Ising critical behavior of a non-Hamiltonian lattice system

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(Received 21 March 1994)

We study steady states in d-dimensional lattice systems that evolve in time by a probabilistic majority rule, which corresponds to the zero-temperature limit of a system with conflicting dynamics. The rule satisfies detailed balance for $d=1$ but not for $d>1$. We find numerically nonequilibrium critical points of the Ising class for $d=2$ and 3.

PACS number(s): 05.20.−y, 05.50.+q, 64.60.Cn

Impure models have been proposed that incorporate diffusion of disorder of the kind that takes place in, e.g., dilute metallic alloys when magnetic ions diffuse [1–3]. They are kinetic Ising-like models evolving with time by a dynamical rule, which is a stochastic superposition of conflicting Hamiltonian evolutions (each of which would by itself ultimately lead to a canonical distribution). In particular, a *nonequilibrium spin-glass model* (NSGM) has recently been introduced [4] that simulates fast random diffusion of (nearest-neighbor, NN) exchange interactions [1]. A conflict arises in this system because whenever a spin flip is attempted, its NN exchange interactions are all set equal to $+J$ or $-J$ with probabilities p and $1-p$, respectively, independently of spin configurations. Some exact results have been obtained for dimension $d=1$ [4], and (rather limited) mean-field [5] and Monte Carlo (MC) [6] descriptions are available for $d>1$.

Previous studies [4–6] have indicated that the steady states of the NSGM depend on the details of the non-Hamiltonian dynamical rule, which exhibits their noncanonical nature. So far, only a rule which combines two Metropolis rates [7], one for each sign of J, has been used in MC experiments. This is an exceptionally simple case in the sense that it satisfies detailed balance for $d=1$ (but not for $d>1$). More specifically, it admits an Ising-like Hamiltonian for $d=1$, namely, $H(\mathbf{s})=K\sum_x s_x s_{x+1}$, where $\mathbf{s}=\{s_x=\pm1; x\in\mathbb{Z}^1\}$ and K has a complicated dependence on p and J, and on the temperature T of the relevant heat bath. Consequently, this one-dimensional case is equivalent to the ordinary Ising model at some *effective temperature* [4]. The study of this simple version of NSGM by the MC method for $d=2$ and 3 (i.e., where detailed balance does not hold) reveals the existence of steady states of the ferromagnetic (antiferromagnetic) type below a d-dependent line of critical points, $T_C(p)$, when p is large (small) enough. No counterpart to the *freezing phenomena* (i.e., extremely slow relaxation) found in quenched spin-glass models occurs, however. Therefore, the NSGM is amenable to numerical analysis, which has enabled us to obtain some information on its critical behavior near $T_C(p)$. The latter seems to be of the Ising type, in general, although a departure from this has been previously reported, which might indicate crossover phenomena at a low enough temperature [6]. (The available mean-field approximation is not sufficient to provide any direct clue on this, but it reveals, for example, that a modification of the rule may transform continuous into discontinuous phase transitions for some values of p [5].)

This situation has motivated us to study systematically for $T\to0$ the simple version of the NSGM described above. In that limit, the dynamical rule of the NSGM becomes simpler [4]. Time evolution may then be implemented by choosing an arbitrary (e.g., random) initial configuration and performing the following MC step reiteratively: First, a site $\mathbf{r}$ of the *simple-cubic* lattice $\mathbb{Z}^d$ is selected at random. Then, one attempts the *flip* $s_\mathbf{r}\to -s_\mathbf{r}$. Let

$$\epsilon_\mathbf{r}\equiv s_\mathbf{r}\sum_{|\mathbf{r}-\mathbf{r}'|=1} s_{\mathbf{r}'}\ , \tag{1}$$

where the sum is over all NN of $\mathbf{r}$; the flip takes place with probability

$$c(s_r\to -s_r|\mathbf{s})=\begin{cases} p & \text{if } \epsilon_r<0\\ 1 & \text{if } \epsilon_r=0\\ 1-p & \text{if } \epsilon_r>0\ . \end{cases} \tag{2}$$

