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Statistical Mechanics of Neural Networks

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CONTENTS

KINETICALLY DISORDERED LATTICE SYSTEMS[1]

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1.- Introduction and some definitions.

This lecture deals with a series of Ising-like model systems in which a relatively complex kinetic process induces the presence of non-equilibrium steady states which are characterized, even dominated by a kind of (dynamic) disorder. It thus results a situation having some connection with the case of more familiar disordered systems [see, for instance, Ziman 1979, Chowdhury 1987], e.g. spin glasses, random-field systems and diluted magnets; some of our considerations may also bear some relevance in relation to the study of certain cellular automata, and perhaps to that of neural networks, which is the topic of this meeting. The material we are presenting has essentially been published before in two recent papers by Garrido and Marro [1989] and by López-Lacomba, Garrido and Marro [1990], although we shall also describe some unpublished results obtained in collaboration with J.J. Alonso, J.M. González-Miranda and A. Labarta. Further related efforts may be found in recent papers by Künsch [1984], Grinstein et al. [1985], Browne and Kleban [1989] and Droz et al. [1989], for instance.

Consider for simplicity a regular lattice $\Omega=\{\mathbf{r}_i;\ i=1,2,\ldots,N\}$ in a d-dimensional space with spin variables $s_i\equiv s_{\mathbf{r}_i}=\pm 1$ at each lattice site. Denote by $s\equiv\{s_{\mathbf{r}};\mathbf{r}\in\Omega\}$ any of the (2^N) possible configurations, and by $P(s;t)$ the probability of s at time t. The latter evolves in time as implied by a Master Equation, namely

$$(1.1)\quad dP(s;t)/dt = \sum_{s'} c(s|s')P(s';t)$$

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where $c(s|s')$ are the elements of a matrix $\mathbf{c}$ of transition rates per unit time from s' to s. The matrix $\mathbf{c}$ involves a series of competing, local spin-flip mechanisms each producing stochastically, as in the so-called Glauber [1963] dynamics, the change $s_r \to -s_r$ of the variable at a site r. This generates a new configuration s', to be denoted specifically either as s^r or else as s^i, from s with a given probability per unit time, $c(s^r|s)$. That is, we shall assume in the following that

$$(1.2)\quad c(s'|s) = -\sum_{l=1}^{m} \sum_{\mathbf{r}} \prod_{\mathbf{x}\neq\mathbf{r}} \delta_{s_x,s'_x}\, c_{\varphi_l}(s^r|s)\, s_r s'_r ,$$

where $r\in\Omega$, and φ_l represents the value of a given parameter such as temperature, chemical potential, magnetic field, sign or strength of interactions, or any combination of these, etc.

The asymptotic behavior of $P(s;t)$ as implied by (1.1) and a given kinetic process (1.2) may be very varied in principle. This may be investigated by solving each specific model [see, for instance, Glauber 1963 and Garrido et al. 1987]. It sounds however more interesting *a priori*, in an effort to construct a theory of nonequilibrium phenomena for instance, to introduce the concept of an appropriate effective Hamiltonian [cf. Garrido and Marro 1989]. That will be our main approach here.

Let us denote by $P^{st}(s)$ the stationary solution or limit of $P(s;t)$ as $t\to\infty$, and assume this is positively defined for all s. When P^{st} exists, one may introduce an object $E(s)$ such that

$$(1.3)\quad P^{st}(s) = Z^{-1}\exp(-E(s)) ,\quad Z \equiv \sum_{s} \exp(-E(s)) .$$

It follows that $E(s)$ is analytic, and one may write quite generally that

$$(1.4)\quad E(s) = \sum_{k=1}^{N} \sum'_{(i_1\ldots i_k)} J^{(k)}_{i_1\ldots i_k}\, s_{i_1}\ldots s_{i_k}$$

where Σ' sums over every set of k lattice sites in the system. We shall only be interested in the following on objects $E(s)$ such that

$$(1.5)\quad J^{(k)}_{i_1\ldots i_k} = 0 \quad \text{for all} \quad k\geq k_o$$

where k_o is independent of N, at least for $N>N_o$. When a unique stationary distribution function P^{st} exists such that (1.3)-(1.5) hold, the resulting object $E(s)$ will be termed the effective Hamiltonian of the system. While that short-ranged object $E(s)$ will differ essentially from the actual Hamiltonian of the system under consideration, it in fact represents an effective Hamiltonian in the sense of equations (1.3), e.g. it may be used to study nonequilibrium stationary states and phase transitions by applying standard methods of equilibrium statistical mechanics.

