EUROPHYSICS LETTERS 15 June 1991
Europhys. Lett., 15 (4), pp. 375-380 (1991)

A Nonequilibrium Version of the Spin-Glass Problem (*).

P. L. GARRIDO (**) and J. MARRO
Departamentos de Física Moderna y de Física Aplicada, Facultad de Ciencias Universidad de Granada - E-18071 Granada, España

(received 14 February 1991; accepted in final form 12 April 1991)

PACS. 05.20 – Statistical mechanics.
PACS. 05.50 – Lattice theory and statistics: Ising problems.
PACS. 75.10N – Spin-glass models.

Abstract. – We present exact results for Ising-like model systems in which a competing kinetic process induces the presence of steady states dominated by a kind of *dynamical* randomness and frustration. The models, which may be relevant to the spin-slass problem, undergo interesting critical phenomena. Unlike more familiar cases, the latter and other properties of steady states reveal nonuniversal behaviour.

The unusual macroscopic behaviour first discovered in diluted metallic alloys such as CuMn with only a few percent of magnetic Mn ions [1] was soon attributed to the structural impossibility (*frustration*) to satisfy all the interactions at low temperature due to microscopic disorder [2, 3]. Despite much effort along that line, however, exact results remain scarce and, consequently, the present understanding of spin-glasses is not totally satisfactory [4]. In particular, no solvable microscopic model is admitted to capture all the essential features of ideal spin-glasses, and full agreement on the meaning of macroscopic *spin-glass behaviour* seems to be lacking. The most outstanding result from the recent study of the spin-glass problem is perhaps the recognition that this may be a nonequilibrium problem, *e.g.*, frustration may avoid a system to reach a single (equilibrium) steady state and cause the macroscopic behaviour to be determined by dynamics.

We here present Ising-like systems in which a competing kinetic process may induce the presence of nonequilibrium steady states which are dominated by a kind of randomness. This is relevant to the general study of nonequilibrium steady states, phase transitions and critical phenomena, and it represents an effort trying to clarify the possible influence of dynamics on spin-glass behaviour. Our kinetically disordered systems are solved for one-

(*) Partially supported by Dirección General de Investigación Científica y Técnica, Project PB88-0487, Plan Andaluz de Investigación (Junta de Andalucía), and Commission of the European Communities.
(**) Partially supported by NSF grant DMR86-12369.

dimensional lattices under certain conditions; even though those situations are apparently canonical, dynamical frustration induces a behaviour which essentially differs from the familiar one in equilibrium phenomena. We also present exact results for two- and three-dimensional lattices, where the system has full nonequilibrium behaviour, and a simple representation of dynamics at zero temperature for arbitrary dimension.

Consider a d-dimensional lattice Ω which is in contact with a thermal bath at temperature T. The probability of any configuration $\boldsymbol{s} \equiv \{s_x = \pm 1; \boldsymbol{x} \in \Omega\}$ at time t satisfies the usual Markovian master equation [5, 6]:

$$\mathrm{d}P(\boldsymbol{s}; t)/\mathrm{d}t = \sum_{x} [w(\boldsymbol{s}^x; \boldsymbol{x}) P(\boldsymbol{s}^x; t) - w(\boldsymbol{s}; \boldsymbol{x}) P(\boldsymbol{s}; t)]. \tag{1}$$

This describes stochastic changes $s_x \to -s_x$ of the (spin) variable at site $\boldsymbol{x}$ which generate a new configuration $\boldsymbol{s}^x$ from $\boldsymbol{s}$ with probability $w(\boldsymbol{s}; \boldsymbol{x})$ per unit time. Unlike in the familiar Glauber case [5], however, microscopic dynamics here involve n competing spin-flip or Glauber mechanisms. That is

$$w(\boldsymbol{s}; \boldsymbol{x}) = \sum_i p_i w_i(\boldsymbol{s}; \boldsymbol{x}), \qquad \sum_i p_i = 1 \qquad (i = 1, \ldots, n), \tag{2}$$

