Physica 144A (1987) 585–603
North-Holland, Amsterdam

ISING MODELS WITH ANISOTROPIC INTERACTIONS: STATIONARY NONEQUILIBRIUM STATES WITH A NONUNIFORM TEMPERATURE PROFILE*

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Received 12 September 1986
Revised 12 February 1987

We study a two-dimensional Ising model with different interaction strengths along each principal axis and a nonuniform temperature profile along one of them, as well as some variations of it. The model is solved analytically and/or numerically on different assumptions to reveal a variety of (nonequilibrium) stationary states and phase transitions; we also investigate the system relaxation in some typical cases.

1. Introduction

The present status of the statistical mechanics of nonequilibrium phenomena allows no general use of the formalism which is most standard and powerful in equilibrium phenomena, the Gibbs ensemble theory. Actually, there is no general (e.g. valid for a system with interacting elements) prescription for choosing appropriate ensembles even for the simplest nonequilibrium case, i.e. that of stationary nonequilibrium states in which external agents maintain steady particle or heat currents throughout the system[1]). Instead, the study of stationary nonequilibrium states is nowadays based on a collection of *ad hoc* methods, most of them approximate, for particular problems. As a consequence, the consideration of mathematically well-defined (e.g. lattice) model systems having nonequilibrium states bears great interest, specially when they are amenable to simple, nontrivial analytical solutions; see refs. 2–4 for some recent examples.

We present in this paper some simple generalizations of the usual kinetic

* Partially supported by the US–Spanish Cooperative Research Program under Grant CCB-8402-025, and by the CAICYT, Project PB85-0062.

0378-4371/87/$03.50

Ising model[5]) showing phase transitions in the local equilibrium conditions which are usually taken as a characteristic of stationary nonequilibrium states. The nonequilibrium condition is a consequence of the action of some external agents inducing, just along one of the principal directions of the lattice, a nonuniform local temperature in the thermal bath which is supposed to interact with the spin system. A simple version of the model then occurs in two dimensions when the interactions in the direction perpendicular to the temperature gradient are assumed to have a mean-field nature. This and further variations of the model are studied with some detail to consider some questions such as the nature of the system relaxation and of the steady state produced by different temperature distributions, the corresponding critical behavior, which happens to be always classical, the influence of the choice for the transition probabilities on the properties of the steady state, the case of "impure" sites, etc. When the temperature is assumed to be constant throughout the system, the basic two-dimensional model reduces to an equilibrium Ising model with anisotropic interactions which was solved previously by us[6]).

2. The basic model system

The basic model of interest consists of a $L_X \times L_Y$ lattice (see fig. 1) with spin variables $s_{ij} = \pm 1$ $(i = 1, \ldots, L_X;\ j = 1, \ldots, L_Y)$ at each lattice site. Some external agent induces a given, nonconstant temperature profile along the X direction, as if the spins s_{ij} $(i = 1, \ldots, L_X;\ j = \text{const})$ at each row were in contact with different thermal baths at temperatures T_i, respectively (or with a single "thermal bath" with local temperatures T_i), while there is a constant temperature along the Y direction for each value of the index i. The interac-

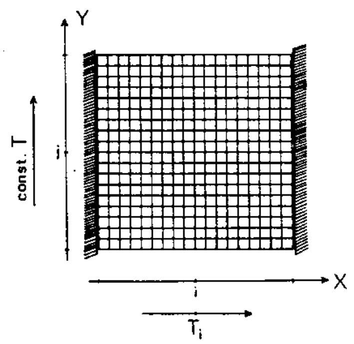

Fig. 1. Schematic representation of the two-dimensional lattice model with a nonuniform temperature along the X axis. There is an occupation or spin variable $s_{ij} = \pm 1$ at each lattice site in contact with a thermal bath at local temperature T_i.

tions will be assumed anisotropic in the sense that there is a nearest-neighbor coupling along X which is characterized by the exchange energy J_X, and a nearest-neighbor (or other) interaction along Y which is characterized by J_Y which may differ from J_X. In addition, the configurational density probability function changes with time according to a Markovian master equation,

$$\frac{\mathrm{d}P(s,t)}{\mathrm{d}t} = -\left[\sum_{ij} w_{ij}(s_{ij})\right]P(s,t) + \sum_{ij} w_{ij}(-s_{ij})P(\ldots, -s_{ij}, \ldots; t) \qquad (2.1)$$

with *local* transition probabilities per unit time given by

$$w_{ij}(s_{ij}) = \frac{\alpha_i}{2}\,[1 - \tfrac{1}{2}\gamma_i^X s_{ij}(s_{i-1,j} + s_{i+1,j})][1 - \tfrac{1}{2}\gamma_i^Y s_{ij}(s_{i,j+1} + s_{i,j-1})]\,, \qquad (2.2)$$

where

$$\gamma_i^Z = \tanh(2J_Z/kT_i)\,, \quad Z = X, Y\,, \qquad (2.3)$$

and α_i simply describes the local time scale on which the transitions take place.

A sufficient condition which may be used to interpret our choice (2.2) is the local detailed balance condition

$$w_{ij}(s_{ij})p_{ij}(s_{ij}) = w_{ij}(-s_{ij})p_{ij}(-s_{ij})\,, \qquad (2.4)$$

where

$$p_{ij}(s_{ij}) \propto \exp(-E_{ij}/kT_i) \qquad (2.5)$$

with the definition

$$E_{ij} = -s_{ij}[J_X(s_{i-1,j} + s_{i+1,j}) + J_Y(s_{i,j-1} + s_{i,j+1})]\,; \qquad (2.6)$$

here $p_{ij}(\pm 1)$ represents the probability of the state $s_{ij} = \pm 1$ of the ijth spin (the others remaining fixed). This amounts to assume local equilibrium in the stationary nonequilibrium state.

