UNIVERSALITY TEST FOR CRITICAL AMPLITUDES IN TWO DIMENSIONAL PERCOLATION

J. MARRO*

Departamento de Física Teórica, Universidad de Barcelona, Barcelona-14, Spain
and Belfer Graduate School of Science, Yeshiva University, New York, N.Y. 10033, USA

Received 14 September 1976

From series expansions estimates of Sykes et al. it is concluded that the ratio $B^{\delta-1}\Gamma E^{-\delta}$ for the critical amplitudes corresponding to the critical exponents β, γ and δ, respectively, behaves like a universal quantity, within reasonable bounds, for the site and bond percolation problem.

The phenomenological theories of scaling and universality have played, together with renormalization-group procedures, an important role in the study of thermodynamic behavior near the critical point, through the use of critical exponents and, very recently, amplitudes to characterize universality classes.

Following standard notation in magnetic language [1], one may write the dimensionless equation of state near the critical point T_c:

$$H = M|M|^{\delta-1}f(x), \quad x = T_c^{-1}|M|^{1/\beta}(T - T_c), \tag{1}$$

where H stands for the magnetic field, M for the magnetization, and T for the absolute temperature. From eq. (1) one can then derive the asymptotic critical relations

$$M = B[(T_c - T)/T_c]^{\beta}, \quad T \lesssim T_c;$$
$$\chi = \Gamma[(T - T_c)/T_c]^{-\gamma}, \quad T \gtrsim T_c, \tag{2a}$$

at $H = 0$, and

$$H = DM|M|^{\delta-1}, \quad T = T_c, \tag{2b}$$

with χ the susceptibility; β, γ and δ represent the usual critical exponents, and B, Γ and D the corresponding critical amplitudes, defined through (2). After introducing the non-universal constants $f_0 = f(0)$ and x_0 such that $f(-x_0) = 0$, and rescaling the function $f(x)$, $\tilde{f}(\tilde{x}) = \tilde{f}(x/x_0) = f_0^{-1}f(x)$, universality implies [1] that the new scaled function $\tilde{f}(\tilde{x})$ has to be the same for all systems belonging to a given equivalence class. In particular, $\tilde{f}(\tilde{x})$ is expected to be invariant under lattice structure, but dependent on dimensionality.

Now, from eqs. (1) and (2) one obtains $B = x_0^{-\beta}$,

$$\Gamma = x_0^{\gamma}f_0^{-1} \lim_{x\to\infty} [\tilde{x}^{\gamma}/\tilde{f}(\tilde{x})],$$

where the limit term is a universal quantity, and $D = f_0$. It immediately follows, given the scaling relation between universal critical exponents, $\gamma = \beta(\delta - 1)$, that

$$R \equiv \Gamma D B^{\delta-1}, \tag{3}$$

is a universal ratio between critical amplitudes whenever the universality hypothesis holds.

This possibility has been analysed very recently for a number of model systems and real materials [1] often with rather crude data, and in connection with the two-dimensional site percolation problem [2] also inconclusively due to inaccuracies in D and, possible, in δ. It is the purpose of this note to give a more stringent test for the universality hypothesis both for site and bond two-dimensional percolation based on new, more accurate data.

In percolation theory a "cluster" is defined as a set of nearest-neighbor-connected sites occupied by particles surrounded by empty sites. The particles are assumed to be independent and randomly distributed on the sites of a given lattice with probability p that a given site is occupied (with an obvious extension to the case of bond percolation). The percolation problem is then characterized by the onset of an infinite size cluster above a critical concentration of particles, p_c, and the phenomenon is known to bear a close

* Partially supported by AFOSR Grant No. 73-2430B.

Table 1
Numerical estimates for the supposed universal, ratio R' (see eq. (6)) between critical amplitudes, as computed from data by Sykes et al. [6].