where, for simplicity, w_i is assumed to satisfy a kind of individual detailed balance condition, namely that

$$w_i(\boldsymbol{s}; \boldsymbol{x}) = w_i(\boldsymbol{s}^x; \boldsymbol{x}) \exp[-\delta H_i], \qquad \delta H_i \equiv H_i(\boldsymbol{s}^x) - H_i(\boldsymbol{s}), \tag{3}$$

with respect to some specific «Hamiltonian», *e.g.*,

$$k_B T \cdot H_i(\boldsymbol{s}) = -J_i \sum_{\text{n.n.}} s_x s_{x'} - h \sum_x s_x. \tag{4}$$

The family of models we have just defined has a simple interpretation. Each Glauber mechanism acts with probability p_i at each kinetic step, as if the interaction strengths had a given value J_i chosen at random from some distribution $p(J)$. This is precisely the kinetic Ising model with nonconserved magnetization [5], excluding the fact that the coupling constant in the «Hamiltonian» changes randomly at each step according to $p(J)$. When $p(J) = \delta(J - \text{const})$, any spin-flip rate satisfying (3) drives the system to the Gibbs (equilibrium) state corresponding to temperature T and energy $H(\boldsymbol{s}) = H_i(\boldsymbol{s})$ whose nature is well known, *e.g.*, the system with $h = 0$ undergoes a continuous phase transition at critical temperature $T_c \geq 0$ for $d \geq 1$, respectively. Thus, the model has two well-defined limits for $p(J) = \delta(J \pm J_0)$, $J_0 > 0$, corresponding, respectively, to the familiar antiferromagnetic and ferromagnetic cases. The interest here is on the crossover between those two limits, a case where the (kinetic) competition between J's will in general drive asymptotically the system towards a nonequilibrium steady state, as if it were acted on by some external (non-Hamiltonian) agent. One would like to comprehend the dependence of that state on $p(J)$, T, h and $w(\boldsymbol{s}; \boldsymbol{x})$, and the possible relevance of that situation to understanding some peculiarities of disordered systems. Concerning the latter objective, one may note that, given the local nature of Glauber kinetic processes, one may interpret either that the interaction strength is changed at each step to take the same value J_i all over the system, or else that it is only changed to J_i around the involved spin(s); moreover, except for energy fluctuations, thermodynamics is the same for those two interpretations. A little thought then suggests that the latter case asymptotically produces a situation *at each* t which is identical in a sense to the random spatial quenched distribution of J's in the model by

Edwards and Anderson [2]. This is interesting because one may argue that some of the reported unusual observations in real disordered materials might also be related to the diffusion of disorder (*e.g.*, due to atomic migration); thus, one may conceive the existence of *dynamical frustration* in the real world, which is an effect contained in our model, to be denoted *nonequilibrium spin-glass model* (NSGM) in the following. Notice also that our system has some other obvious realizations, *i.e.* one may consider as well the competition between several T's, h's or, in general, a Hamiltonian distribution $p(H)$ with an expression for H more realistic than (4) [7]; although those cases are amenable to a treatment which is formally close to the one in this letter, they correspond to other physical situations, behave quite differently, and consequently will be described separately.

