Pendulum, elliptic functions, and relative cohomology classes

J.-P. Françoise,¹,a P. L. Garrido,²,b and G. Gallavotti³,c

¹Laboratoire J.-L. Lions, Université P.-M. Curie, UMR 7598 CNRS, 175 Rue de Chevaleret, Paris 75252, France
²Institute Carlos I for Computational and Theoretical Physics, Universidad de Granada, E-18071, Granada, Spain
³Dipartimento di Fisica and INFN, Università di Roma “La Sapienza,” 00185, Roma, Italia

(Received 15 September 2009; accepted 4 January 2010; published online 5 March 2010)

Revisiting canonical integration of the classical pendulum around its unstable equilibrium, normal hyperbolic canonical coordinates are constructed and an identity between elliptic functions is found whose proof can be based on symplectic geometry and global relative cohomology. Alternatively it can be reduced to a well known identity between elliptic functions. Normal canonical action-angle variables are also constructed around the stable equilibrium and a corresponding identity is exhibited. © 2010 American Institute of Physics. [doi:10.1063/1.3316076]

I. PENDULUM NEAR THE SEPARATRIX

The theory of Jacobian elliptic functions, for reference, see Ref. 1, yields a complete calculation for the motion of a pendulum as a function of time. This is revisited here, to exhibit a few interesting properties of the elliptic integrals.

Write the pendulum energy, with inertia moment $I$ and gravity constant $g$ (rather than the usual $g$), in the canonical coordinates $(B, \beta)$ or as

$$\frac{B^2}{2I} - Ig^2(1 - \cos \beta) \equiv H(B, \beta),$$

where the origin in $\beta$ is set at the unstable equilibrium: the definition implies that $g$ has dimension of inverse time and the Lyapunov exponents of the unstable equilibrium are $\pm g$. $B = I\beta$ and $\beta$ are canonical coordinates for the motions.

It is well known that near the unstable equilibrium of the pendulum $B=0$, $\beta=0$, it is possible to define a canonical transformation, mapping the origin into itself, introducing new local coordinates $(p, q)$, such that

$$B = R(p, q), \quad \beta = S(p, q),$$

with $R, S$ holomorphic in a polidisk $|p|, |q| < \kappa$ with $\kappa > 0$, and in terms of which the motion near $B=\beta=0$ is described by a Hamiltonian $G$ depending on the product $pq$ only, of the form $\dot{u}(p, q) = H(B, \beta)$ with $[dH/d(pq)](0) = g$.

The purpose of this paper, which includes an unpublished note² where a proof of the latter statement via the theory of elliptic functions was derived, is to provide an alternative proof of the main formula, Eq. (5.1). The interest of the new derivation is that it is deductive in nature and it

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¹Electronic mail: jpf@math.jussieu.fr.
²Electronic mail: garrido@onsager.ugr.es.
³Electronic mail: giovanni.gallavotti@romal.infn.it.
II. SOLUTION IN TERMS OF ELLIPTIC INTEGRALS

Motions near the unstable equilibrium have a quite different nature depending on the sign of the total energy \( H(B, \beta) = U \): the ones with \( U < 0 \) are “oscillations” (their motions do not encompass the full perimeter of the circle) while the ones with \( U > 0 \) are “librations.” Therefore, it will not be possible to find global action-angle coordinates: motions near the separatrix (which with our conventions has \( U = 0 \)) require other coordinates to be expressed in a simple way.

Introduce the variables that appear in the theory of Jacobi’s elliptic functions,

\[
k' = \sqrt{1 - k^2}, \quad h' = \frac{k}{\sqrt{1 + k^2}}, \quad h = \sqrt{1 - h'^2},
\]

\[
U = 2g^2 \frac{l}{k^2}, \quad u = t \sqrt{\frac{U}{2l}}, \quad g_0 = g \frac{\pi}{2h' K(h)},
\]

where \( K(k) = \int_0^\pi \frac{1}{\sqrt{1 - k^2 \sin^2 \alpha}} d\alpha \). Hence the separatrix has \( k = +\infty \) and \( U = 0 \); and the data above the separatrix correspond to \( U > 0 \) (or \( k > 0 \)). Note that \( g_0(0) = g \) because \( K(0) = \pi/2 \) (as \( U = 0 \) corresponds to \( k = \infty \) and \( h' = 1, h = 0 \)); the following formulas become singular as \( U \rightarrow 0 \), but the singularity is only apparent and it will disappear from all relevant formulas derived or used in the following.

