



# Influence of topology on the performance of a neural network

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## Abstract

We studied the computational properties of an attractor neural network (ANN) with different network topologies. Though fully connected neural networks exhibit, in general, a good performance, they are biologically unrealistic, as it is unlikely that natural evolution leads to such a large connectivity. We demonstrate that, at finite temperature, the capacity to store and retrieve binary patterns is higher for ANN with scale-free (SF) topology than for highly random-diluted Hopfield networks with the same number of synapses. We also show that, at zero temperature, the relative performance of the SF network increases with increasing values of the distribution power-law exponent. Some consequences and possible applications of our findings are discussed.

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## 1. Introduction

There is a growing interest in evolving complex networks, in particular, networks with scale-free (SF) topology [1,2,5]. SF occurs in many different contexts, including the www and the Internet, e-mail and scientific-citation networks, ecological, protein and gene interaction networks, etc. In these examples, the degree  $k$  of a vertex, i.e., the number of arcs linking it to other vertices, is power-law distributed,  $P(k) \sim k^{-\gamma}$  (see Fig. 1). This implies that the network includes a relatively large number of nodes with small connectivity, defining what we call the network *boundary*, and

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a few nodes, the *hubs*, with a large connectivity, comparable to the network size  $N$ . As a consequence, SF networks exhibit the interesting *small-world* property, that is, the average path length between two nodes is very small compared to the network size.

Evolving networks with such complex topology are also common in biology. Neuronal networks, for instance, seem to exhibit the small-world property. This was recently demonstrated in a set of in vitro experiments of growing cultured neurons [6]. Although an impressive amount of work has been done in the last few years concerning SF networks, it has only recently been reported on the specific consequences of such an architecture on the performance of auto-associative neural networks [3,7]. The authors in [7] show that a SF neural network is able to store and retrieve  $P$  patterns with a lower computer-memory cost than the fully connected Hopfield neural network. They also find a similar performance with a (biologically unrealistic) nearest-neighbor hypercubic Ising lattice. The authors in [3] study the zero temperature behavior of different topologies, namely, the Barabási–Albert (BA) SF, small-world and random diluted networks, and a better performance for the random diluted case than for the other topologies is reported. However, for the relative large mean connectivity these authors use ( $\langle k \rangle = 50$ ), the BA network has not the SF property [7], so that this result lacks interest.

We here report on the influence of topology on the associative-memory task of the network as a function of temperature. In particular we focus on two main issues, namely, on the robustness of the network performance against thermal noise for varying topology, and on the effect of varying the SF connectivity distribution  $P(k)$  on the network performance.

## 2. Definition of models

Consider the BA *evolving* network [1] with  $N$  nodes and  $\eta(N - \eta_0)$  links. Here,  $\eta_0$  is the initial number of nodes generating the network,  $\eta \leq \eta_0$  is the number of links that are added during the *evolution* at each time step, and  $N$  is the final number of nodes in the network. This will latter be generalized to consider other SF networks. In order to have a neural system with the chosen topology, we place a binary neuron,  $s_i = 1$  or 0, at each node  $i$ , and then “store”  $P$  binary random patterns,  $\xi^\mu \equiv \{\zeta_i^\mu = 1 \text{ or } 0\}$ ,  $\mu = 1, \dots, P$ , with mean activity level  $\langle \zeta_i^\mu \rangle = \frac{1}{2}$ . This is done in practice by associating a synaptic intensity  $\omega_{ij}$  at each link according to the Hebbian learning rule,

$$\omega_{ij} = \frac{1}{N} \sum_{\mu=1}^P (2\zeta_i^\mu - 1)(2\zeta_j^\mu - 1). \quad (1)$$

A meaningful direct comparison of this SF neural network (SFNN) and the standard Hopfield neural network cannot be made because the second case involves  $N^2$  synaptic connections. A hypercubic Ising lattice has the same number of synapses than the SFNN; however, real neural systems are known to exhibit more complex neuron connectivity than the Ising network. Consequently, we compare the performance of the

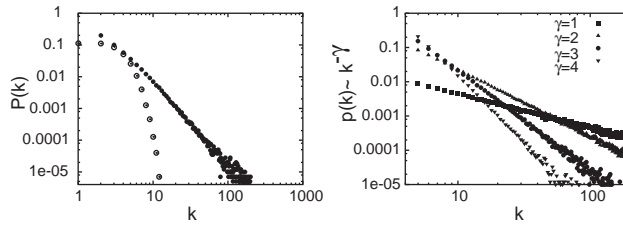


Fig. 1. Left: Connectivity probability distributions for a SF network with  $\gamma=3$  (filled circles) and for a HDH network (open circles). Right: Connectivity probability distributions generated for a SF network by tuning the exponent  $\gamma$ . Each data point corresponds to an average over 100 networks with  $N=4000$  neurons each.

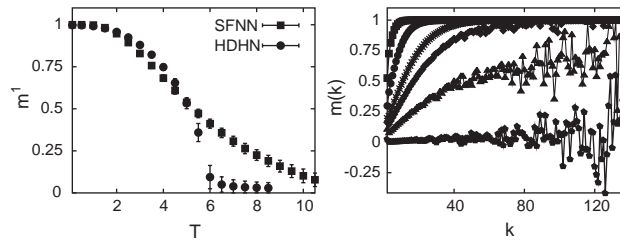


Fig. 2. Left: Overlap curves for varying temperature (in arbitrary units) for a BA network (filled squares) and a HDHN (filled circles). Data correspond to an average over 100 realizations of a network with  $N=1600$  neurons,  $P=1$  and  $\eta=\eta_0=3$ . Right: Local overlap  $m(k)$  for a BA network as a function of the connectivity  $k$  for increasing values of the temperature (successive decreasing curves).