The conceptually simplest realization of the general model defined above are the kinetic Ising models studied by Glauber [1963] and others. Those systems, which correspond to having $m=1$ in Eq. (1.2), consist of any lattice Ω whose configurations, assuming for the moment that there is no external magnetic field or chemical potential, have a potential or configurational energy given by

$$(1.6)\quad H(s) = -J \sum_{nn} s_{\mathbf{r}_1} s_{\mathbf{r}_2} ,$$

where the sum is over nearest neighbour (nn) pairs of sites. Moreover, the lattice is in contact with a thermal bath at temperature T which induces changes in s. Namely, the bath provokes spin-flips with a prescribed rate depending on the change of the energy (1.6) which would cause the flip:

$$(1.7)\quad c(s^r|s) = f_r(s)\exp[-Ks_r \sum_{q} s_{r'}] = f_r(s)\exp[-\tfrac{1}{2}\delta H]$$

where $\delta H\equiv[H(s^r)-H(s)]/k_BT$, $K\equiv J/k_BT$, the sum is over the q nearest neighbours of site r and $f_r(s)=f_r(s^r)$ (>0) as required by the detailed balance condition:

$$(1.8)\quad c(s^r|s)/c(s|s^r) = \exp(-\delta H) .$$

The transition rates used before by Glauber [1963] and others [see, for instance, López-Lacomba et al. 1990] to deal with various problems correspond to different realizations for the function $f_r(s)$. Each of those choices drives the system asymptotically towards the same stationary state, the canonical or Gibbs equilibrium state characterized by the distribution $P^{st}(s)=\text{const.}\exp(-H(s)/k_BT)$ corresponding to the temperature T and to the energy $H(s)$. It follows simply in those (trivial) cases that $E(s)=H(s)/k_BT$, and the system properties may be obtained in principle, and some times also in practice, from the computation of Z in (1.3).

The situation is however more complex, and more interesting and

challenging nowadays, for the dynamics (1.2) with $m \geq 2$. The homogeneous Markov process (1.1) now results from a competition between several stochastic spin-flip mechanisms, each acting as if it were associated to a different value of a given parameter (e.g. corresponding to a different temperature), and occurring at a rate (1.7) satisfying (1.8) which guarantees it would individually drive the system to the corresponding Gibbs equilibrium state. Nevertheless, that competition essentially complicates the system dynamics, e.g. the resulting transition rate (1.2) will not satisfy (1.8) in general and this may induce the presence of nonequilibrium steady states, as occurs when a system is not isolated but acted on by some external agent. Consequently, the existence of P^{st} , and that of an effective Hamiltonian $E(s)$, is so an open question in general.

It may be mentioned that the above concept of an effective Hamiltonian may also be applied to the study of the so-called probabilistic (spin-flip) cellular automata [Lebowitz et al. 1990] which have no configurational energy similar to (1.6) defined, but a certain dynamical process. For instance, the one-dimensional case ($\Omega \equiv \mathbb{Z}$)

$$(1.9) \qquad P(s|s';\delta t) = \prod_{i \in \Omega} \text{const.}[1+s_{r_i} w(s'_{r_i})]$$

where w is any function such that $1 \pm w \geq 0$. We shall not consider that possibility explicitly in the following but refer, for instance, to the paper by Grinstein et al. [1985].

2.- On the existence of an effective Hamiltonian.

The conditions for the existence of an effective Hamiltonian (EH) for the general system defined above were investigated before by Garrido and Marro [1989] and by López-Lacomba et al. [1990]. The main general results in those studies are summarized in this section and in the next one; they provide a framework to study those systems which is simultaneously simple, systematic and powerful.

One may easily prove that, under some mild conditions [see, for instance, Ligget 1985], a system with a finite number of sites, N , and any dynamics c starting from any configuration may reach any other configuration in a finite number of steps. That is, there exists a unique, non-zero stationary probability distribution $P^{st}(s)$, one for each rate, and the system will tend asymptotically to it independently of the initial distribution. Thus, the system admits a unique object $E(s)$, independent of the initial probability distribution, for each choice of transition rates $c(s'|s)$.

Now, in order to solve the problem of computing the coefficients $J^{(k)}$ in $E(s)$ efficiently, it is convenient (see, however, section 4) to consider those cases in which the resulting transition rate (1.2), say

$$(2.1) \qquad c(s'|s) = \sum_{l=1}^{m} p_l \, c_l(s'|s) \ , \qquad \sum_{l=1}^{m} p_l = 1 \ ,$$

satisfies a global detailed balance (GDB). That is, unless otherwise indicated we shall assume that (2.1) satisfies that

$$(2.2) \qquad c(s|s') \, P^{st}(s') = c(s'|s) \, P^{st}(s) \quad \text{for all} \quad s \quad \text{and} \quad s'$$

or $c(s'|s) = f_r(s) \exp(-\frac{1}{2}\delta E)$ where $\delta E \equiv E(s')-E(s)$ and $f_r(s)=f_r(s') > 0$, the latter implying that $f_r(s)$ has no dependence on the value of s_r . When this holds, we just have:

$$(2.3) \qquad \ln[c(s'|s)/c(s|s')] = -\delta E = 2s_r \sum_{k=1}^{N} \sum'_{(r,i_2 \ldots i_k)} J^{(k)}_{r,i_2,\ldots i_k} s_{i_2} \ldots s_{i_k}$$

and the unknowns $J^{(k)}$ follow by identifying coefficients. Moreover, it also follows that the necessary and sufficient condition for any dynamics c to fulfil the GDB condition (2.2) is that any set of k spin variables satisfies

$$(2.4) \qquad \sum_{s} s_{i_1} s_{i_2} \ldots s_{i_j} s_{i_*} \ldots s_{i_k} \ln[c(s^{i_j}|s) c(s|s^{i_*}) / c(s|s^{i_j}) c(s^{i_*}|s)] = 0$$

for all i_j and i_* .

It also follows from above that, in the infinite-volume limit ($N \to \infty$), $E(s)$ may not be unique. Nevertheless, in order for $E(s)$ to be useful it needs to have some short-ranged nature such as the one specified in (1.5). That is, one is only interested in cases in which the number of coefficients $J^{(k)}$ in $E(s)$ remains constant, independent of N , and their expressions are also independent of the system size, and one may show this will certainly occur in many cases of interest. As a matter of fact, there is a large and interesting class of situations having that property when the system dynamics satisfies GDB. Namely, the object $E(s)$ is the EH of the system when the resulting transition rate (1.2) or (2.1) both satisfies the condition (2.2) of GDB and depends only on a constant number of sites that is independent of N.

3.- General application to one-dimensional systems.

The previous facts indeed suggest an efficient formalism to investigate the existence of an EH for one-dimensional systems and, when it exists, to find the corresponding explicit expression [López-Lacomba et al. 1990j]. Consider the system defined in section 1 with d=1 whose dynamics consist of a competition as in (2.1) with

$$(3.1)\quad c_i(s^i|\mathbf{s}) = f_i^{(1)}(\mathbf{s})\,\exp\{-\tfrac{1}{2}[H_i(s^i)-H_i(\mathbf{s})]\}\ .$$

Here, the functions f_i are assumed to be analytic, positive defined and independent of the variable s_i, so that each individual transition rate $c_i(s^i|\mathbf{s})$ satisfies the condition of detailed balance (1.8). It is further assumed that the individual physical Hamiltonians involved by the rates contain a term which corresponds to the action of an external magnetic field, i.e.

$$(3.2)\quad H_i(\mathbf{s}) = -K_1^{(i)}\sum_{i=1}^{N} s_i - K_2^{(i)}\sum_{i=1}^{N} s_i s_{i+1}\ ,$$

and that the functions f_i have the following properties:

(i) They are invariant under the interchange of s_{i-1} and s_{i+1}; consequently, they may be written as

$$(3.3)\quad f_i^{(1)}(\mathbf{s}) = g_0^{(1)}(\mathbf{s}_{i1}) + g_1^{(1)}(\mathbf{s}_{i1})\sigma_1^i + g_2^{(1)}(\mathbf{s}_{i1})\sigma_2^i$$

where $\sigma_1^i \equiv \tfrac{1}{2}(s_{i-1}+s_{i+1})$, $\sigma_2^i \equiv s_{i-1}s_{i+1}$, and $\mathbf{s}_{i1}$ represents the set of occupation variables in $f_i^{(1)}$ excluding s_{i-1} and s_{i+1}.

(ii) They are homogeneous in the sense that the coefficients g in (3.3) are independent of i, as is reflected in our notation.

(iii) They are symmetrical in the sense that $s_{i+m}^1 \in \mathbf{s}_{i1}$, where $m^1 < \tfrac{1}{2}(N-1)$, implies that $s_{i-m}^1 \in \mathbf{s}_{i1}$, and $f_i^{(1)}$ has the same dependence on both, s_{i+m}^1 and s_{i-m}^1.

(iv) Each individual dynamics has a few-body nature in the sense that it does not involve "many" neighbours of site i, i.e. $\max_{\{1\}}\{m^1 | s_{i+m}^1 \in \mathbf{s}_{i1}\} \equiv M << \tfrac{1}{2}(N-1)$ as certainly occurs in the familiar rates.