Other important quantities in the elliptic functions theory are, see Ref. 1, (8.198.1), (8.198.2), and (8.146),

\[
\chi' = \xi(h) = e^{-\pi K(h')/K(h)} = \lambda + 2\lambda^5 + 15\lambda^9 + \cdots \)

\[
\lambda = \frac{1}{2} \frac{1 - \sqrt{h'}}{1 + \sqrt{h'}} = \frac{\sum_{n=0}^{\infty} \xi(h)(2n + 1)^3}{1 + 2\sum_{n=1}^{\infty} \xi(h)^{2n+1}},
\]

where \( \xi(k') \) denotes here what in Ref. 1 would be \( q(k') \) [Ref. 1, (8.146.1) and (8.194.2)].

In terms of the above conventions we have, directly from the definitions of am, en, sn, dn (Jacobi’s elliptic functions [Ref. 1, (8.141)]), and from the equations of motion,

\[
\beta(t) = 2\text{am}(u, ik), \quad u = \frac{18g}{k} \quad B(t) = I\beta = \frac{2Ig}{k} \text{dn}(u, ik)
\]

[Ref. 1, (8.143) and (8.141)]. So that the action \( B \) is given as a function of time,
\[ B(t) = \frac{2I_g}{k \operatorname{dn}(\frac{u}{h}, h')} = 2I_g \frac{\operatorname{cn}(-i\frac{u}{h}, h)}{k \operatorname{dn}(-i\frac{u}{h}, h)} \]  

(2.4)

[Ref. 1, (8.153.9) and (8.153.3)] assuming that initial data are assigned with \( \beta = 0 \).

The \( t \) dependence of \( B(t) \) is naturally expressed via the argument \( u/h = gt/khK(h) \), if the second of Eq. (2.4) is used, since \( kh = h' \), see Eq. (2.1). This explains the important role that the quantity

\[ g_0(x') \overset{\text{def}}{=} g_0 = \frac{\pi}{2khK(h)} = \frac{\pi}{2h'K(h)} \]

(2.5)

will play in the following analysis. The \( g_0(x') \) admits a rather simple product expansion, [Ref. 1, (8.197.1) and (8.197.4)],

\[ g_0(x') = g \prod_{n=1}^{\infty} \left( \frac{1 + x'^n}{1 - x'^n} \right)^2 \]

(2.6)

and its logarithmic derivative is \( 4\sum_{n=1}^{\infty} \left( n\xi'^n - 1 - x'^{2n} \right) \) so that \( x'(d/dx') \log g_0(x') \) is (1/2) \( \times(d^2/dz^2) \log \theta_4(z, x') \) for \( \theta_4 \) (Ref. 4, pp. 463 and 489) is a Jacobi’s theta function.

It is also convenient to remark that in a motion with energy \( U \) it will be

\[ H(B(t), \beta(t)) = U = 2g^2 I \frac{1}{k^2}. \]

(2.7)

### III. POWER SERIES REPRESENTATION

From the theory of elliptic functions the evolution \( B(t), \beta(t) \) with any initial data above the separatrix [i.e., with \( \beta(0) = 0 \) and \( B(0) = 1/\beta(0) \) corresponding to a given value of \( h \), with \( U > 0 \)], can be expressed as \( B(t) = \bar{R}(\gamma, \delta) \) and \( \beta(t) = \bar{S}(\gamma, \delta) \) with \( \gamma = e^{i\theta t}, \delta = e^{-i\theta t} \) and, taking into account

Ref. 1, (8.146.11),

\[ \bar{R}(\gamma, \delta) = -4g_0 \sum_{n=1}^{\infty} \left[ \frac{(-1)^n \xi'^{n-1/2} (\delta^{2n-1} + \delta^{2n-1})}{1 - \xi^{2n-1}} \right] \]

(3.1)

with \( \xi = \xi(h) \). Definitions in Eqs. (2.1) and (2.2) yield

\[ g_0(\xi) = g \frac{\pi}{2} \frac{1}{\sqrt{1 - h^2K(h)}} = g \left( 1 + \frac{1}{4} h^2 + \cdots \right) \]

(3.2)

which is analytic in \( h^2 \) by Ref. 1, (8.113.1) near \( h = 0 \).