The local magnetization,

$$q_{ij}(t) \equiv \langle s_{ij} \rangle = \sum_s s_{ij}P(s,t)\,, \qquad (2.7)$$

satisfies

$$\frac{\mathrm{d}q_{ij}(t)}{\mathrm{d}t} = -2\langle s_{ij}w_{ij}(s_{ij})\rangle\,. \qquad (2.8)$$

Thus, when one characterizes the stationary regime by the condition $\mathrm{d}q_{ij}/\mathrm{d}t = 0$ and assumes translational invariance along the Y axis in the sense that $\langle s_{ij}\rangle = \langle s_{i,j\pm1}\rangle$, it follows

$$\langle s_i\rangle(1-\gamma_i^Y) - \tfrac{1}{2}\gamma_i^X(\langle s_{i-1}\rangle + \langle s_{i+1}\rangle) \\ -\tfrac{1}{4}\gamma_i^X\gamma_i^Y J_Y^{-1}\langle [E_{ij} + J_X s_i(s_{i-1}+s_{i+1})](s_{i-1}+s_{i+1})\rangle = 0\,, \tag{2.9}$$

where we dropped most dependence on j for clarity, and E_{ij} is defined in eq. (2.6).

3. One-dimensional and mean-field cases

3.1. *Nearest-neighbor interactions*

The simplest case occurs when $J_Y = 0$ corresponding to a one-dimensional system with nearest-neighbor interactions along X. It then follows from eq. (2.9) that

$$q_i = \tfrac{1}{2}\gamma_i[q_{i-1} + q_{i+1}]\,, \quad i = 1,\ldots,N\,, \tag{3.1a}$$

where $\gamma_i \equiv \gamma_i^X$, for N interior spins and

$$q_0 = \gamma_0 q_1\,, \quad q_{N+1} = \gamma_{N+1}q_N \tag{3.1b}$$

for the two spins at the ends, $N+2 = L_X$. The only solution of the system (3.1) of $(N+2)$ equations is $q_i = 0$ for all i when the corresponding determinant is nonzero; this can be written as a function of

$$A_n \equiv \begin{vmatrix} 1 & -\gamma_n/2 & 0 & & \cdots & & 0 \\ -\gamma_{n+1}/2 & 1 & -\gamma_{n+1}/2 & & & & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & & & & -\gamma_N/2 & 1 & -\gamma_N/2 \\ 0 & & \cdots & & 0 & -\gamma_{N+1} & 1 \end{vmatrix}, \tag{3.2}$$

$n = 1,\ldots,N-1$; $A_N = 1 - \tfrac{1}{2}\gamma_N\gamma_{N+1}$, which satisfy the recurrence relations $A_n = A_{n+1} - \tfrac{1}{4}\gamma_n\gamma_{n+1}A_{n+2}$. It then follows for the system determinant:

$$\det \mathbf{A} = A_1 - \tfrac{1}{2}\gamma_0\gamma_1 A_2 \tag{3.3}$$

which may be readily evaluated in some limiting conditions. Namely, in the case of weak coupling, i.e. small J_X so that $2J_X/kT_i$ is small enough for all i, one obtains

$$\det \mathbf{A} = 1 - \left(\frac{J_X}{k}\right)^2 \left[\frac{1}{T_0 T_1} + \frac{1}{T_1 T_2} + \cdots + \frac{1}{T_N T_{N+1}}\right]. \tag{3.4}$$

Also, when the temperature profile is linear, $T_i = T_0 + i\nabla t$, with a small gradient ∇T it follows

$$\det \mathbf{A} = 1 - \left(\frac{J_X}{kT_0}\right)^2 \left[1 - (N+1)^2 \frac{\nabla T}{T_0}\right] \tag{3.5}$$

for any value of J_X. Both expressions, (3.4) and (3.5), are nonzero in general implying the absence of a phase transition, as in the equilibrium counterpart, for rather arbitrary temperature profiles.

3.2. *Coherent-field coupling*

As in equilibrium, however, the above model system may be forced to present in general a (nonequilibrium) phase transition at finite temperatures by introducing a mean-field coupling. That is, we shall assume now that the transition probabilities are given by

$$w_i(s_i) = \frac{\alpha_i}{2} [1 - s_i \tanh(\tilde{E}_i/kT_i)], \tag{3.6}$$

where T_i represents the temperature profile along a principal direction, say X, and

$$\tilde{E}_i \equiv h + \sum_{l=1}^{L_X} J_{il} s_l . \tag{3.7}$$

Here h represents an external magnetic field contribution and every spin is supposed to interact with the rest via J_{ij}; we are also assuming a macroscopic system. A simple hypothesis is then that $J_{ij} = J/N$ at each site, and the consistency condition $s_l = \langle s_l \rangle$; this is the so-called coherent-field approximation by Braggs and Williams[7]). The corresponding stationary regime can be seen to be characterized by the condition

$$M = L_X^{-1} \sum_i \tanh[(h + JM)/kT_i], \tag{3.8}$$

where

$$M \equiv L_X^{-1} \sum_i \langle s_i \rangle \tag{3.9}$$

is the mean or global magnetization.