It follows from the results in the previous section that the object $E(\mathbf{s})$ always exists for the one-dimensional system just defined. In order to show under what conditions GDB is satisfied so that $E(\mathbf{s})$ is the system EH, one first notices that

$$(3.4)\quad \ln[c(\mathbf{s}|\mathbf{s}^1)/c(\mathbf{s}^1|\mathbf{s})] = s_i[D_0(\mathbf{s}_i^*) + 2D_1(\mathbf{s}_i^*)\sigma_1^i + D_2(\mathbf{s}_i^*)\sigma_2^i]$$

where

$$(3.5)\quad D_0(\mathbf{s}_i^*)\equiv\tfrac{1}{4}(b+d+2a)\ ,\quad D_1(\mathbf{s}_i^*)\equiv\tfrac{1}{4}(b-d)\ ,\quad D_2(\mathbf{s}_i^*)\equiv\tfrac{1}{4}(b+d-2a)$$

with

$$(3.6a)\quad a(\mathbf{s}_i^*) \equiv \ln\{\Sigma_i[g_0^{(1)}-g_2^{(1)}]\exp[K_1^{(1)}]/\Sigma_i[g_0^{(1)}-g_2^{(1)}]\exp[-K_1^{(1)}]\}\ ,$$

$$(3.6b)\quad b(\mathbf{s}_i^*) \equiv \ln\left[\frac{\Sigma_i[g_0^{(1)}+g_1^{(1)}+g_2^{(1)}]\exp[K_1^{(1)}+2K_2^{(1)}]}{\Sigma_i[g_0^{(1)}+g_1^{(1)}+g_2^{(1)}]\exp[-K_1^{(1)}-2K_2^{(1)}]}\right]\ ,$$

$$(3.6c)\quad d(\mathbf{s}_i^*) \equiv \ln\left[\frac{\Sigma_i[g_0^{(1)}-g_1^{(1)}+g_2^{(1)}]\exp[K_1^{(1)}-2K_2^{(1)}]}{\Sigma_i[g_0^{(1)}-g_1^{(1)}+g_2^{(1)}]\exp[-K_1^{(1)}+2K_2^{(1)}]}\right]\ ,$$

$l = 1,2,\ldots m$. Notice that $\mathbf{s}_i^*$ represents the set of spin variables appearing in a, b and d, and that $\mathbf{s}_i^*$ may differ from the set $\mathbf{s}_{i1}$; in any case, one still has the properties (i)-(iv) above. Thus, a necessary and sufficient condition in order to have GDB here is that, for all j and $k\neq j$:

$$(3.7)\quad \sum_{\mathbf{s}} s_{i_1}\ldots s_{i_m} G^{j,k}(\mathbf{s}) = 0\ ,\quad i_l\neq j,k,\quad l=1,\ldots,m\ ,\quad m\geq 0\ ,$$

where the functions $G^{j,k}(\mathbf{s}) \equiv s_j s_k \ln[c(\mathbf{s}^j|\mathbf{s})c(\mathbf{s}|\mathbf{s}^k)/c(\mathbf{s}|\mathbf{s}^j)c(\mathbf{s}^k|\mathbf{s})]$ are given for the model in this section as

$$(3.8)\quad G^{j,k}(\mathbf{s}) = s_j[D_0(\mathbf{s}_k^*)+2D_1(\mathbf{s}_k^*)\sigma_1^k+D_2(\mathbf{s}_k^*)\sigma_2^k] - s_k[D_0(\mathbf{s}_j^*)+2D_1(\mathbf{s}_j^*)\sigma_1^j+D_2(\mathbf{s}_j^*)\sigma_2^j]\ .$$

Then, taking k as a nn of j, e.g. $k=j-1$, the result is that the condition of GDB implies that

$$(3.9)\quad D_1(\mathbf{s}_{j-1}^*)+D_2(\mathbf{s}_{j-1}^*)s_{j-2}=D_1(\mathbf{s}_j^*)+D_2(\mathbf{s}_j^*)s_{j+1}\ ,$$

and the only way to satisfy (3.8) is by requiring that

$$(3.10)\qquad D_1(\mathbf{s}_{j-1}^*)=D_1(\mathbf{s}_j^*)=\text{const.}\ ,\quad D_2(\mathbf{s}_{j-1}^*)=D_2(\mathbf{s}_j^*)=0\ .$$

The consequences of those facts are noteworthy: The necessary and sufficient condition for having GBD in the one-dimensional model is that

(3.11) $$\ln[c(\mathbf{s}|\mathbf{s}')/c(\mathbf{s}'|\mathbf{s})] = s_i[D_0(\mathbf{s}_i^*) + 2D_1\sigma_i^z] \ ,$$

where D_1 is a constant and

(3.12) $$D_0(\mathbf{s}_j^*) = \sum_{m\geq 0} \sum_{(i_1\ldots i_n)}{}' A^*_{i_1\ldots i_n} s_{j+i_1}\cdots s_{j+i_n} \ ,$$

and that the coefficients A^* satisfy

(3.13) $$A^*_{i_1\ldots i_1\ldots i_n} = A^*_{i_1-i_1\ldots -i_1\ldots i_n-i_1} \ .$$

Thus, the coefficients in (1.4) for the EH of the one-dimensional system are given by

(3.14) $$J^{(k)}_{i_1\ldots i_1\ldots i_k} = (2^{m+1})^{-1} \sum_{\mathbf{s}} s_{i_1}\ldots s_{i_1}\ldots s_{i_k} \ [D_0(\mathbf{s}^*_{i_1}) + 2D_1\sigma_i^z] \ .$$

For instance, $J^{(2)}_{1,1+1}=-\frac{1}{2}D_1$.