Illustrative of that family of models is the simple case of competing J's with $h=0$ and $d=1$. In addition, we shall only consider rates $w_i(\boldsymbol{s};\boldsymbol{x})=\varphi(\delta H)$, where $\varphi(X)$ is an arbitrary function with properties $\varphi(X)=e^{-X}\varphi(-X)$, $\varphi(0)=1$, and $\varphi(X)\to 0$ as $X\to\infty$. Actually, most familiar choices in different problems correspond to that with $\varphi(X)=1-\mathrm{tgh}((1/2)X)$ [5, 8], $\varphi(X)=\min(1,e^{-X})$ [9] or $\varphi(X)=e^{-(1/2)X}$ [6]. The resulting system is still interesting and rather general, and may be the goal of some concepts and theorems developed before [10, 11]. That is, the stationary solution of (1) or limit of $P(\boldsymbol{s};t)$ as $t\to\infty$ may in principle be written as $P^{\mathrm{st}}(\boldsymbol{s})=Z^{-1}\exp[-E(\boldsymbol{s})]$, $Z\equiv\sum_{\boldsymbol{s}}\exp[-E(\boldsymbol{s})]$, where $E(\boldsymbol{s})$ will in fact play the role of an *effective Hamiltonian*, roughly as far as it involves only a finite number of items. The simplest situation occurs when the latter and the effective rate (2) are related by $w(\boldsymbol{s};\boldsymbol{x})\exp[-E(\boldsymbol{s})]=w(\boldsymbol{s}^x;\boldsymbol{x})\exp[-E(\boldsymbol{s}^x)]$. Previous theorems [10] then allow to conclude that, under the conditions enumerated in this paragraph, one simply has $E(\boldsymbol{s})=-K_{\mathrm{e}}\sum_x s_x s_{x+1}$ for the NSGM. The effective coupling constant is, however, rather complex: $K_{\mathrm{e}}=-(1/4)\ln[\langle\!\langle\varphi(4K)\rangle\!\rangle\langle\!\langle\varphi(-4K)\rangle\!\rangle^{-1}]$, where $K\equiv J/k_{\mathrm{B}}T$ and $\langle\!\langle\cdot\rangle\!\rangle$ represents an average with the (normalized) distribution $p(J)$.

Though it is clear that one may conceive more involved and perhaps interesting situations by varying the choices for $w_i(\boldsymbol{s};\boldsymbol{x})$ and/or $H_i(\boldsymbol{s})$, the above simple case is far from trivial. It may be noticed, in particular, that the annealed version of the Edwards-Anderson [2] spin-glass model, where impurities have reached equilibrium with the other degrees of freedom instead of remaining frozen in, may be characterized by an effective Hamiltonian $K_{\mathrm{e}}=-(1/2)\ln[\langle\!\langle e^{-K}\rangle\!\rangle\langle\!\langle e^{K}\rangle\!\rangle^{-1}]$. Consequently, there is a similarity between the annealed and NSGM systems only in some particular cases when $d=1$ and $h=0$. Namely, the two expressions for K_{e} are alike for all $p(J)$ when $\varphi(X)=e^{-(1/2)X}$, and for all $\varphi(X)$ when $p(J)=(1-q)\,\delta(J-J_0)+q\delta(J+J_0)$. Even then, however, those two systems differ essentially in some respects. Consider, for instance, the latter case with $J_0>0$. It follows for the NSGM that $K_{\mathrm{e}}=-(1/4)\ln\{[1-q+qe^{4K_0}]/[q+(1-q)\,e^{4K_0}]\}$, $K_0\equiv J_0/k_{\mathrm{B}}T$, independent of dynamics (contrary to the usual situation far from equilibrium [7], by the way). Moreover, $T\to\infty$ leads to $K_{\mathrm{e}}\approx K_0(1-2q)$, and $T\to 0$ produces