3.3. *Different temperature profiles*

To make evident the general existence of nonequilibrium phase transitions in the model of section 3.2, one may consider in eq. (3.8) $h = 0$ and the rather general profile $T_i = T_0[1 + \alpha f(i)]$ with $|f(i)|$ bounded for all i. In the case of small α it follows that

$$M \simeq \tanh x - \frac{\alpha x}{\cosh^2 x} [a(L_X) + \alpha(1 - x \tanh x) b(L_X)] \tag{3.10}$$

with the notation $x \equiv JM/kT_0$ and

$$a(L_X) = L_X^{-1} \sum_i f(i) , \quad b(L_X) = L_X^{-1} \sum_i f(i)^2 . \tag{3.11}$$

The critical temperature (with respect to the first, $i = 0$, spin) is then given by

$$K_0^c = 1 - \alpha a(L_X) + \alpha^2 b(L_X) + \cdots , \tag{3.12}$$

where $K_0 \equiv kT_0/J$ and one should notice that α will depend in general on temperature.

The above may refer in particular to the linear behavior $T_i = T_0 + i\nabla T$ with a small gradient ∇T; one has $f(i) = i$, $\alpha = \nabla T/T_0$, and

$$K_0^c = 1 - \Delta K/2 + (\Delta K)^2/3 + \cdots , \tag{3.13}$$

where $\Delta K \equiv K_{N+1} - K_0$ and L_X is assumed to be large enough.

Figs. 2–4 correspond to the linear temperature profile with an arbitrary gradient ∇T, i.e. they were prepared numerically from eq. (3.8) avoiding the approximation, small α, leading to (3.13). Fig. 2 reveals the monotonous decreasing of the local spontaneous magnetization with increasing i; this is clearly implied by the above equations, e.g. $\langle s_i \rangle = \tanh(JM/kT_i)$, and it is to be expected on physical grounds. As shown by fig. 3, there are important qualitative differences with ∇T in the phase diagram $M(T_0)$, where T_0 is the temperature corresponding to the first spin to the left of the line, while M scales near a mean critical temperature when it is represented as in fig. 4, namely when one uses

$$T \equiv L_X^{-1} \sum_i T_i , \quad T_c \equiv L_X^{-1} \sum_i T_i^c , \tag{3.14}$$

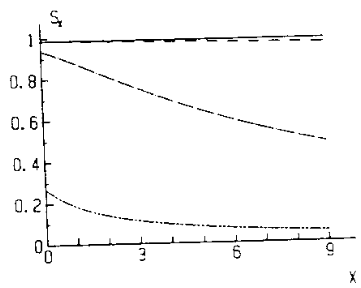

Fig. 2.

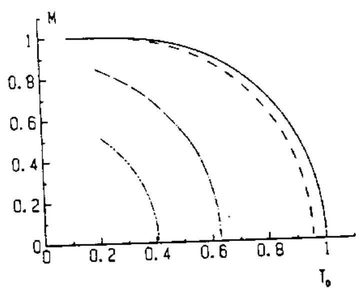

Fig. 3.

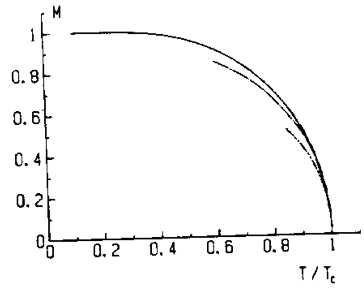

Fig. 4.

Fig. 2. Local spontaneous magnetization as a function of the position along the X axis in the case of a one-dimensional system with 10 spins and a linear temperature profile $T_i = T_0 + i\nabla T$, $T_0 = 0.4$ ($J/k = 1$ unit) for different values of the gradient ∇T; cf. section 3.3. The solid line is for $\nabla T = 0$ (equilibrium); the dashed lines from the top to the bottom are, respectively, for $\nabla T = 0.01$, 0.1 and 0.2.

Fig. 3. The mean magnetization $M = L_X^{-1} \Sigma_i \langle s_i \rangle$ for the system in fig. 2 as a function of T_0 (the temperature of the first spin). The curves are from the top to the bottom for $\nabla T = 0$, 0.01, 0.1 and 0.2, respectively; the corresponding critical temperatures are $T_0^c = 1$, 0.944, 0.630 and 0.405, respectively ($J/k = 1$ units).

Fig. 4. Same data as in fig. 3 plotted versus T/T_c; cf. eq. (3.14). The solid line is for $\nabla T = 0$ and $\nabla T = 0.01$; the other two lines are from the top to the bottom for $\nabla T = 0.1$ and 0.2, respectively.

where $T_i^c \equiv T_0^c + i\nabla T$, as natural variables. Fig. 4 suggests a common critical behavior near T_c; actually the numerical analysis of the phase diagrams in fig. 4 near $T = T_c$ shows that the critical exponent for the mean magnetization is $\beta = \frac{1}{2}$, independent of ∇T. One may also get convinced analytically by oneself from eq. (3.8) and by expanding for small M that $L_X^{-1} \Sigma_i T_i^{c^{-1}} = 1$ and $\beta = \frac{1}{2}$. Note however that, as is also suggested by fig. 4, the width of the critical region decreases with increasing ∇T. Note also that the equilibrium version of the model, $\nabla T = 0$, is already characterized by classical critical exponents[6]) so that this fact is not a peculiarity of the nonequilibrium condition here.