Summing up, when a system satisfies the condition of GDB, a situation which sounds appealing and is also very frequent in practice, there is always a unique short-ranged object $E(\mathbf{s})$, and we found the necessary and sufficient conditions for GDB to hold when $d=1$. Namely, a one-dimensional system has an EH which is given by (3.14) if and only if D_0 fulfils (3.12) and (3.13), D_1 is a constant, and D_2 is zero. Moreover, $D_2=0$ implies among other facts that the EH for a system satisfying GDB will have the familiar nearest-neighbour Ising structure with only pair interactions. The latter fact may also be concluded for $d=2$ [Garrido and Marro 1989]. As a matter of fact, although the applications become very involved as one increases the system dimension, the formalism in this section may in principle be generalized for $d>1$.

4.- Steady states near equilibrium.

Notice that in the absence of the GDB condition the system may still have an EH and one may devise alternative methods, usually more indirect and specific ones, to determine the unique short-ranged object $E(\mathbf{s})$. For instance, one may relate in some cases the coefficients $J^{(k)}$ to some relevant correlation functions to be computed independently [see, for instance, Browne and Kleban 1989]. Alternatively, one may follow the philosophy and formalism we described before; e.g. when $D_2\neq 0$ is small enough one may attempt an expansion around $D_2=0$. Such procedures, however, will usually show a strong dependence on the details of the system of interest or involve lengthy algebraic manipulations. Instead, we shall focus in this section on a general method which demonstrates the existence of a "linear regime" where a nonequilibrium system may still retain most canonical features.

Consider the competing dynamics (2.1) where each $c_i(\mathbf{s}'|\mathbf{s})$ is in the form (3.1) so that it satisfies the (individual) detailed balance condition (2.2) with respect to a given Hamiltonian $H(\mathbf{s};\boldsymbol{\varphi}_i)$. Here $\boldsymbol{\varphi}_i$ represents the set of parameters characterizing a class of Hamiltonians; for instance, one may imagine $H(\mathbf{s};\boldsymbol{\varphi}_i)$ of the nn Ising type with an external field, (3.2), i.e. $\boldsymbol{\varphi}_i=\{K_i^{(1)},K_i^{(1)}\}$. Moreover, we shall assume that each set of values $\boldsymbol{\varphi}_i$ differ by a small enough amount from some reference values, $\boldsymbol{\varphi}$, say $\boldsymbol{\varphi}_i=\boldsymbol{\varphi}+\boldsymbol{\delta}_i$ for every i , so that one may expand the Hamiltonian as

(4.1) $$H(\mathbf{s};\boldsymbol{\varphi}_i) = H(\mathbf{s};\boldsymbol{\varphi}) + (dH/d\boldsymbol{\varphi})\cdot\boldsymbol{\delta}_i + \tfrac{1}{2}(d^2H/d\boldsymbol{\varphi}d\boldsymbol{\varphi}):\boldsymbol{\delta}_i\boldsymbol{\delta}_i + \ldots$$

By using this expansion into condition (2.2), one obtains:

(4.2) $$c_i(\mathbf{s}'|\mathbf{s}) = c^0(\mathbf{s}'|\mathbf{s}) + c_i^{(1)}(\mathbf{s}'|\mathbf{s}) + c_i^{(2)}(\mathbf{s}'|\mathbf{s}) + \ldots$$

where the contributions at each order, $c^0(\mathbf{s}'|\mathbf{s})$ and $c_i^{(n)}(\mathbf{s}'|\mathbf{s})$ with $n=1,2,\ldots$, satisfy a kind of detailed balance condition, namely that

(4.3) $$c^0(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\} = c^0(\mathbf{s}|\mathbf{s}')\exp\{-H(\mathbf{s}';\boldsymbol{\varphi})\} \ ,$$

(4.4) $$c^0(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\}\ [\Omega_i(\mathbf{s})-\Omega_i(\mathbf{s}')] + c_i^{(1)}(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\} = c_i^{(1)}(\mathbf{s}|\mathbf{s}')\exp\{-H(\mathbf{s}';\boldsymbol{\varphi})\} \ ,$$