$$K_{\mathrm{e}}\approx(1/4)\ln[(1-q)/q]-(1/4)[(1-2q)/q(1-q)]\,e^{-4K_0},$$

which remains finite. The latter indicates that the pure (*i.e.* the one for $q=0$ or 1) critical point is washed out by the additional disorder. Other system properties follow from $u\equiv\mathrm{tgh}\,K_{\mathrm{e}}$. For instance, the correlation function is given by $\langle s_0 s_x\rangle=e^{-r/\xi}=u^x$. This implies in particular that the correlation length ξ remains finite at $T=0$ except for $q\to 0$ where it diverges according to $\xi\approx Aq^{-1/2}$. The susceptibility $\chi=\sum_n\langle s_0 s_n\rangle=(1+u)/(1-u)$ is also diverging at T, $q\to 0$ with exponent $\gamma_q=1/2$. That is, $\nu_q=\gamma_q=1/2$ instead of 1 as in the annealed version. In the case of our first interpretation of the NSGM given two paragraphs above, the energy density is naturally defined as $e \equiv -\langle\langle J \rangle\rangle u = (2q-1) J_0 u$, and one has for its temperature derivative and squared mean fluctuations, respectively, that $\partial e/\partial T = k_B(2q-1) K_0(1-u^2)(\partial K_e/\partial K_0)$ and $\langle e^2 \rangle - \langle e \rangle^2 = J_0^2(1-u^2)N^{-1} + 4J_0^2 q(1-q)u^2$, where N represents the volume. This uncovers the presence of specific fluctuations for the NSGM having a finite relative magnitude in the thermodynamic limit. Consequently, there is no fluctuation-dissipation theorem. Of course, one might define instead $C_H \equiv \partial e/\partial T_e$ with $T_e \equiv \langle\langle J \rangle\rangle (k_B K_e)^{-1}$, but this is not appropriate here, *e.g.*, such «specific heat» cannot be interpreted as a response function. Our conclusions remain qualitatively the same when dealing with the second interpretation of the NSGM, except for the definition of the energy and magnitude of its fluctuations, as one may easily infer by himself. One may also note that the cluster distribution is $P_n = 2^{-(n+1)}(1-u)^2(1+u)^{n-1}$, $\sum_n P_n = (1/2)(1-u)$, and that the mean cluster size is $\langle n \rangle = 2/(1-u)$.

To illustrate the role played in general by dynamics in this problem, one may consider any distribution such that $p(J) = g(J - J_0)$ with $g(X) = g(-X)$. Again, $\varphi(X) = e^{-(1/2)X}$ reveals itself singular: this is the only rate in the family we have considered for which any distribution having that symmetry produces $K_e = J_0/k_B T$. (As a trivial corollary, $K_e = 0$ for such dynamics when $p(J)$ is symmetrical around zero, *i.e.* $J_0 = 0$.) It also follows, for instance, that the steady properties when $\varphi(X) = e^{-(1/2)X}$ and the distribution is Gaussian, *i.e.* $p(J) \propto \exp[-(J-J_0)^2/2\sigma^2]$, are independent of the parameter σ.