The cases $T_i = T_0 + \nabla T \omega^{-1} f(\omega i)$ with $f(z) = \sin(z)$ or $\cos(z)$ also bear some interest. The corresponding local spontaneous magnetization can be seen to present in this case oscillations around a constant value, reminding stationary waves on a line, whose amplitude and frequency strongly depend both on ∇T and ω. The curves $M(T)$, on the other hand, lie one onto the other for all values of T, independently of ∇T and ω. In particular, for $f(\omega i) = \sin(\omega i)$, $\alpha = \nabla T / T_0 \omega$, one finds to first order in α that

$$\langle s_i \rangle \simeq \tanh(M/K_0) - \frac{\nabla K M \sin(\omega i)}{\omega K_0^2 \cosh^2(M/K_0)} . \tag{3.15}$$

Some representative critical temperatures (with respect to the first spin) when $f(z) = \sin(z)$ are as follows: $T_0^c(\omega = \pi/10,\ \nabla T = 0.1) = 0.99$, $T_0^c(\omega = \pi/10,\ \nabla T = 0.1) = 0.91$, $T_0^c(\omega = 2\pi/10,\ \nabla T = 0.1) = 1.11$, $T_0^c(\omega = 2\pi/10,\ \nabla T = 0.01) = 1.0$.

Finally, we consider explicitly the case of two competing temperatures. A simple situation is that with a fraction n of the lattice sites with fixed spins, e.g. as a consequence of their contact with a thermal bath at zero temperature. It follows immediately that

$$M = n + (1 - n) \tanh[(h + JM)/kT] \tag{3.16}$$

when every "active" spin is at the same temperature T; one also has

$$T^c = (1 - n) J/k \tag{3.17}$$

as in the usual mean-field case with vacancies. More interesting is the case in which half the system (say, $i \leqslant 0$) is at temperature T and the other half ($i > 0$) at temperature T'; it follows then

$$2M = \tanh[(h + JM)/T] + \tanh[(h + JM)/T'] \tag{3.18}$$

and

$$T'^c = T[2kT/J - 1]^{-1} , \tag{3.19}$$

where T needs to be greater than $J/2k$. Notice in particular that half the system ($i > 0$) may still present order when the other half is completely disordered, i.e. for $T \to \infty$.

3.4. *System relaxation*

The system relaxation towards the stationary nonequilibrium state is described by the equations

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -M + L_X^{-1} \sum_i \tanh(JM/kT_i) , \tag{3.20}$$

where M is defined in eq. (3.9), so that there is a separate equation for each $\langle s_i \rangle$ following from eq. (2.8) (we set $h = 0$ for simplicity): we also need to assume $\alpha_i = 1$, i.e. equal local time scales at each lattice site.

For small derivations from the stationary state and for temperatures near the critical one, eq. (3.20) may be approximated by the Bernouilli differential equation[8,5]):

$$\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}t} = (1 - A)y + B , \tag{3.21}$$

where $y = M^{-2}$ and

$$A \equiv JL_X^{-1} \sum_i (kT_i)^{-1} , \quad B \equiv J^3(3L_X)^{-1} \sum_i (kT_i)^{-3} . \tag{3.22}$$

This leads in general to the exponential behavior

$$M = \sqrt{1 - A}\{[(1 - A)M_0^{-2} + B]\,\mathrm{e}^{(1-A)2t} - B\}^{-1/2} , \quad A \neq 1 . \tag{3.23}$$

At the critical temperature, however, one has $A = 1$ (cf. the discussion following eq. (3.14)) and it follows the slower relaxation

$$M = [2Bt + M_0^{-2}]^{-1/2} , \quad A = 1 , \tag{3.24}$$

$M_0 \equiv M(t = 0)$. The local order parameter, on the other hand, relaxes according to

$$\langle s_i \rangle = \mathrm{e}^{-t}\left[q_i^0 + \int_0^t \mathrm{d}t'\, \mathrm{e}^{t'}(\beta_i JM - \tfrac{1}{3}\beta_i^3 J^3 M^3) \right] , \tag{3.25}$$

where $M = M(t)$ is given either by (3.23) when $A \neq 1$ or by (3.24) when $A = 1$, $\beta_i \equiv 1/kT_i$ and q_i^0 represents the initial ($t = 0$) value for $\langle s_i \rangle$.

The above equations involve a general description of the system relaxation in a nonequilibrium phase transition. A comparison between different cases is made in fig. 5 when the system is characterized by a linear temperature profile; further details may be worked out easily.

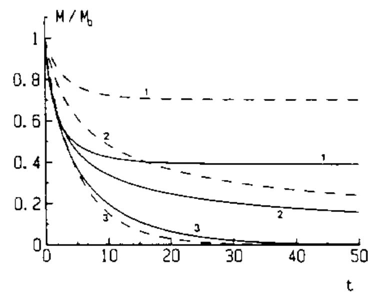

Fig. 5. Time relaxation of the global order parameter, as given by eq. (3.20), normalized to the initial ($t = 0$) value in the case of a linear temperature profile $T = T_0 + i\nabla T$. The solid lines correspond to $\nabla T = 0.2$, i.e. relaxation towards a nonequilibrium state, for different temperatures of the first, $i = 0$, spin: $T_0 < T_0^c$ (curve labelled 1), $T_0 = T_0^c$ (curve 2) and $T_0 > T_0^c$ (curve 3). The dashed lines correspond to $\nabla T = 0$, i.e. relaxation towards the equilibrium state, for the same temperatures.

3.5. *Influence of transition probabilities*

Stationary nonequilibrium states may in principle depend on the transition probabilities one considers in the master equation (2.1); as a matter of fact, this is the observation in some recent Monte Carlo experiments on the stationary nonequilibrium states in a fast ionic conductor model system[9]). That possibility may be analyzed in the present model.

