = 0.68$.

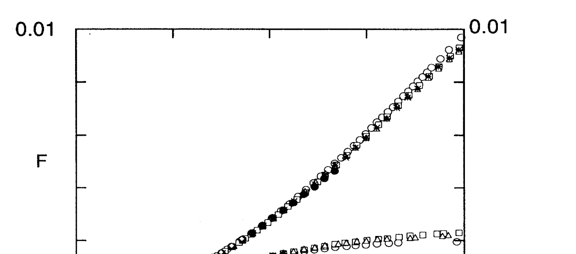

FIG. 5. Scaling plot of the (dimensionless) function $F(x)$ in (16) for both equilibrium (lower curve) and nonequilibrium (upper curve) clusters for $-\epsilon = 0.052\,(\circ)$, $0.048\,(\square)$, $0.039\,(\triangle)$, $0.030\,(*)$, $0.026\,(\bullet)$, $0.021\,(\diamond)$, $0.017\,(\times)$, and $0.012\,(+)$, and $-\epsilon = 0.144\,(\circ)$, $0.110\,(\square)$, $0.075\,(\triangle)$, $0.062\,(*)$, $0.041\,(\bullet)$, $0.027\,(\diamond)$, $0.014\,(\times)$, and $0.007\,(+)$, respectively.

conforms to

$$\frac{q_C(L)}{q_C(\infty)} - 1 = AL^{-1}\left(1 + BL^{-\omega}\right), \tag{17}$$

and we have obtained $A = 0.142\,(5)$, $B = 184.5\,(1)$, $\omega = 1.367\,(7)$, and $q_C(\infty) = 0.146\,(0)$. (A simpler *effective* fit is $q_C = 0.14607 + 1.6924L^{-2}$.) This reflects the importance for the nonequilibrium system of the second-order contribution beyond the expected term $\sim L^{-\nu}$, $\nu = 1$; cf. Ref. [29].

V. DISCUSSION

We have studied the class of d-dimensional Ising-like systems defined by (1)–(4) which, in general, exhibit nonequilibrium steady states. A well-defined relation exists between these systems and some familiar ones in the literature; furthermore, an experimental realization has been suggested for some of them. We have identified a subclass of these systems having the quasicanonical solution (8) in terms of an *effective Hamiltonian*. For $d = 1$, the systems in this subclass correspond to (10) with $\alpha_2 = 0$, and the steady state is particularly simple, namely, it is characterized by zero *effective field*, i.e., $\mu_e = 0$, and by an *effective temperature* given by (11) or (12). Otherwise, the situation is expected to be more complex; then one may use (13), for example, to guess about details of the phase diagram.

The analytical study of (1)–(4) has stimulated us to study numerically two particular two-dimensional cases, both corresponding to $\lambda_0 = d = 2$ in (7), that are good candidates for complex properties. In spite of their correspondence to a full nonequilibrium situation, the MC experiments performed depict a situation that is perhaps simpler than expected (but interesting and a bit surprising). That is, two different dynamical rules lead to different steady states [e.g., phase transitions occur for $q = q_C = 0.146\,(0)$ and $0.21\,(5)$, respectively] that have the same critical properties. The latter seem to be precisely of the Onsager type that characterizes the equilibrium case (for $d = 2$). However, the data suggest that the *region of influence* of the $2-d$ Ising critical point is larger for the nonequilibrium systems (and larger for one of the nonequilibrium cases—denoted above as *M rule*— than for the other). This is puzzling; it reflects some essential difference in the nature of correlations between the two cases. In particular, it seems that the divergence of the correlation length occurs much closer to the critical point for the nonequilibrium system than in equilibrium. This also produces a peculiar size dependence of the critical parameter, $q_C(L)$.

The detailed study of the cluster distribution confirms the above. In addition, we have obtained clear evidence (for both equilibrium and nonequilibrium cases) for the scaling proposal (15) and (16), and give accurate estimates of the parameters involved for $d = 2$ that turn out to be model independent. In particular, we find $\sigma = 0.68 \pm 0.04$, $\tau = 2.054 \pm 0.005$, and $y = 0.44 \pm 0.01$ (to be compared with the estimates in Ref. [15] of $\tau = 2.05 \pm 0.03$ and $y = 0.40 \pm 0.02$ for the equilibrium case). The value of β that follows then from the scaling law $\beta = (\tau - 2)\,y^{-1}$ is close to the equilibrium value of $\frac{1}{8}$. As indicated in Fig. 5, the scaling function in (16) varies essentially from equilibrium to nonequilibrium conditions. Finally, also remarkable is the fact that the nonequilibrium system (unlike the equilibrium counterpart) does not exhibit the large and very ramified, say, percolating clusters (of aligned spins) just below criticality that have previously systematically hampered accurate confirmation of scaling hypothesis (which has motivated introducing alternate definitions of a *cluster* in previous studies of lattice systems).

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