The numerical study of (2) is also motivated by the fact

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that it corresponds to a *majority vote* rule. Two variations of it have been considered before. Whereas the probability that s_r flips when the neighboring configuration has an equal number of plus and minus signs is 1 here, it is $1-p$ and $\frac{1}{2}$ for majority-vote models in Liggett [8] and in Garrido and Marro [9], respectively, the three models have the same transition probability otherwise. It is easy to check that, just as for the NSGM, detailed balance is satisfied in these cases for $d=1$ but not for $d>1$. de Oliveira [10] has studied numerically (for $d=2$) a variation of (2) in which $c(s_r \to -s_r|\mathbf{s})=\frac{1}{2}$ for $\epsilon_r=0$. Unfortunately, no precise criteria are known to predict the effect of the details of the rule on critical behavior and other steady-state properties of the system [11–14].

Our MC results for periodic boundary-conditions are summarized in Figs. 1–4, for lattices of 128^2 and 64^3 sites (all steady-state averages involve approximately 10^6 MC steps per lattice site after equilibration). The system exhibits a sharp phase transition at $p_C(d)$ in $d=2$ and $d=3$, where $p_C(2)\approx 0.928$ and $p_C(3)\approx 0.835$; the former value agrees with the one obtained by de Oliveira [10]. Note that the phase diagram $T_C(p)$ must be symmetrical about $p=\frac{1}{2}$, given that such a symmetry is contained in the dynamical rule; consequently, a similar phenomenon must occur at $1-p_C(d)$. The transition is clearly of

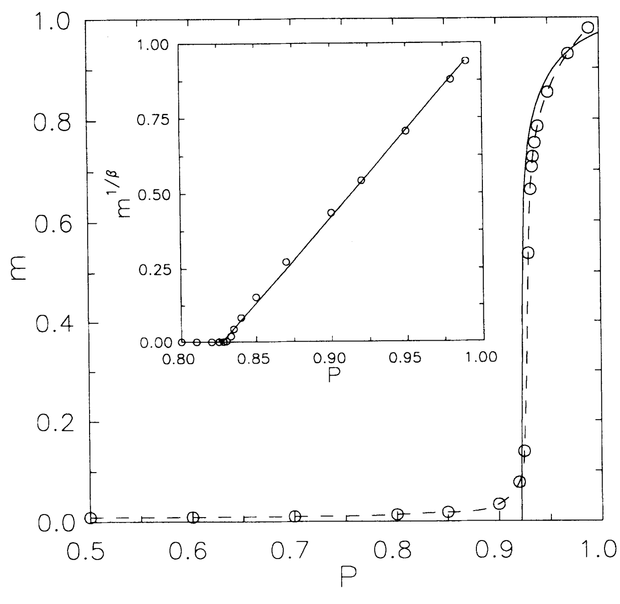

FIG. 1. The mean value of the order parameter (i.e., the *magnetization*) $m \equiv N^{-1}\sum_r s_r$, where N is the number of lattice sites, for $d=2$ as a function of the parameter p in (2). The solid in the main graph corresponds to the Onsager solution of the Ising model for $N\to\infty$ shifted (arbitrarily) by $p=aT+b$ ($a=-0.133$, $b=1.23$). The inset is a plot of $\langle m\rangle^{1/\beta}$, with $\beta=\frac{5}{16}$, versus p for $d=3$. The data within the latter have approximately linear behavior (solid line) that extrapolates to a value for p_C, which is consistent with the rest of the data; varying the value of β by 7% or more produces a curvature of the data points that may be noticed even by direct inspection in a similar plot (and the data cannot be extrapolated then to the value of p_C estimated by other means).

