Spontaneous Symmetry Breaking at the Fluctuating Level

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Phase transitions not allowed in equilibrium steady states may happen however both at the fluctuating level or in nonequilibrium settings. We confirm this surprising and general phenomenon studying current fluctuations in an isolated diffusive system. While small current fluctuations result from uncorrelated local events, for currents above a critical threshold the system is unable to sustain such an unlikely fluctuation by just superimposing incoherent events, and self-organizes into a coherent traveling wave which facilitates the current deviation, thus breaking translation invariance. This results in Gaussian statistics for small fluctuations but non-Gaussian tails above the critical current. Our observations, which match perfectly the predictions of hydrodynamic fluctuation theory, strongly suggest that rare events are generically associated with coherent, self-organized patterns.

Fluctuations characterize most physical phenomena, and their study has proven once and again to be a fruitful endeavour. The first example is probably Einstein's determination of molecular scales on the basis of the fluctuating behavior of a mesoscopic particle inmersed in a fluid [1], which opened the door to an experimental verification of the molecular hypothesis. Other examples range from fluctuations in the local density of an equilibrium system, which are directly related to the free-energy density [2] (a central object in equilibrium statistical physics), to the study of fluctuations of space-time correlations in glasses and other disordered materials, which has revealed the universal existence of dynamic heterogeneities in these systems [3]. In all cases the statistics of fluctuations encodes essential information to understand the physics of the system of interest. Even further, fluctuations reflect the symmetries of the microscopic world at the macroscale. This is the case for instance of the Gallavotti-Cohen fluctuation theorem [4–6] or the recently introduced isometric fluctuation relation [7], which express the subtle but enduring consequences of microscopic time-reversibility at the macroscopic level. Special attention is due to large fluctuations which, though rare, play a dominant role in many fields of science as they drastically affect the system behavior.

The study of fluctuating behavior provides an alternative way to derive thermodynamic potentials from which to calculate the properties of a system, a path complementary to the usual ensemble approach. This is most relevant for systems far from equilibrium [2–10], where no bottom-up approach exists yet connecting microscopic dynamics with macroscopic properties. The large-deviation function (LDF) controlling the fluctuations of the relevant macroscopic observables plays in nonequilibrium systems a role akin to the equilibrium free energy, although it may exhibit pathologies typical of nonequilibrium physics (e.g. non-local and non-convex behavior [2, 6, 8–11]). The hydrodynamic fluctuation theory (HFT), which has been recently introduced [8–10] to characterize large dynamic fluctuations in diffusive systems, offers explicit predictions for both the LDF and the optimal path in phase space responsible of a given fluctuation, which can be in general time-dependent [9]. However, it has been conjectured [12] and confirmed in simulations [13] that this optimal path is in fact time-independent for a wide range of fluctuations, an hypothesis known as additivity principle. This scenario may eventually fail for extreme fluctuations in some cases, and it is this additivity breakdown which in fact corresponds to a dynamic phase transition *at the fluctuating level* [14].

In this paper we test these ideas in a very general yet simple model of transport defined on a one-dimensional (1D) ring, where we study fluctuations of the time-averaged current. We find that small current fluctuations result indeed from the superposition of random local events, uncorrelated in space and time, thus giving rise to Gaussian statistics as dictated by the central limit theorem, see Fig. 1.a. However, for large enough currents, the system is unable to sustain such a large fluctuation by a superposition of random events, and self-organizes into a coherent traveling wave which facilitates this unlikely deviation, see Fig. 1.b, with a critical

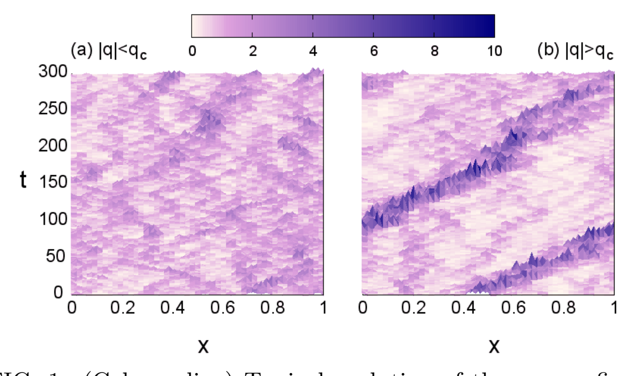

FIG. 1. (Color online) Typical evolution of the energy field for different current fluctuations in the 1D KMP model on a ring. (a) Small current fluctuations result from incoherent local events. (b) However, for $|q| > q_c$ the system facilitates these unlikely deviations by forming a traveling wave.