Let the generalized (local) transition probabilities per unit time be

$$\omega_i(s) = f_i(s) \exp[-A_i(s)s_i] , \tag{3.26}$$

where $f_i(s)$ is an even function, $f_i(s_1, \ldots, s_i, \ldots, s_L) = f(s_1, \ldots, -s_i, \ldots, s_L)$ and the explicit form for $A_i(s)$ depends on the specific assumption on the interactions; for instance,

$$A_i = \frac{J_X}{kT_i} (s_{i-1} + s_{i+1}) \tag{3.27}$$

for nearest-neighbor interactions as in section 3.1, and

$$A_i = JM/kT_i \tag{3.28}$$

for a coherent-field coupling as in section 3.2. The stationary regime is then characterized by

$$\sum_s P(s, t) f_i(1) \cosh(A_i)(1 - s_i \tanh A_i) = 0 \tag{3.29}$$

and

$$\sum_s P(s,t)s_is_l[f_i(1)\cosh(A_i)(1-s_i\tanh A_i) + f_l(1)\cosh(A_l)(1-s_l\tanh A_l)] = 0\,, \tag{3.30}$$

as follows, respectively, from the stationarity of the local order parameter $\langle s_i\rangle$ and the correlations $\langle s_is_l\rangle$. That is, there is indeed a nontrivial dependence of the stationary nonequilibrium state on the choice for ω_i. For instance, the simple choice $f_i(s_i)\cosh A_i = \alpha_i$ produces

$$\langle s_i\rangle = \langle\tanh A_i\rangle \tag{3.31}$$

and

$$(\alpha_i + \alpha_l)\langle s_is_l\rangle = \alpha_i\langle s_l\tanh A_i\rangle + \alpha_l\langle s_i\tanh A_l\rangle \tag{3.32}$$

while other choices may obviously produce a quite different behavior including a different critical temperature. Of course, those differences wash out in the equilibrium cases, $T_i = T$ for all i, as one may easily prove from eqs. (3.31) and (3.32). A related crucial point, which can only be addressed properly in a more general context than the present one, is to find the conditions on the transition probabilities to obtain a given nonequilibrium universality class; as suggested by previous Monte Carlo work[9]), chances are that one should obtain the same critical exponents (even though the critical temperature differs) at least for the most familiar choices for w_i, e.g. for the ones in refs. 2 and 5.

4. Two-dimensional cases

4.1. *Zeroth order solution*

The stationary nonequilibrium states for the two-dimensional Ising lattice with a nonuniform temperature distribution, T_i, along one of the principal axes are characterized by the condition (2.9), i.e.,

$$\langle s_{ij}\rangle(1-\gamma_i^Y) - \tfrac{1}{2}\gamma_i^X(\langle s_{i-1,j}\rangle + \langle s_{i+1,j}\rangle) + \tfrac{1}{4}\gamma_i^X\gamma_i^Y\langle s_{ij}(s_{i-1,j}+s_{i+1,j})(s_{i,j-1}+s_{i,j+1})\rangle = 0\,, \tag{4.1}$$

where γ_i^X and γ_i^Y are defined in eq. (2.3).

In order to proceed further, however, one needs to decouple the three-spins correlations in the last term of eq. (4.1). One of the simplest assumptions is to write $\langle s_{ij}s_{kj}s_{i,j+1}\rangle = \langle s_{ij}s_{kj}s_{i,j-1}\rangle = \langle s_{ij}\rangle^2\langle s_{kj}\rangle$, both for $k = i-1$ and for $k = i+1$; at equilibrium ($T_i = T$ for all i) this produces the solutions

$$\langle s\rangle = \pm[(\gamma^X + \gamma^Y - 1)/\gamma^X\gamma^Y]^{1/2} \tag{4.2}$$

and the magnetization critical exponent $\beta = 1/2$. The dynamics also follows easily in this case; for instance, one finds for a homogeneous temperature after some algebra that

$$\langle s\rangle = [4\gamma^X\gamma^Y t + q_0^{-2}]^{-1/2}\,, \quad T = T_c \tag{4.3}$$

and

$$\langle s\rangle = \left[\frac{(\gamma^X + \gamma^Y - 1)q_0^2 A}{1 + \gamma^X\gamma^Y q_0^2 A}\right]^{1/2}, \quad T \neq T_c\,, \tag{4.4}$$

where

$$A \equiv (\gamma^X + \gamma^Y - 1 - \gamma^X\gamma^Y q_0^2)^{-1}\exp[4(\gamma^X + \gamma^Y - 1)t]\,, \tag{4.5}$$

which leads to the solutions (4.2) as $t \to \infty$.