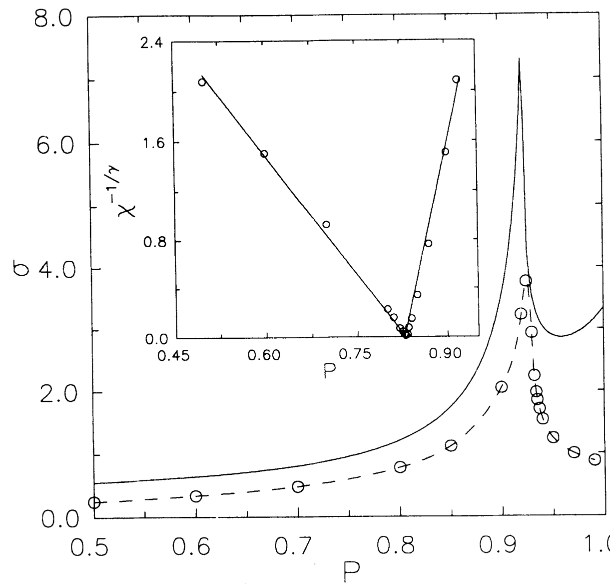

FIG. 2. The short-range order parameter, $\sigma \equiv \langle N_{++}N_{--}(N_{+-})^{-2}\rangle$ (N_{++},N_{--}, and N_{+-} are the number of particle-particle, hole-hole, and particle-hole NN pairs, respectively, and $\langle\cdots\rangle$ stands for the Monte Carlo average), as a function of p for $d=2$. The solid line in the main graph is the Ising result (shifted as in Fig. 1) estimated from the values for the magnetization and energy assuming $\sigma=\langle N_{++}N_{--}\rangle\langle N_{+-}\rangle^{-2}$. The inset is a plot of $\chi^{-1/\gamma}$ versus p for $d=3$; here, $\chi\equiv\langle(m-\langle m\rangle)^2\rangle$, and the assumption $\gamma=\frac{5}{4}$ is made to obtain linear behavior on both sides of the estimated value of p_C.

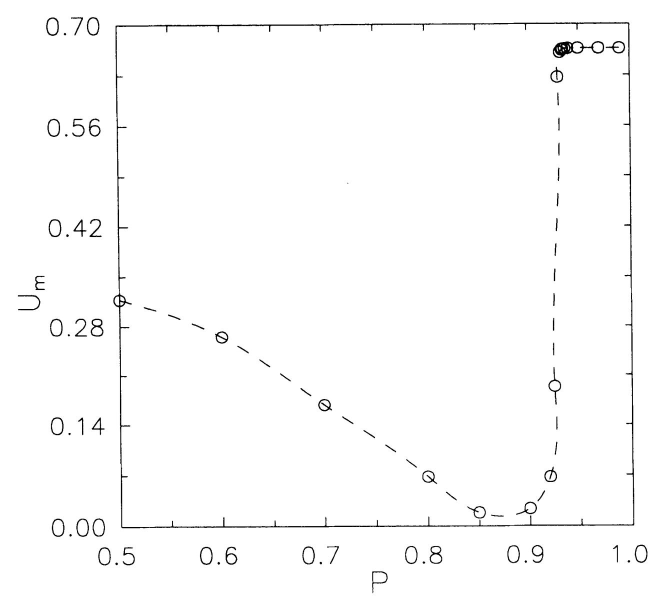

FIG. 3. The cumulant $U_m=1-\frac{1}{3}\langle m^4\rangle/\langle m^2\rangle^2$ for $d=2$. One has that $U_m\approx\frac{2}{3}$ for $p>p_C$, as for the pure Ising model at low temperature, while a more intriguing distribution of m values is revealed by the data for $p<p_C$; cf. the main text.