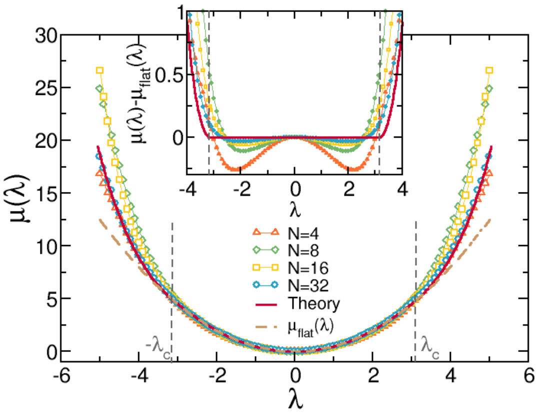

FIG. 2. (Color online) Main: Measured $\mu(\lambda)$ for the 1D KMP model on the ring and increasing values of N, together with the HFT prediction and the Gaussian approximation. Inset: $\mu(\lambda) - \mu_{\rm flat}(\lambda)$ for the same N. Data converge to the HFT prediction as N increases.

current q_c separating both regimes. This phenomenon, which is predicted by HFT, is most striking as it happens in an isolated equilibrium system in the absence of any external field, breaking spontaneously a symmetry in 1D. This is an example of the general observation that symmetry-breaking instabilities forbidden in equilibrium steady states can however happen both at the fluctuating level or in nonequilibrium settings [15]. Such instabilities may help explaining puzzling asymmetries in nature [15], from the dominance of left-handed chiral molecules in biology to the matter-antimatter asymmetry in cosmology.

Our model system is the paradigmatic 1D Kipnis-Marchioro-Presutti (KMP) model of transport on a ring [16]. This is a very general model of transport which represents at a coarse-grained scale the physics of many quasi-1D systems of theoretical and technological interest characterized by a single locally-conserved field which diffuses across space. In this sense our results are of great generality and could have important implications in actual experiments. Moreover, this model acts as a benchmark to test theoretical advances in nonequilibrium physics. The model is defined on a 1D lattice of N sites with periodic boundary conditions. Each site $i \in [1, N]$ is characterized by an energy e_i, and models an oscillator which is mechanically uncoupled from its nearest neighbors but interacts stochastically with them via a random energy redistribution process which conserves total energy. We are interested in the statistics of the total current q flowing through the system, averaged over a long time τ. For $\tau \to \infty$ this time average converges toward the ensemble average $\langle q\rangle$, which is of course zero because the system is isolated and in equilibrium. However, for long but finite τ we may still observe fluctuations $q \neq \langle q\rangle$, and their probability $\mathrm{P}_\tau(q)$ obeys a large deviation principle in this limit [11], $\mathrm{P}_\tau(q) \sim \exp[+\tau N G(q)]$. This means that the probability of observing a current fluctuation decays exponentially as both τ and N increase, at a rate given by the current LDF $G(q) \le 0$, with $G(\langle q\rangle) = 0$.

In order to study in depth current statistics, we performed extensive simulations of the 1D KMP model using an advanced Monte Carlo method which allows to explore the tails of the current LDF [17, 18]. This method implies a modification of the stochastic dynamics so that the rare event responsible of a large current fluctuation is no longer rare, and requires the simulation of multiple *clones* of the system [17]. In this work we used $M = 10^4$ clones. The method yields the Legendre transform of the current LDF, $\mu(\lambda) = \max_q[G(q) + \lambda q]$, with λ a parameter conjugated to the current, and Fig. 2 shows simulation results for $\mu(\lambda)$ and increasing values of N, together with the HFT result (see below). Gaussian current statistics corresponds to a quadratic behavior in $\mu(\lambda) \approx \mu_{\rm flat}(\lambda) = -\lambda^2/2$, which is fully confirmed in Fig. 2 for $|\lambda| < \lambda_c$, with $\lambda_c = \pi$, and different values of N. This means that small and intermediate current fluctuations have their origin in the superposition of uncorrelated local events, giving rise to Gaussian statistics as dictated by the central limit theorem. However, for fluctuations above a critical threshold, $|\lambda| > \lambda_c$, deviations from this simple quadratic form are apparent, signalling the onset of a phase transition. In fact, as N increases a clear convergence toward the HFT prediction is observed, with very good results already for $N = 32$. Strong finite size effects associated with the finite population of clones M prevent us from reaching larger system sizes [19], but fortunately $N = 32$ is already *close* enough to the asymptotic hydrodynamic behavior. Still, small corrections to the HFT predictions are observed which quickly decrease with N, see inset to Fig. 2.