(4.5) $$\begin{aligned} c_i^{(2)}(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\}\ [\Gamma_i(\mathbf{s})-\Gamma_i(\mathbf{s}')-\Omega_i(\mathbf{s}')(\Omega_i(\mathbf{s})-\Omega_i(\mathbf{s}'))] + \\ + c_i^{(1)}(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\}\ [\Omega_i(\mathbf{s})-\Omega_i(\mathbf{s}')] + \\ + c_i^{(2)}(\mathbf{s}'|\mathbf{s})\exp\{-H(\mathbf{s};\boldsymbol{\varphi})\} = c_i^{(2)}(\mathbf{s}|\mathbf{s}')\exp\{-H(\mathbf{s}';\boldsymbol{\varphi})\} \ , \end{aligned}$$

where $\Omega_i(\mathbf{s}) \equiv -[dH(\mathbf{s};\boldsymbol{\varphi}_i)/d\boldsymbol{\varphi}]\cdot\boldsymbol{\delta}_i$ and $\Gamma_i(\mathbf{s}) \equiv \frac{1}{2}[(dH(\mathbf{s};\boldsymbol{\varphi}_i)/d\boldsymbol{\varphi})\cdot\boldsymbol{\delta}_i]^2 - \frac{1}{2}[d^2H(\mathbf{s};\boldsymbol{\varphi}_i)/d\boldsymbol{\varphi}d\boldsymbol{\varphi}]:\boldsymbol{\delta}_i\boldsymbol{\delta}_i$. Then, after using the result (4.2) in the expression defining the effective Hamiltonian, one obtains

(4.6) $$E(\mathbf{s}) = H(\mathbf{s};\boldsymbol{\varphi}) + [dH(\mathbf{s};\boldsymbol{\varphi})/d\boldsymbol{\varphi}]\cdot[\Sigma_i p_i\boldsymbol{\delta}_i] + O(\delta^2) \ .$$

That is, one may define a "linear regime", where the relevant parameters (temperature, interaction strength, external magnetic field, etc.) appearing in the individual Hamiltonians which are associated to each

transition rate (via a detailed balance condition) have values close enough to a given set of values, such that one has both i) that GDB is satisfied by the system, and ii) that an EH always exists which is given by (4.6). Therefore, a system defined in such a linear regime, while being capable of a full nonequilibrium behavior which may differ qualitatively from the equilibrium one, is expected to suffer "small" departures from the canonical equilibrium associated with the reference Hamiltonian $H(\mathbf{s};\varphi)$ in the sense that it fits most relevant canonical qualities, say i) and ii).

5.- Several one-dimensional examples.

In this section we shall initiate the application of the formalism to several one-dimensional systems with a practical interest. Consider first the (trivial) case in which every rate is defined with respect to the same Hamiltonian, say (3.2). That is, $H_i(s)\equiv H(s)$, $K_i^{(1)}\equiv h/k_BT$ and $K_i^{(2)}\equiv K$. It follows that $\ln[c(s|s^i)/c(s^i|s)] = 2s_i[h/k_BT+2K\sigma_i^i]$, so that GDB is (trivially) satisfied, $J^{(1)}_i=-h/k_BT$, $J^{(2)}_{i,i\pm1}=-K$, and the rest of coefficients are zero. Consequently, $E(s)=H(s)$ independent of the specific choice for the rate function.

More interesting is the situation in which the dynamics is a mixture of two or more rates each defined with respect to a different "Hamiltonian", $H_i(s)$. As an illustration, López-Lacomba et al.[1990] considered within the present formalism a case solved before by Garrido et al. [1987] which may be characterized by $K_i^{(1)}\equiv0$, $K_i^{(2)}\equiv\varphi_iK$ with $\varphi_i = [k_B(T-\delta T)]^{-1}$ and $\varphi_2=[k_B(T+\delta T)]^{-1}$, and $c_i(s^i|s)=\{\cosh[2K_i^{(1)}\sigma_i^i]\}^{-1}\exp\{2K_i^{(1)}\sigma_i^is_i\}$. It follows rather straightforwardly from the results in section 3 that $D_0=D_2=0$ and $D_1=\frac{1}{2}b$, with $b = \ln\{[1+p\varphi_1+(1-p)\varphi_2]/[1-p\varphi_1-(1-p)\varphi_2]\}$, $\varphi_i\equiv\tanh(2K\varphi_i)$, and, consequently, that $J^{(2)}_{i,i\pm1}=-\frac{1}{2}b$ is the only non-zero coefficient. Thus, by defining an "effective temperature" T_{eff} such that $J^{(2)}_{i,i\pm1}=-J/k_BT_{eff}$, one obtains $\tanh(2J/k_BT_{eff})=p\varphi_1+(1-p)\varphi_2$. This is precisely the result which was only obtained before [Garrido et al. 1987] after solving the model explicitly. The relative simplicity of the present method also becomes evident by considering different choices for $f_i^{(1)}(s)$.