Problem	Lattice Triangular	Simple quadratic	Honeycomb
Site	1.251 (±0.008)	1.255 (±0.031)	1.262 (±0.019)
Bond	1.258 (±0.010)	1.253 (±0.005)	1.250 (±0.007)

formal analogy [3] with the magnetic critical behavior discussed above. In fact, scaling ideas are applied and universality is postulated. One can postulate [2] that the cluster size distribution n_l (i.e. density of clusters of a given size l is given by

$$n_l = c_0 l^{-\tau} f(x), \quad x = c_1 l^{\sigma}(p_c - p), \tag{4}$$

where c_0 and c_1 are positive constants and $\sigma = 1/\beta\delta$, $\tau = 2 + 1/\delta$ in the usual notation [3, 4]. Here one takes $s_0 \equiv \Sigma_{l=1}^{\infty} l\, n_l = 1$ at $p < p_c$, and the infinite size cluster excluded from this and the following sums, i.e., p_c is defined by the relation $s_0 < 1$ at $p \geqslant p_c$. The function $f(x)$ in (4) is assumed, as in the thermodynamic problem, to be analytic and universal and the analogous to the critical thermodynamic amplitudes in (2) are defined as follows. (Note the use here of symbols used before for different quantities.) The asymptotic critical behavior of the percolation probability P, i.e., the fraction of lattice sites belonging to the infinite percolating cluster, which corresponds to the zero field magnetization in the thermodynamic problem, is

$$P = 1 - \sum_l l\, n_l = B(q - q_c)^{\beta}, \quad q \lesssim q_c, \tag{5a}$$

where $q = 1 - p$. The so-called "mean cluster size", corresponding to the zero field susceptibility, behaves as

$$S = \sum_l l^2 n_l = \Gamma(q - q_c)^{-\gamma}, \quad q \gtrsim q_c, \tag{5b}$$

and for the "critical isotherm" behavior at $p = p_c$ one can introduce [5] a field variable H so that

$$1 - \sum_l l\, n_l e^{-lH} = D^{-1/\delta} H^{1/\delta} \propto E(1 - e^{-H})^{1/\delta},$$

$$p = p_c, \quad H \to 0^+. \tag{5c}$$

The critical behavior (5) is consistent with new data on series expansion for the critical exponents and for the value of p_c for different two-dimensional lattices [6].

Using (4) in (5) and the usual scaling relations for critical exponents, it easily follows (note $E \propto D^{-1/\delta}$) that the ratio $R \propto \Gamma E^{-\delta} B^{\delta - 1}$, analogous to (3), between critical amplitudes for the percolation problem (both for site and bond percolation due to the bond-to-site transformation) or, more conveniently,

$$R' = \Gamma^{1/\delta} E^{-1} B^{1-1/\delta} \propto R^{1/\delta}, \tag{6}$$

has to define a universal quantity if the hypothesis of universality is valid (assuming scaling). Note that the same can be concluded avoiding [7] the use of eq. (4).

The best current estimates for the critical amplitudes, B, Γ and E are those by Sykes et al. using standare Padé approximants techniques on high and low density and field expansions, respectively [6]. The corresponding errors are assumed to be independent from each other in our calculations (in any case, the error in B strongly dominates the others). Thus we obtain for R' corresponding to the triangular, simple square and honeycomb lattices, both for site and bond percolation, the values in table 1, with the accompanying errors. The value 1.253 (±0.006) is the best universal estimate from that data, and it is well within the quoted ranges for each case. So it seems reasonable to conclude that, for the lattices considered, R' behaves like a universal quantity and that bond and site percolation belong to the same universality class.

The author would like to thank D. Stauffer for suggesting him the problem and M. Kalos for valuable discussion and facilities through ERDA Contract No. E(11-1)-3077.

References

[1] A. Aharony and P.C. Hohenberg, Phys. Rev. B 13 (1976) 3081, and references therein.
[2] D. Stauffer, Z. Phys. B (1976), in press.
[3] P.W. Kasteleyn and C.M. Fortuin, J. Phys. Soc. Japan Suppl. 26 (1969) 11;
V.K.S. Shante and S. Kirkpatrick, Adv. Phys. 20 (1971) 325;
J.W. Essam, in: Phase transitions and critical phenomena, vol. 2, eds. C. Domb and M.S. Green (Academic Press, New York, 1972).
[4] D. Stauffer, Phys. Rev. Lett. 35 (1975) 394.
[5] A.G. Dunn, J.W. Essam and J.M. Loveluck, J. Phys. C 8 (1975) 743;
D.S. Gaunt and M.F. Sykes, J. Phys. A 9 (1976) 1109.
[6] M.F. Sykes, D.S. Gaunt and M. Glen, J. Phys. A9 (1976) 87, 97, 715, 725;
D.S. Gaunt and M.F. Sykes, ref. [5].
[7] J.W. Essam and K.M. Gwilym, J. Phys. C 4 (1971) L228.