SFNN with that of a *highly diluted* Hopfield network (HDHN). The HDHN is obtained from the standard Hopfield network by randomly suppressing synapses until only  $\eta(N - \eta_0)$  of them remain, i.e., the number of synapses scales as  $N$  and not as  $N^2$ . To maintain the SF behavior in the BA network, the value of  $\eta$  must be very small compared to the network size, that is,  $\eta \ll N$  [5]. The connectivity distribution of the HDHN is illustrated in Fig. 1 (left). The main differences between this distribution and the corresponding one for a SF network is that the latter has no typical connectivity value. More specifically, the SF network distribution is a power-law while the HDHN distribution has a maximum and an Gaussian decay and, consequently, may be characterized by a (typical) mean connectivity.

A relevant magnitude to monitor in order to compare the performance of different topologies is the overlap function, defined for pattern  $v$  as

$$m^v \equiv \frac{2}{N} \sum_i (2\xi_i^v - 1)s_i. \tag{2}$$

The performance of the two networks is compared in Fig. 2 for  $P=1$  and  $\eta=3$ . This clearly shows that, excluding very low temperature, the retrieval of information

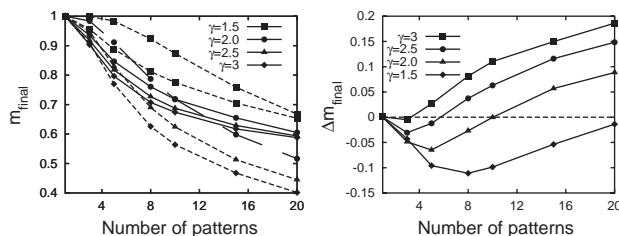


Fig. 3. Left: Zero temperature performance of a Molloy–Reed SFN for  $\gamma=1.5, 2, 2.5, 3$  (solid lines) compared to the performance of different HDHN with the same number of synapses (dashed lines), and as a function of the number of stored patterns  $P$ . Right: Relative difference in the performance for the cases showed in the left panel.

as characterized by  $m^\nu$  is better for the SFNN than for the HDHN. In both cases, the retrieval of information deteriorates as  $P$  is increased. However, we also observe that, at finite temperature, the performance of the SFNN increases significantly if one considers only the retrieval of information concerning neurons with a connectivity degree higher than certain value,  $k_0$ , i.e., the hubs [8]. This can be understood on simple grounds by computing the local overlap  $m_i^1$  with one pattern for neuron  $i$  and averaging over all neurons with the same connectivity  $k$ . The resulting mean overlap for a given connectivity  $m(k)$  is plotted in Fig. 2 (right) for different temperatures. This shows that, even at high temperature, the local overlap for hubs is close to one whereas it is very small for the boundary (fluctuations in the lower curves are due to the small number of hubs present in the network for the relative small network size we are using). This finding reflects the “negative” role of the boundary and the “positive” role of the hubs on the SFNN performance during each retrieval experiment when thermal fluctuations are considered. This observation is in agreement with the  $T = 0$  behavior reported in [3].

Another important issue is how the exponent of the distribution influences the performance of SF network for associative memory tasks. In order to analyze this, we studied networks characterized by different power-law exponents. With this end, we used a Molloy–Reed (MR) SF network [4] with  $P(k) \sim k^{-\gamma}$ , where  $\gamma$  is a tunable parameter. As illustrated in Fig. 1 (right), the number of neurons in the network with a high connectivity increases as  $\gamma$  is decreased.

Even more interesting is when one compares the behavior of the MR network with that of the HDHN in the limit  $T = 0$  (cf. Fig. 3). As thermal fluctuations are then suppressed, the network performance is only perturbed by the interference among the stored patterns. In order to consider the limit of interest, we started with one of the stored patterns, and computed the state of each neuron according to the deterministic rule  $s_i(t+1) = \Theta(h_i(t))$ . Here,  $\Theta(x)$  is the Heaviside step function, and  $h_i \equiv \sum_j^{(i)} w_{ij}s_j$  is the local field associated with neuron  $i$  with the sum over all neurons  $j$  connected to it. At the end of the network evolution, we recorded the overlap,  $m_{\text{final}}$ , with the starting pattern as a function of the total number of stored patterns  $P$ . In order to visualize the difference in performance between the two types of networks, we defined

$\Delta m_{\text{final}} \equiv m_{\text{final}}^{\text{SFNN}} - m_{\text{final}}^{\text{HDHN}}$ . Fig. 3 shows how this difference in performance varies with the number of stored patterns. The graph illustrates that the SFNN has a better and better performance as compared with the HDHN as the number of stored patterns is increased. This effect is enlarged as  $\gamma$  is increased. This can be understood by considering the different decays of  $P(k)$  for large  $k$  in both SFNN and HDHN, and the fact that for increasing values of  $\gamma$ , the relative position of  $\langle k \rangle$  moves to the left for both topologies, but due to the power-law decay the effect of the hubs remains for the SFNN.

Summing up, the topology of a neural network has a key role in the processes of memorization and retrieval of patterns. In particular, neural networks with scale-free topology may exhibit a better performance than Hopfield-like networks with the same number of synapses distributed randomly over the network. Our study can be useful to understand the role of regions with different connectivity degrees in real neural systems during memorization and retrieval of information. In particular, it may improve our understanding of how fluctuations or perturbations in the typical number of synapses of some brain areas can affect the processing of information and memorization in these regions. Our study also suggests the convenience of developing new methods to store the more relevant information into the hubs, increasing in this way the effective network-performance and efficiency. It would be desirable to check our findings against experimental observations on real neural systems focusing on the topology which is built up by natural selection.

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