4.2. *Coherent field along Y*

More interesting is to analyze the model system defined in section 2 by assuming a coherent-field coupling along the Y axis and treating exactly the nearest neighbor interactions along the X axis. The transition probabilities (2.2) reduce in this case to

$$w_{ij}(s_{ij}) = \frac{\alpha_i}{2}\left[1 - s_{ij}\,\frac{s_{i-1,j} + s_{i+1,j}}{2}\,\gamma_i\right](1 - s_{ij}\bar{\gamma}_i)\,, \tag{4.6}$$

where the last bracket in eq. (2.2) suffered the treatment discussed in section 3.2, and we introduced the notation $\gamma_i \equiv \gamma_i^X$ and $\bar{\gamma}_i \equiv \tanh(M_i J/kT_i)$. The stationary regime is then characterized by

$$\langle s_i\rangle = \bar{\gamma}_i + \tfrac{1}{2}\gamma_i(\langle s_{i-1}\rangle + \langle s_{i+1}\rangle) + \tfrac{1}{2}\gamma_i\bar{\gamma}_i\langle e_i\rangle J_X^{-1}\,, \tag{4.7}$$

where we dropped the trivial dependence on j and use the notation

$$\langle e_i\rangle = -J_X\langle s_i(s_{i-1} + s_{i+1})\rangle\,, \quad M_i = \frac{1}{N}\sum_j \langle s_{ij}\rangle\,. \tag{4.8}$$

In order to proceed further with this equation we may assume that the correlations $\langle e_i \rangle$ are given locally by their equilibrium value[10]). That is, when the temperature distribution is homogeneous throughout the system, $T_i = T$ for all i, one may compute the system canonical partition function[6]); it then follows in particular from that partition function the correlations which are given as

$$\langle s_{ij}(s_{i-1,j} + s_{i+1,j}) \rangle = 2kT \frac{\partial}{\partial J_X} \ln \lambda_+ , \tag{4.9}$$

where

$$\lambda_+ = \exp(J_X/kT) \cosh(JM/kT) + \tilde{\alpha}(T, M, J_X, J) \tag{4.10}$$

with the notation

$$\tilde{\alpha} \equiv [\exp(2J_X/kT) \sinh^2(JM/kT) + \exp(-2J_X/kT)]^{1/2} , \tag{4.11}$$

and the magnetization which is given as the solution of the self-consistency equation:

$$M = \tilde{\alpha}^{-1} \exp(J_X/kT) \sinh(JM/kT) . \tag{4.12}$$

The general solution on that assumption (local thermodynamic equilibrium) is thus given by eq. (4.7) where each $\langle e_i \rangle$ follows by combining eqs. (4.9) and (4.12). More specifically, the solution for the inhomogeneous case is

$$M_i - \tfrac{1}{2}\gamma_i(M_{i-1} + M_{i+1}) = \overline{\gamma}_i[1 + \gamma_i(2J_X)^{-1}\langle \tilde{e}_i(T_i) \rangle] , \tag{4.13}$$

where $\langle \tilde{e}_i(T_i) \rangle$ means the combined local solution (by eliminating M) of

$$\langle e \rangle = -2J_X\tilde{\alpha}^{-1}[\exp(J_X/kT) \cosh(JM/kT) - 2\cosh(2J_X/kT)\lambda_+^{-1}] , \tag{4.14}$$

which follows from eqs. (4.8)–(4.10) and eq. (4.12) for each temperature T_i. We shall come back to these expressions in section 5.

4.3. *Weak coherent-field coupling*

The above equations allow the explicit analysis of a number of interesting situations. The simplest one corresponds to the case $J = 0$ which reduces exactly to the inhomogeneous one-dimensional system with nearest-neighbor interactions discussed in section 3.1. It thus seems interesting to study now small enough values of J, e.g. such that one is allowed to write $\bar{\gamma}_i = \tanh(JM_i/kT_i) \approx JM_i/kT_i$.

We shall refer explicitly to the case of an infinite system with only two temperatures, $T_i = T_1$ for $i \geq 0$, $T_i = T_2$ for $i < 0$, small differences $|T_1 - T_2|$ and a weak coherent-field coupling in the sense specified above. One has immediately from eq. (4.13) the two sets of equations (notice $\gamma_i = \tanh(2J_X/kT_i)$):

$$M_i - \frac{\gamma_1}{2}(M_{i-1} + M_{i+1}) = \frac{JM_i}{kT_1}\left[1 + \frac{\gamma_1}{2J_X}\langle e_i(T_1)\rangle\right], \quad i \geq 0, \tag{4.15a}$$

$$M_i - \frac{\gamma_2}{2}(M_{i-1} + M_{i+1}) = \frac{JM_i}{kT_2}\left[1 + \frac{\gamma_2}{2J_X}\langle e_i(T_2)\rangle\right], \quad i \leq -1, \tag{4.15b}$$

The only general solution of eqs. (4.15) and (4.14) with a physical relevance is

$$M_i = B_1[\gamma_1^{-1}(1 - \sqrt{1-\gamma_1^2})(1 + J/kT_1)]^{i-1}, \quad i \geq 1, \tag{4.16a}$$

and

$$M_i = B_2[\gamma_2^{-1}(1 - \sqrt{1-\gamma_2^2})(1 + J/kT_2)]^{-i-2}, \quad i < -1; \tag{4.16b}$$

and one also has from eqs. (4.15) for $i = 0, 1$ that

$$M_0 = \frac{2}{\gamma_2}[1 - (J/kT_2)\sqrt{1-\gamma_2^2}]M_{-1} - B_2 \tag{4.16c}$$

and

$$M_{-1} = \frac{2}{\gamma_1}[1 - (J/kT_1)\sqrt{1-\gamma_1^2}]M_0 - B_1, \tag{4.16d}$$

where B_1 and B_2 cannot be determined within the present theory (one would need to know the value of M_i at two points), i.e. the theory only predicts the spatial variations of M_i.