The above cases suggest looking for more striking behaviour when $d = 1$ and $h = 0$ by considering, for instance, $p(J) = [|J_2|/(J_1 + |J_2|)]\,\delta(J - J_1) + [J_1/(J_1 + |J_2|)]\,\delta(J - J_2)$, $J_2 < 0$. This is a relatively general, nontrivial case with $\langle\langle J \rangle\rangle = 0$. It follows at high T that $K_e \approx -8(1/3)[1/2 - 3\varphi''(0)]\mu(1-\mu)(J_1/k_B T)^3 + O(J_1/k_B T)^4$, where $\mu \equiv |J_2|/J_1 \in [0, \infty]$. The maximum value of $|K_e|$ occurs for $\mu = 1/2$, while $\mu = 0, 1$ correspond to the completely disordered systems with $p(J) = \delta(J)$ and $p(J) = (1/2)\,\delta(J - J_1) + (1/2)\,\delta(J + J_1)$, respectively, considered before. In any case, the behaviour at high T is inspired by the curvature of φ near the origin, and the effective temperature appears proportional to T^3, in contrast, respectively, to universality with respect to φ, and to linear dependence, both found above for $p(J) = (1-q)\,\delta(J - J_0) + q\delta(J + J_0)$. When $T \to 0$, it is interesting to distinguish *case* i) *rates* characterized by the property $\varphi(-X) \to a$ as $X \to \infty$, where it follows that $K_e \approx (1/4)\ln\mu$, and *case* ii) *rates* for which $\varphi(-X) \to e^{\pi X}$, $\pi < 1$, as $X \to \infty$, where $K_e \approx \pi(1-\mu) J_1/k_B T + (1/4)\ln\mu$; $\varphi(X)$ is assumed to be differentiable at the origin (in which the Metropolis rate [9] is excluded). Those cases essentially differ from each other: in case i), there is no zero-T critical point, and the effective ground state is ferromagnetic (antiferromagnetic) when the antiferromagnetic (ferromagnetic) interaction is stronger than the ferromagnetic (antiferromagnetic) one, $\mu > 1$ ($\mu < 1$). Even when one allows, say, that $|J_2| \to \infty$, the situation remains the same because it then decreases the probability of J_2. That is, the probability coefficient dominates the value of the impurity strength in determining the state. In case ii), on the contrary, there is a critical point at $T = 0$, and the ground state is antiferromagnetic (ferromagnetic) for $\mu > 1$ ($\mu < 1$). That is, the impurity strength now plays the dominant role. Notice also that it simply follows from the expression for K_e above that for both rates of type i) with $2\varphi''(0) > 1/3$ or rates of type ii) with $2\varphi''(0) < 1/3$, K_e changes sign at some finite T revealing a (continuous!) changeover between ferro and antiferromagnetic macroscopic behaviours. The relevance of the dynamics shows up again when considering the Metropolis rates $\varphi(X) = \min(1, e^{-X})$. This produces $K_e = -(1/4)\ln\{[1 + \mu\exp[-4J_1/k_B T]][\mu + \mu\exp[-4J_1\mu/k_B T]]^{-1}\}$ implying again a novel behaviour as $T \to \infty$, namely the effective temperature becomes proportional to T^2, while it reduces to case i) above when $T \to 0$.

The critical behaviour for case ii) is very interesting. Thermal critical exponents (defined for $T \to T_c$) differ from the pure ones at equilibrium in a factor $(1-\mu)\pi$, *e.g.*, one obtains here $\nu = 2(1-\mu)\pi$ and $\alpha = 2(1-\mu)\pi$. This is surprising: critical exponents not only depend on asymptotic features of dynamics, such as π, which one might have expected would here play a role similar to the one by the Hamiltonian in familiar problems, but they also depend on an apparently irrelevant parameter such as μ. This is a rare example where one can prove that universality is lost. It seems that the specific system we are analysing has a relevant symmetry parameter, μ, such that $\mu = 1$ corresponds to $p(J)$ with a kind of symmetry, while $\mu \neq 1$ denounces lack of that symmetry, and that fact is unambiguously reflected by the critical exponents which are zero for $\mu = 1$, while they have a continuum of *impure* values when the symmetry is broken.

Under the same conditions as for $d = 1$ above, the NSGM with $d \geqslant 2$ has no effective Hamiltonian [11]. We may still bound the temperature locating a phase transition, however. Concerning, for instance, case $h = 0$ with $\varphi(X) = \min(1, e^{-X})$ and $p(J) = (1-q)\,\delta(J-J_0) + q\delta(J+J_0)$ with $J_0 > 0$, one may prove from the positivity property of transition rates [6] the existence for $d = 2$ of a unique phase at any temperature when $7 \leqslant 16q \leqslant 9$; also, one may guarantee that for $9/16 \leqslant q \leqslant 1$ and $0 \leqslant q \leqslant 7/16$ there exists a unique phase as far as $T > T_0$, where $T_0^{-1} \equiv -(1/4)\ln\{(2/3)(2|2q-1|-1) + (1/3)[16(2q-1)^2 - 40|2q-1| + 1]^{1/2}\}$. Consequently, any sharp phase transition may only occur when $T < T_0$ as far as $7 > 16q$ or $16q > 9$. The case $d = 3$ presents no similar gap: for any q, there exists $T_0 \neq 0$ which is the solution of $(1-4|2q-1)\alpha^3 + 30\alpha^2 + (15 + 60|2q-1|)\alpha - 14 - 56|2q-1|$, $\alpha \equiv \exp[-2T_0]$. It also seems valuable to note that the above version of the NSGM may be represented at $T = 0$ for any d by a simple random cellular automaton which is close to the so-called voter model [6]: $w(s; x) = q$, $1-q$ and 1 for $s_x \sum_{\text{n.n.}} s_y > 0$, < 0 or $= 0$, respectively, where the sum is over all n.n. of site x. It then follows, in particular, that the system at $T = 0$ is always ergodic when $d = 1$, is ergodic for $7 \leqslant 16q \leqslant 9$ when $d = 2$, and we cannot claim when $d = 3$ of any region where it is necessarily ergodic.