The same formalism may be applied to the study of a system in which the dynamics involves a competition between several interactions, a case which may be more relevant in relation to the topic of this meeting given that it may include a kinetic version of the familiar spin glass systems. Consider, for instance, the case characterized by $K_i^{(1)}\equiv0$ and $K_i^{(2)}\equiv J_i/k_BT$, $J_1=K$ and $J_2=K+\delta K$. Clearly, one obtains the same formal results as for the two-temperatures model except that the need is now for an "effective interaction" $J_{eff}=J^{(2)}_{i,i\pm1}=-\frac{1}{2}b$. In spite of that formal similarity, the present case bears a novel physical significance. This becomes evident when one considers the possibility of an external magnetic field h, i.e. $K_i^{(1)}\neq0$. The two-temperatures model would then require $K_i^{(1)}=h/(T-\delta T)\neq K_i^{(2)}=h/(T+\delta T)$, implying that GDB is not satisfied, while we now have $K_i^{(1)}=K_i^{(2)}=h/T$, so that GDB holds and an EH exists whose only non-zero coefficients are $J^{(1)}_i=-h/k_BT$ and $J^{(2)}_{i,i\pm1}$, the latter being the same as for $h=0$.

One may consider in the same way kinetic versions of random field systems, e.g. the simple case of a mixture of two dynamics such that the associated "Hamiltonians" are characterized instead by $K_i^{(1)}=h/k_BT$, $K_i^{(2)}=0$, and $K_2^{(1)}=K_2^{(2)}=K$, i.e. one acts with probability p as if the external magnetic field were h, and the other acts with probability $1-p$ as if there were no external field. The study of this problem for different rate functions evidences in particular the outstanding role played by the details of the dynamics on the qualitative features of the (nonequilibrium) steady state.

One may also consider with little effort the so-called voter model [see, for instance, Kernstein 1986] which belongs to the class of systems whose definition does not involve any Hamiltonian but a certain dynamical process. Namely, any configuration s evolves via spin-flips with a rate $c(s^i|s)$ satisfying, for $d=1$:

$$(5.1)\quad \ln[c(s|s^i)/c(s^i|s)] = s_i \ln\left|\frac{1+(1-1)\sigma_i^i-1(1-2p)}{1-(1-1)\sigma_i^i+1(1-2p)}\right|$$

where $0\le1,p\le1$. It follows that GDB is only fulfilled either for $1=1$, when $E(s) = -\frac{1}{2}\ln[p/(1-p)]\Sigma_is_i$, or else for $p = 1/2$, when $E(s) = -\frac{1}{2}\ln[(2-1)/1]\Sigma_is_is_{i+1}$.

Notice that the above four simple models depict an interesting situation. In particular, each example satisfies GDB only for some range of values of the system parameters or for certain families of transition rates. When that is the case, we obtain explicit expressions for the EH, and it follows that the nonequilibrium system can be mapped onto an equivalent equilibrium one with some "effective" value for the relevant parameter, say T, J or h.

Finally, once the general method has been illustrated, we turn to the detailed study, including critical behavior, of a one-dimensional system having a considerable physical interest. Namely, we are interested now in a kinetic random-field model with rates $c_i(s;x)$ depending on $\delta H_i=H_i(s^i)-H_i(s)$ with $H_i(s)=-K\Sigma_ss_is_{s-1}-h_i\Sigma_is_s$ where h_i is a

random variable. Given that it follows from above that there is an E with the familiar Ising structure, we write from the start $H_{ef}=-K_{ef}\Sigma_i s_i s_{i+1} - h_{ef}\Sigma_i s_i$. We first notice that different rates may imply different expressions for both, K_{ef} and h_{ef}. For instance, $c(s;x)=a\cdot\exp(\delta H/2)$ produces $K_{ef}=J/k_BT$ and $h_{ef}=\frac{1}{2}\ln[\langle\langle e^{J/k_BT}\rangle\rangle/\langle\langle e^{-J/k_BT}\rangle\rangle]$ where $\langle\langle\cdot\rangle\rangle$ represents an average with respect to the distribution $g(h)$ for the random variable h_i, which we shall assume to have the symmetry $g(h)=g(-h)$, whereas $c(s;x)=a[2\cosh(\delta H/2)]^{-1}\cdot\exp(\delta H/2)$ produces $K_{ef}=J+\frac{1}{4}\ln(\langle\langle c_+\rangle\rangle/\langle\langle c_-\rangle\rangle)$, where $c_\pm\equiv e^{h/k_BT}/\cosh[(2J\pm h)/k_BT]$, and $h_{ef}=0$.