The above equations allow us to consider, for instance, the situation in which half the system is in contact with a heat bath at a very high temperature, namely $T_2 = \infty$ and a finite value for T_1. It follows in this case $\gamma_2 = 0$, $B_2 = 0$,

$$M_0 = \tfrac{1}{2}B_1\gamma_1\left(1 + \frac{J}{kT_1}\sqrt{1-\gamma_1^2}\right) \tag{4.17}$$

and M_i, $i \geqslant 1$, given by eq. (4.16a). The nature of the steady state in this case depends on the functional dependence of B_1 on T_1 and J; one may get convinced in particular that, in order to have a phase transition, $B_1(T_1, J)$ needs to be such that $B_1^c = 0$ defines a critical temperature T_1^c: $B_1 \neq 0$ for $T_1 < T_1^c$, $B_1 = 0$ for $T_1 \geqslant T_1^c$. Further cases of interest are considered in the following within a more general context.

4.4. *Impurities*

Our basic two-dimensional model also allows the study of the influence of impure sites on the details of a number of steady states, either equilibrium or nonequilibrium. In order to illustrate this fact we first mention the case of fixed spins, $s_{ij} = 1$, for all even i when there is a coherent-field coupling along the Y axis as described in section 4.2; one has immediately from eqs. (4.7) and (4.8) that the steady state is characterized by

$$M_i = \tanh[(2J_X + JM_i)/kT_i] \tag{4.18}$$

for odd i, and $M_i = 1$ for even i. When $T_i = T$ and $M_i = M$ for all i, eq. (4.18) describes in particular the case of a homogeneous temperature distribution and fixed, impure spins at alternate columns.

In order to generalize the above situation, one may consider the Ising square lattice with nearest-neighbor interactions along both principal directions, X and Y, and a local temperature T_{ij} at each lattice site; that is, we have now instead of eqs. (2.5) and (2.2):

$$p_{ij}(s_{ij}) \propto \exp[J_X s_{ij}(s_{i-1,j} + s_{i+1,j})/kT_{ij} + J_Y s_{ij}(s_{i,j-1} + s_{i,j+1})/kT_{ij}] \tag{4.19}$$

and

$$w_{ij}(s_{ij}) = \frac{\alpha_{ij}}{2}\left[1 - s_{ij}\gamma_{ij}^X \frac{s_{i-1,j} + s_{i+1,j}}{2}\right]\left[1 - s_{ij}\gamma_{ij}^Y \frac{s_{i,j-1} + s_{i,j+1}}{2}\right], \tag{4.20}$$

respectively, and it follows the stationary condition

$$\begin{aligned} &\langle s_{ij}\rangle - \frac{\gamma_{ij}^X}{2}(\langle s_{i-1,j}\rangle + \langle s_{i+1,j}\rangle) - \frac{\gamma_{ij}^Y}{2}(\langle s_{i,j-1}\rangle + \langle s_{i,j+1}\rangle) \\ &\quad + \tfrac{1}{4}\gamma_{ij}^X\gamma_{ij}^Y\langle s_{ij}(s_{i-1,j} + s_{i+1,j})(s_{i,j-1} + s_{i,j+1})\rangle = 0 \end{aligned} \tag{4.21}$$

instead of eq. (2.9).

Let us apply this equation to the case of two coupled sublattices, i.e. like the ones characterizing the ground state of the square antiferromagnetic Ising model. The first sublattice is occupied by fixed, impure spins so that one has $s_{ij} = 1$, say when both i and j are either even or odd, e.g. the corresponding spins are in contact with a bath at zero temperature; on the contrary, the spins at the second sublattice are free, so that they can take any of the two possible values $s_{ij} = \pm 1$ (with i and j having different parity) as a consequence of their interactions with a single heat bath at temperature T inducing transitions $s_{ij} \to -s_{ij}$ with the probabilities per unit time (4.20) with $\alpha_{ij} = \alpha$, $\gamma^X_{ij} = \gamma^X$ and $\gamma^Y_{ij} = \gamma^Y$ for i and j belonging to the second sublattice. When the site (i, j) belongs to the first sublattice, one has $s_{ij} = 1$, $T_{ij} = 0$, $\gamma_{ij} = 1$, and it follows from eq. (4.21) that

$$\langle s_{i-1,j} s_{i,j-1} \rangle = 2M - 1 . \tag{4.22}$$

When the site (i, j) belongs to the second sublattice, one has $\langle s_{ij} \rangle = M$, $T_{ij} = T$, and

$$M = \tanh[2(J_X + J_Y)/kT] \tag{4.23}$$

to be combined with eq. (4.22).

4.5. *Coherent-field coupling along the two directions*

The basic two-dimensional model system in section 2 may also be approximated by a much simpler version, namely assuming coherent-field couplings along both the X and Y principal directions of the lattice. That is, we write now with an obvious notation:

$$P_{ij}(s_{ij}) \propto \exp[(J_X M_i + J_Y \tilde{M}_j) s_{ij}/kT_i] ; \tag{4.24}$$

here

$$M_i = N^{-1} \sum_j \langle s_{ij} \rangle , \quad \tilde{M}_j = N^{-1} \sum_i \langle s_{ij} \rangle \tag{4.25}$$

and

$$w_{ij}(s_{ij}) = \frac{\alpha_i}{2} [1 - s_{ij}\gamma^X_i][1 - s_{ij}\gamma^Y_{ij}] , \tag{4.26}$$

where

$$\gamma^X_i = \tanh(J_X M_i/kT_i) , \quad \gamma^Y_{ij} = \tanh(J_Y \tilde{M}_j/kT_i) . \tag{4.27}$$

The stationary state is characterized then by

$$\langle s_{ij}\rangle = \tanh[(J_X M_i + J_Y \tilde{M}_j)/kT_i]\,, \tag{4.28}$$

and

$$M_i = N^{-1}\sum_j \tanh[(J_X M_i + J_Y \tilde{M}_j)/kT_i]\,, \tag{4.29}$$

$$\tilde{M}_j = N^{-1}\sum_i \tanh[(J_X M_i + J_Y \tilde{M}_j)/kT_i]\,. \tag{4.30}$$

Different temperature profiles may easily be worked out from these equations.