Our model belongs to a large class of diffusive systems characterized at the macroscale by a single locally-conserved field which evolves in time according to a rescaled continuity equation

$$\partial_t \rho(x,t) = \partial_x \Big(D[\rho] \partial_x \rho(x,t) + \xi(x,t) \Big) . \qquad (1)$$

Here $\rho(x,t)$ is the density field, with $x \in [0,1]$, $j(x,t) \equiv -\left(D[\rho]\partial_x \rho(x,t) + \xi(x,t)\right)$ is the fluctuating current, and $D[\rho]$ is the diffusivity (a functional of the density profile in general). The (conserved) noise term $\xi(x,t)$, which accounts for microscopic random fluctuations at the mesoscopic level, is Gaussian and white with $\langle \xi(x,t)\rangle = 0$ and $\langle \xi(x,t)\xi(x',t')\rangle = 2N^{-1}\sigma[\rho]\delta(x-x')\delta(t-t')$, being $\sigma[\rho]$ the mobility functional and N the system size. In particular, for the KMP model $D[\rho] = 1/2$ and $\sigma[\rho] = \rho^2$ [16]. We now study the above equation under periodic boundary conditions, $\rho(0,t) = \rho(1,t)$ and $j(0,t) = j(1,t)$, so the total *mass* in the system is conserved, $\rho_0 = \int_0^1 \rho(x,t)dx$. The probability of observing a particular history $\{\rho(x,t), j(x,t)\}_0^\tau$ of duration τ for the density and current fields can be written as a path integral over all possible noise realizations, $\{\xi(x,t)\}_0^\tau$, weighted

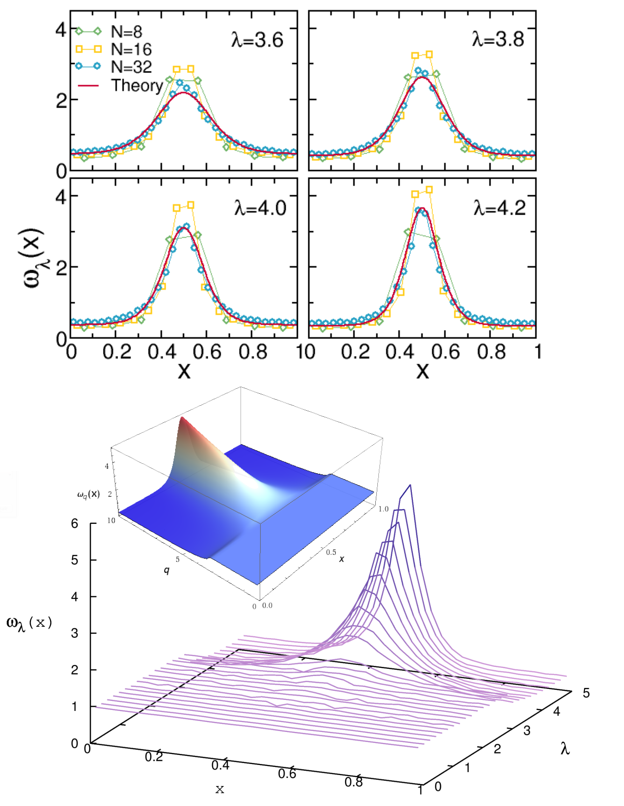

FIG. 3. (Color online) Top: Supercritical profiles for different λ and varying N, and HFT predictions. Bottom: Measured profiles as a function of λ for $N=32$. Inset: HFT prediction for the optimal profile $\omega_q(x)$. In both cases the profile is flat up to the critical current, beyond which a nonlinear wave pattern develops.