In order to go further, we select the latter rate for simplicity and the distribution $g(h)=\frac{1}{2}p[\delta(h-h_o)+\delta(h+h_o)]+(1-p)\delta(h)$ which, in addition to an obvious intrinsic interest, has the merit of generalizing a case previously solved by Grinstein and Mukamel [1983]. It follows that $K_{ef}=bJ+\frac{1}{4}\ln([\frac{1}{2}p(c_+^{+}+c_+^{-})+p']/[\frac{1}{2}p(c_-^{+}+c_-^{-})+p']$ where we write $c_\pm^{+}\equiv\exp(bh_o)/\cosh(2bJ\pm bh_o)$, $c_\pm^{-}\equiv\exp(-bh_o)/\cosh(2bJ\pm bh_o)$, $p'\equiv(1-p)/\cosh(2bJ)$ and $b\equiv 1/k_BT$. Consequently, by making $J=1$ and $b_{ef}\equiv K_{ef}/J$ we have $b_{ef}\approx b+O(b^3)$ for $b\to 0$, i.e. the system tends to present Ising equilibrium behavior when T is high enough. Now, if one defines $t\equiv e^{-b}$, $b_{ef}\equiv -\ln t_{ef}$, it follows that $t_{ef}=t([1+t^{4h_o}+2t^{4+2h_o}]/[t^4+2t^{2h_o}+t^{4+4h_o}])^{\frac{1}{4}}$ for $p=1$ implying four different behaviors:

1) When $h_o=0$, we have $t_{ef}=t$ and $T_{ef}=T$, i.e. equilibrium behavior, as one should expect.

2) When $0<h_o<2$, we have $t_{ef}\approx 2^{-\frac{1}{4}}t^{1-\frac{1}{4}h_o}+O(t^n)$ with $n>0$, or either $b_{ef}\approx\frac{1}{4}\ln 2+(1-\frac{1}{2}h_o)b+O(t^n)$ as $b\to\infty$, i.e. there is a simple linear relation near $T=0$.

3) When $h_o=2$, we have $b_{ef}\approx\frac{1}{4}\ln 3+O(t^n)$ for $b\to\infty$, i.e. the system is "hot", and it remains so even as $T\to 0$, as a consequence of the competition between local random fields and thermal fluctuations.

4) When $h_o>2$, we have $b_{ef}=O(t^n)\approx 0$ for $b\to\infty$, i.e. the randomness induced by the fields is so strong that the system is completely disordered.

Also interesting is the case $p\neq 1$ with $h_o=\infty$ which is characterized by $t_{ef}=([1+p+2pt^{-4}]/[1+p+2pt^4])^{\frac{1}{4}}$.

The behavior of the system energy follows as $e=N^{-1}\Sigma_i\langle s_i s_{i+1}\rangle = -N^{-1}(d/dK_{ef})\Sigma_s\exp[-H_{ef}(s)] = -\tanh(b_{ef}) = (t_{ef}^2-1)/(t_{ef}^2+1)$, and the mean square fluctuations of the energy are given by $(\delta E)^2 = \langle E^2\rangle-\langle E\rangle^2 = -(d/db_{ef})\langle E\rangle = -N(db/db_{ef})(de/db)$. In the present nonequilibrium situation one may define two different "specific heats": $C_1\equiv de/dT = b^2(db_{ef}/db)[\cosh b_{ef}]^{-2}$ and $C_2\equiv b^2N^{-1}\delta E^2 = -b^2(de/db_{ef})=^2[\cosh b_{ef}]^{-2}$. It thus follows that $C_1/C_2 = db_{ef}/db=(t/t_{ef})(dt_{ef}/dt)$, i.e. there is no fluctuation-dissipation theorem in general.

Now let us describe the critical behavior of the kinetic random-field model of interest; this results quite novel. For instance, the correlations are $C(r)\equiv\langle s_i s_{i+r}\rangle = (\tanh b_{ef})^r$ and, when one defines a correlation length, say α, such that $C(r)\equiv\exp(-r/\alpha)$, it follows that $\alpha=-1/\ln\tanh b_{ef}$. Now, define a critical index ν such that $\alpha\equiv t^{-\nu}$, one has a critical behavior for $b\to\infty$ which depends on the value of h_o:

a) When $h_o=0$, one has $\alpha\approx\frac{1}{2}\exp(2b)$ diverging with $\nu=2$, i.e. the equilibrium behavior, as expected.

b) When $0<h_o<2$, one has instead $\alpha\approx 2\exp[b(2h_o)]$ whose divergence depends on the value of h_o, namely $\nu=2-h_o$.

c) When $h_o=2$ and $h_o>2$ one has $\alpha\approx\frac{1}{3}3^{\frac{1}{4}}$ and $\alpha\approx\frac{1}{2}$ respectively, i.e. $\nu=0$ corresponding to the absence of a critical point, even for $b=\infty$, due to the randomness induced by (relatively) strong local fields. It also follows that, for $h_o=\infty$, one has $\alpha\approx\frac{1}{2}[(1+p)/2p]^{\frac{1}{4}}$ which only diverges as $p\to 0$, the equilibrium case.

The same models are also being studied at present by means of mean-field and Monte Carlo techniques. Some interesting, preliminary results will be presented graphically which we hope will convince you, together with the above facts, that such a series of nonequilibrium systems with competing dynamics has an amazingly rich and interesting physical behavior which deserves a more detailed and systematic study.

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