Summing up, the NSGM defined by way of eqs. (1)-(4) presents a variety of nonequilibrium steady states and critical phenomena. This is illustrated here by solving the simple case $h = 0$, $d = 1$ and $w_i(s; x) = \varphi(\delta H)$, when system and external constraint may be represented by a simple effective Hamiltonian (involving an *effective* detailed condition), and by deducting exact upper bounds locating possible phase transitions for $d > 1$. For $d = 1$, when $p(J)$ is the sum of two symmetric delta-functions with $\langle\!\langle J \rangle\!\rangle \neq 0$, there follows the lack of a fluctuation-dissipation theorem due to the presence of some excess fluctuations, and the divergence of the correlation length and susceptibility as $T \to 0$ *and* $q \to 0$ which is characterized by universal critical exponents, namely $\nu = \gamma = 1/2$. When $p(J)$ has zero mean, however, even the behaviour at high T depends on the details of transition rates. Moreover, there is then a family of rates for which the zero-T critical point has exponents depending both on the asymptotic features of dynamics and the details of $p(J)$. Such nonuniversal behaviour for $d = 1$ seems basically a consequence of the *dynamical* frustration involved by the model. For $d > 1$, this will add up to other nonequilibrium effects; a very rich phase diagram should thus be expected. Finally, the study here for $d \geqslant 1$ indicates that further studies of the NSGM by computer simulations and other techniques may indeed help the understanding both of disordered systems and nonequilibrium phase transitions and critical phenomena.

* * *

We acknowledge useful discussions with J. J. ALONSO, J. F. FERNÁNDEZ and A. I. LÓPEZ-LACOMBA.

REFERENCES

[1] CANNELLA V. and MYDOSH J. A., *Phys. Rev. B*, 6 (1972) 4220.

[2] EDWARDS S. F. and ANDERSON P. W., *J. Phys. F*, 5 (1975) 965; 6 (1976) 1927.

[3] TOULOUSE G., *Commun. Phys.*, 2 (1977) 115; PARISI G., *Phys. Rep.*, 67 (1980) 97.

[4] See, for instance, MÉZARD M., PARISI G. and VIRASORO M. A., *Spin Glass Theory and Beyond* (World Scientific, Singapore) 1987.

[5] GLAUBER R. J., *J. Math. Phys.*, 4 (1963) 294.

[6] LIGGET T. M., *Interacting Particle Systems* (Springer-Verlag, New York, N.Y.) 1985.

[7] MARRO J. and GARRIDO P. L., in *Statistical Mechanics of Neural Networks*, edited by L. GARRIDO (Springer-Verlag, Berlin) 1990.

[8] KAWASAKI K., in *Phase Transitions and Critical Phenomena*, Vol. 4, edited by C. DOMB and M. S. GREEN (Academic Press, London) 1972.

[9] METROPOLIS N., ROSENBLUTH A. W., ROSENBLUTH N. M., TELLER A. H. and TELLER E., *J. Chem. Phys.*, 21 (1953) 1087.

[10] LÓPEZ-LACOMBA A. I., GARRIDO P. L. and MARRO J., *J. Phys. A*, 23 (1990) 3809.

[11] GARRIDO P. L. and MARRO J., *Phys. Rev. Lett.*, 62 (1989) 1929.