5. Small departures from equilibrium

Finally, we present in this section a differential formulation of the model defined in section 4.2 by introducing infinitesimal variations, $T_i = T + \delta T_i$, $M_i = M + \delta M_i$, with respect to the homogeneous, equilibrium solution $M = M(T)$ which was already analyzed before by us[6]); notice that we drop here the dependence on j for simplicity.

One has immediately that

$$\gamma_i^X \equiv \tanh(2J_X/kT_i) = \gamma^X + a^X \delta T_i/T\,, \tag{5.1}$$

where $\gamma^X = \tanh(2J_X/kT)$ and $a^X = -2J_X(1-(\gamma^X)^2)/kT$, and

$$\bar{\gamma}_i = \tanh(JM_i/kT_i) = \gamma + a\left(\frac{\delta T_i}{T} - \frac{\delta M_i}{M}\right), \tag{5.2}$$

where $\bar{\gamma} = \tanh(JM/kT)$ and $a = -JM(1-\bar{\gamma}^2)/kT$. Let us denote by $\tilde{\tilde{M}}_i$ the local solution, corresponding to the local temperature T_i, of the equilibrium self-consistency condition eqs. (4.12) and (4.11), and let us write then $\tilde{\tilde{M}}_i = M + \delta\tilde{\tilde{M}}_i$; it follows after some algebra that

$$\frac{\delta \tilde{\tilde{M}}_i}{M} = A(M, T)\,\frac{\delta T_i}{T} \tag{5.3}$$

with the auxiliary function

$$A \equiv \{2J_X[M^2\,\mathrm{e}^{-4\beta J_X} - (1-M^2)\sinh^2(\beta JM)] - JM(1-M^2)\sinh(2\beta JM)\}$$
$$\times [2kT\sinh^2(\beta JM) - JM(1-M^2)\sinh(2\beta JM)]^{-1}\,, \tag{5.4}$$

$\beta = 1/kT$. In the same way, we may write $\langle \tilde{e}_i(T_i) \rangle = \langle e \rangle + \delta\tilde{e}_i$ where $\langle e \rangle$ is given by eq. (4.14) and $\langle \tilde{e}_i(T_i) \rangle$ represents the local solution (by eliminating M) of eqs. (4.14) and (4.12); it follows that

$$\delta\tilde{e}_i = B(M, T)\,\frac{\delta T_i}{T}\,, \tag{5.5}$$

$$B \equiv -\frac{2J_X}{\gamma^X}\left\{\frac{M(1-\gamma^X)}{\bar{\gamma}}\left[\frac{a^X}{\gamma^X} + \frac{a(1-A)}{\bar{\gamma}} - A\right] + a^X\left(\frac{M}{\bar{\gamma}} - \frac{1}{\gamma^X}\right)\right\} \tag{5.6}$$

by noticing the fact

$$\tilde{\tilde{\gamma}}_i \equiv \tanh(J\tilde{\tilde{M}}_i/kT_i) = \gamma + a(1-A)\,\frac{\delta T_i}{T}\,. \tag{5.7}$$

The above expressions allow to write immediately from eq. (4.13) that

$$\delta M_i = C(M, T)\,\frac{\delta T_i}{T}\,E(M, T)\,, \tag{5.8}$$

where we introduced the assumption $\delta M_{i+1} + \delta M_{i-1} \approx 2\delta M_i$, i.e. smooth variations, and the notation

$$C \equiv (a\gamma^X + a^X\gamma)D(M, T) + a + a^X M + \gamma\gamma^X B/2J_X \tag{5.9}$$

with

$$D \equiv [(1-\gamma^X)M/\gamma - 1]/\gamma^X \tag{5.10}$$

and

$$E \equiv \left[1 - \frac{2a}{M} + \frac{1-\gamma^X}{\gamma}\,a + \gamma^X\right]^{-1}. \tag{5.11}$$

As expected, this implies in particular that $\delta M_i =$ constant for $\delta T_i =$ constant. One also has for the local order parameter that

$$\delta\langle s_i \rangle = C(M, T)\,\frac{\delta T_i}{T}\,(1-\gamma^X)^{-1}\left[1 - \frac{2a}{M} + \gamma^X\right]E(M, T) \tag{5.12}$$

on the assumption $\delta\langle s_{i+1}\rangle + \delta\langle s_{i-1}\rangle \approx 2\delta\langle s_i\rangle$.

The above equations represent a complete solution for small deviations from the equilibrium state. In order to extract from them some concrete information, one may define the quantities:

$$\overline{M} = N^{-1} \sum_i M_i = M + N^{-1} \sum_i \delta M_i \tag{5.13}$$

and

$$\overline{T} = N^{-1} \sum_i T_i = T + N^{-1} \sum_i \delta T_i \,. \tag{5.14}$$

It then follows, for instance, that

$$\overline{M} = M + T^{-1} C(M, T) E(M, T)(\overline{T} - T) \tag{5.15}$$

showing that the critical temperature is given by the condition $\overline{M} = 0$, i.e.

$$\overline{T}^{c} = T[1 - M\{C(M, T)E(M, T)\}^{-1}] \,. \tag{5.16}$$

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