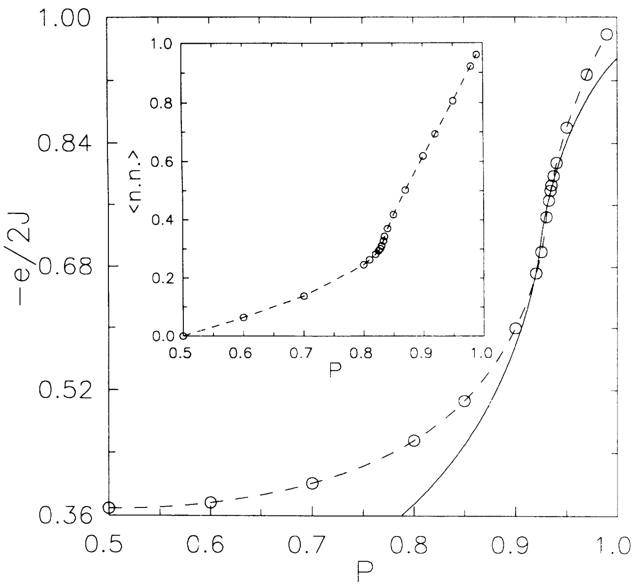

FIG. 4. The mean number of particle-particle NN pairs as a function of p for $d=2$ (main graph) and 3 (inset). The solid line in the main graph is the Onsager result (shifted as for Fig. 1).

second order. In addition, inspection of the neighborhood of p_C also reveals that the critical exponents for both square and simple-cubic lattices have the same values (within statistical error) as the corresponding (thermal) ones for the pure Ising model. Figures 1 and 2 support this statement: The inset in Fig. 1 shows a plot of $m^{1/\beta}$ (for $\beta=0.31$) versus p for $d=3$. Our data points deviate significantly from straight lines when plotted for values of β that differ from 0.31 by more than 0.02. Similar plots (not shown) of our data for $d=2$ are consistent with $\beta=\frac{1}{8}$. The inset of Fig. 2 shows data points for χ [defined as $\langle (m-\langle m\rangle)^2\rangle$], obtained for $d=3$, plotted as $\chi^{1/\gamma}$ versus p for $\gamma=1.25$. Only variations of more than 7% of the latter value of γ lead to significant deviations of our data points from straight-line behavior. The same procedure leads to a value of $\gamma=1.75$ when applied to our data points for $d=2$. Therefore, the values we have obtained numerically for both β and γ in $d=2$ and $d=3$ are consistent with the critical indexes β and γ of the pure Ising model. That is, (2) leads to Ising critical behavior. We remark that errors in our data are exceptionally small, e.g., we have obtained γ as accurately as β. This is because (i) there are no thermal fluctuations, and because it turns out that (ii) all the relevant quantities evolve monotonically and smoothly with time and that (iii) finite-size rounding-off effects are small for the sizes studied (see Fig. 1, for instance). These facts mark an important qualitative difference between rate (2) and, e.g., the Metropolis one for the ordinary Ising model.

Figure 3 shows our data points for the cumulant $U_m \equiv 1-\frac{1}{3}\langle m^4\rangle/\langle m^2\rangle^2$. It follows straightforwardly for macroscopic systems from the definition of U_m that $U_m=\frac{2}{3}$ if $\langle m\rangle \neq 0$ and the system is in a single phase, which we indeed obtain for $p>p_C(\approx 0.93)$. On the other hand, $\langle m\rangle=0$ for $p<p_C$. It does *not* then follow that $U_m=\frac{2}{3}$ for macroscopic systems. A normal distribution of m implies $U_m=0$, since $\langle m^4\rangle=3\langle m^2\rangle^2$ then. Our data points are therefore *not* consistent with a normal distribution of m, in general. This behavior, reminiscent of spin glasses, is, quite likely, related to the disordered nature of the system.

Figure 4 suggests a singularity in the mean number of particle-particle NN pairs at p_C, and a very rapid increase for $p>p_C$; p_C marks the onset of the ordered phase. The curves agree well very near p_C with the equilibrium thermal (e.g., Onsager) ones after scaling of the temperature variable of the Onsager case according to $p=aT+b$ ($a=-0.133$ and $b=1.23$), but the agreement gets systematically worse as one goes away from p_C. That is, it seems that there is agreement in their singular behavior at the critical point, but the behavior that ensues from rule (2) differs, in general, from the thermal behavior of the Ising model for both $d=2$ and $d=3$. Inspection of Fig. 2 for $p>p_C$ leads to the same conclusion. We have found that the slope of the curves in Fig. 4, for specific-heat-like quantities, agrees (within statistical error) with the corresponding mean-square fluctuations; thus, Einstein's fluctuation-dissipation relation holds; analysis of the corresponding critical behavior gives some consistency with $\alpha=0$.