by its Gaussian measure, and restricted to those realizations compatible with eq. (1) at every point in space and time. This results in $\mathrm{P}(\{\rho,j\}_0^\tau) \sim \exp\{+N\,\mathcal{I}_\tau[\rho,j]\}$, with a rate functional defined by the familiar formula [8–10]

$$\mathcal{I}_\tau[\rho,j] = -\int_0^\tau dt \int_0^1 dx \frac{\Big(j(x,t) + D[\rho(x,t)]\partial_x \rho(x,t)\Big)^2}{2\sigma[\rho(x,t)]} \,. \tag{2}$$

with $\rho(x,t)$ and $j(x,t)$ coupled via the continuity equation, $\partial_t \rho(x,t) + \partial_x j(x,t) = 0$. Eq. (2) expresses the locally Gaussian nature of current fluctuations around its average behavior, given by Fourier's law. We are interested in the fluctuations of the time-averaged current q, defined as $q = \tau^{-1}\int_0^\tau dt \int_0^1 dx j(x,t)$. The probability of observing a given q can be in turn obtained from the path integral of $\mathrm{P}(\{\rho,j\}_0^\tau)$ restricted to histories $\{\rho,j\}_0^\tau$ consistent with that value of q. This probability scales as $\mathrm{P}_\tau(q) \sim \exp[+\tau N G(q)]$, and the current LDF $G(q)$ is related to $\mathcal{I}_\tau[\rho,j]$ via a simple saddle-point calculation in the long time limit, $G(q) = \tau^{-1}\max_{\rho,j}\mathcal{I}_\tau[\rho,j]$, such the the *optimal* profiles $\rho_q(x,t)$ and $j_q(x,t)$ solution of this variational problem are compatible with the constraints on ρ_0 and q and are related via the continuity equation. These optimal profiles can be interpreted as the ones the system adopts to facilitate a given current fluctuation.

Small deviations of the empirical current away from its ensemble average $\langle q\rangle = 0$ typically result from uncorrelated local fluctuations. The average density profile associated to these small fluctuations hence corresponds still to the flat, stationary one, $\rho_q(x,t) = \rho_0$. In this case, the optimal current profile is just $j_q(x,t) = q$ and the current LDF is simply quadratic, $G_{\text{flat}}(q) = -q^2/2\sigma(\rho_0)$ [14], resulting in Gaussian current statistics as confirmed is simulations, see Fig. 2. A natural question thus concerns the stability of this flat profile against small perturbations. Bodineau and Derrida have shown, using a linear stability analysis [14], that the flat profile indeed becomes unstable, in the sense that $G(q)$ increases by adding a small time-dependent periodic perturbation to the otherwise constant profiles, whenever

$$8\pi^2 D(\rho_0)^2\sigma(\rho_0) - q^2\sigma''(\rho_0) < 0\,, \tag{3}$$

where σ'' denotes second derivative. This defines a critical current $|q_c| = 2\pi D(\rho_0)\sqrt{2\sigma(\rho_0)/\sigma''(\rho_0)}$ for the instability to kick in. When this happens, the form of the associated relevant perturbation suggests that current fluctuations in this regime are sustained by a traveling wave pattern moving at constant velocity v [14]. We hence write $\rho_q(x,t) = \omega_q(x-vt)$, which results in $j_q(x,t) = q - v\rho_0 + v\omega_q(x-vt)$ via the continuity equation. The variational problem for $G(q)$ can be now written as

$$G(q) = -\min_{\omega_q(x),v}\int_0^1 \frac{\big[q - v\rho_0 + v\omega_q(x)\big]^2 + \omega_q'(x)^2 D[\omega_q]^2}{2\sigma[\omega_q]}\,dx \tag{4}$$

resulting in the following differential equation for the shape of the optimal traveling wave

$$\big[q - v\rho_0 + v\omega_q(x)\big]^2 - \omega_q'(x)^2 D[\omega_q]^2 = 2\sigma[\omega_q]\Big\{C_1 + C_2\omega_q(x)\Big\}\,. \tag{5}$$