Summing up, (2) induces (steady nonequilibrium) ordering. The associated critical phenomena are of the Ising variety for any $1\leq d\leq 3$, even though the underlying mechanism differs fundamentally from the one in the pure Ising model. For example, no simple relationship exists between (2) and the familiar rules of Metropolis or Glauber [7], and relevant quantities exhibit a smooth monotonic behavior as a function of time, which contrasts with the relatively noisy one in ordinary MC experiments. Nevertheless, our findings are consistent with some expectations: A perturbative treatment [15] suggests that the original Ising critical point is stable (even though additional critical points might become dominant [1]) under *small amounts* of irreversibility for any model with short-range interactions respecting the symmetry of the lattice and exhibiting symmetry under spin inversion. This seems to be the case for majority-vote rules such as (2) and for the simple version of the NSGM mentioned above. Ising behavior has been reported also for a model of chemical reactions with desorption [16], for a model of the immune system [17], and for a reaction-diffusion lattice gas [18], for example. The model investigated here is not microscopically equivalent to the Ising model; it seems, however, that their critical behavior does not differ, although their macroscopic behavior does so in general for $d>1$. One would like to know the precise generality of this result. For instance, it is not general enough to cover the case in which one adds a random spin-flip process to a majority rule [13], which has been reported [14] to induce Landau-classical critical behavior for $d=3$.

This work was supported by the DGICYT (Project No. PB91-0709) and PAI (Junta de Andalucia) of Spain by CONICIT and FUNDAYACUCHO of Venezuela, and by the Comisión de las Comunidades Europeas (Contract No. C11.0409).

[1] P. L. Garrido and J. Marro, J. Stat. Phys. **74**, 663 (1994).
[2] A. I. López-Lacomba and J. Marro, Europhys. Lett. **25**, 169 (1994).
[3] A. I. López-Lacomba and J. Marro, J. Phys. A **27**, 1111 (1994).
[4] P. L. Garrido and J. Marro, Europhys. Lett. **15**, 375 (1991).
[5] J. J. Alonso and J. Marro, Phys. Rev. B **45**, 10 408 (1992).
[6] J. M. González-Miranda, A. Labarta, M. Puma, J. F. Fernández, P. L. Garrido, and J. Marro, Phys. Rev. E **49**, 2041 (1994).
[7] Cf. Ref. [1], for instance.
[8] T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985); R. Durrett, *Particle Systems and Percolation* (Wadsworth, Beltmont, 1988), and references therein.
[9] P. L. Garrido and J. Marro, Phys. Rev. Lett. **62**, 1229 (1989).
[10] M. J. de Oliveira, J. Stat. Phys. **66**, 273 (1992).
[11] R. Dickman, in *Physics Computing '92,* edited by R. A. de Groot and J. Nadrchal (World Scientific, Singapore, 1993), p. 45.
[12] A. I. López-Lacomba and J. Marro, Phys. Rev. B **46**, 8244 (1992).
[13] J. L. Lebowitz and H. Saleur, Physica **138A**, 194 (1986).
[14] J.-S. Wang and J. L. Lebowitz, J. Stat. Phys. **51**, 893 (1988), and references therein.
[15] G. Grinstein, C. Jayaprakash, and Y. He, Phys. Rev. Lett. **55**, 2527 (1985).
[16] T. Tomé and R. Dickman, Phys. Rev. E **47**, 948 (1993).
[17] D. Stauffer and M. Sahimi, in *Physics Computing '92* (Ref. [11]), p. 169.
[18] J. J. Alonso, J. Marro, and J. M. González-Miranda, Phys. Rev. E **47**, 885 (1993).