This equation yields a $\omega_q(x)$ which is generically a symetric function with a single minimum $\omega_1 = \omega(x_1)$ and maximum $\omega_2 = \omega(x_2)$ such that $|x_2 - x_1| = 1/2$. The constants C_1 and C_2 can be then related to these extrema, which in turn are fixed by the constraints on the total mass of the system and the distance between extrema,

$$\frac{\rho_0}{2} = \int_{\omega_1}^{\omega_2} \frac{\omega D(\omega)}{Z_v(\omega)}d\omega \quad ; \quad \frac{1}{2} = \int_{\omega_1}^{\omega_2} \frac{D(\omega)}{Z_v(\omega)}d\omega\,, \tag{6}$$

where $Z_v(\omega) = [(q - v\rho_0 + v\omega)^2 - 2\sigma(\omega)(C_1 + C_2\omega)]^{1/2}$. The optimal wave velocity is given implicitely by $v = -q\nu_1^{(v)}/\nu_2^{(v)}$ [14], with

$$\nu_n^{(v)} \equiv \int_{\omega_1}^{\omega_2} \frac{D(\omega)\big(\omega-\rho_0\big)^n}{\sigma(\omega) Z_v(\omega)}\,d\omega \tag{7}$$

In this way, for a constant ρ_0 and a given q, we use eqs. (6)-(7) to compute the profile extrema and its velocity, and use this information to solve eq. (5) for $\omega_q(x)$.

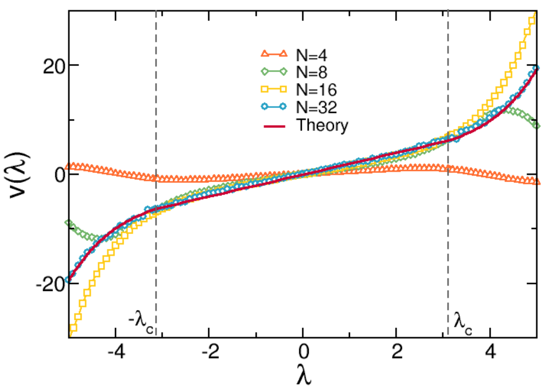

FIG. 4. (Color online) Velocity measured as a function of λ for increasing N, and HFT result.

In order to test the whole picture, we also measured the average energy profile associated to a given current fluctuation in simulations [18], see Fig. 3. Due to the system periodicity, and in order not to blur away the possible structure, we performed profile averages around the instantaneous center of mass. For that, we consider the system as a 1D ring embeded in two-dimensional space, and compute the angular position of the center of mass, shifting it to the origin before averaging. Notice that this procedure yields a spurious weak structure in the subcritical region, equivalent to averaging random profiles around their (random) center of mass. Such spurious profile is of course independent of q, and can be easily substracted. On the other hand, supercritical profiles exhibit a much pronounced structure resulting from the appearance of a traveling wave, see Fig. 1.b. Top panel in Fig. 3 shows the measured $\omega_\lambda(x)$ for different $\lambda > \lambda_c$ and varying N. Again, fast convergence toward the HFT result is observed, with excellent agreement for $N = 32$ in all cases. Bottom panel in Fig. 3 shows the measured profiles for $N = 32$ and different λ, which closely resembles the HFT scenario, see inset. We also measured the average velocity associated to a given current fluctuation by fitting the motion of the center of mass during small time intervals Δt to a ballistic law, $x_{\rm CM}(t+\Delta t)-x_{\rm CM}(t) = v\Delta t$, see e.g. Fig. 1.b, and making statistics for the measured velocity. Fig. 4 shows the mean velocity for $\Delta t = 100$ Monte Carlo steps as a function of λ for increasing values of N, and the agreement with HFT is again very good already for $N = 32$ (other values of Δt yield equally good results). Notice that for subcritical current fluctuations the velocity is simply proportional to the current, while above the critical line the relation becomes nonlinear.

In summary, we have shown that an isolated diffusive system exhibits a phase transition at the fluctuation level. This phenomenon, which is fully captured by hydrodynamic fluctuation theory, is most surprising as it happens in an equilibrium system in the absence of external fields, breaking spontaneously a symmetry in 1D. This illustrates the idea that critical phenomena not allowed in equilibrium steady states may however arise in their fluctuating behavior or under nonequilibrium conditions [15]. Our results strongly support that the phase transition is continuous as conjectured in [14], excluding the possibility of a first-order scenario, and suggest that a traveling wave is in fact the most favorable time-dependent profile once additivity is broken. This observation may greatly simplify general time-dependent calculations, but the question remains to whether this is the whole story or other, more complex solutions may play a dominant role for even larger fluctuations. In any case, it seems clear that rare events call in general for coherent, self-organized patterns in order to be sustained [20].

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