Equation (2.2) implies that \( \xi = \lambda + O(\lambda^5) \) is analytic in \( \lambda \) near \( \lambda = 0 \) so that \( h^2 = 16\lambda + \cdots = 16\xi + \cdots \). Therefore,

\[ g_0 = (1 + 4\xi + 12\xi^2 + \cdots) g \]

(3.3)

is analytic in \( \xi \) near \( \xi = 0 \).

The evolution of \( \phi \) is then a consequence of Eq. (3.1) which leads to an expression for \( \bar{S} \) by the remark that \( \bar{R} = g_0(\gamma\delta - \delta\gamma) \bar{S} \) (just expressing that \( B \) is \( I \) times the derivative of \( \beta \)): namely,
\[
\tilde{S}(\gamma, \delta) = -4 \sum_{n=1}^{\infty} \frac{(-1)^n \gamma^{n+1/2}}{1 - \xi^{2n-1}} \frac{\gamma^{2n-1} - \delta^{2n-1}}{2n-1}
\]  

(3.4)

and, after developing in powers of \(\xi\) the denominators and resuming,

\[
\tilde{S} = 4 \sum_{m=0}^{\infty} \left( \arctan(\xi^m \gamma \xi) - \arctan(\xi^m \gamma \xi) \right),
\]

\[
R = 4Ig_0 \sum_{m=0}^{\infty} \left( \frac{\xi^m \gamma \xi}{1 + (\xi^m \gamma \xi)^2} + \frac{\xi^m \gamma \xi}{1 + (\xi^m \gamma \xi)^2} \right).
\]

(3.5)

The first formula reminds of one found by Jacobi which he commented by saying that “\textit{inter formulas elegantissimas censeri debet}” (Ref. 4, p. 509) (i.e., “it should be counted among the most elegant formulae”).

Note that \(g_0\) depends only on \(\xi\), see Eq. (3.2), which would be surprising if the mechanical interpretation was not taken into account. Equation (3.5) exhibits the convergence of the map \((B, \beta) \leftrightarrow (\xi, \gamma)\) since \(\xi < 1\) in the region above the separatrix: in the latter region Eq. (3.5) provides a convergent expansion of the solution.

**IV. HYPERBOLIC COORDINATES**

Motions with initial coordinate \(\beta(0) \neq 0\) also admit a rather simple representation. Remark that all pendulum motions with \(\beta > 0\) (hence different from the two equilibria) pass at some time through a phase space point with \(\beta = 0\). If \(\dot{\beta}\) is their velocity at that moment we can find a quantity \(\xi\), such that \(\beta, \dot{\beta}\) are given by Eq. (3.5) with \(\gamma = \delta = 1\) Therefore, they can be represented, at least as long as \(U > 0, \dot{\beta} > 0\), by introducing the \textit{dimensionless} variables \(q' = \gamma \sqrt{\xi}, p' = \delta \sqrt{\xi}\) and allowing \(\delta, \gamma\) to be arbitrary. Then the motions will be \(t \rightarrow (p' e^{i\theta_0}, q' e^{-i\theta_0})\) showing that the motion can be represented by the following two functions:

\[
S' = 4 \sum_{m=0}^{\infty} \left( \arctan((p' q')^m q') - \arctan((p' q')^m p') \right),
\]

(4.1)

\[
R' = 4Ig_0 \sum_{m=0}^{\infty} \left( \frac{(p' q')^m p'}{1 + (p' q')^m p'} + \frac{(p' q')^m q'}{1 + (p' q')^m q'} \right).
\]

The motions \(t \rightarrow (p' e^{i\theta_0}, q' e^{-i\theta_0})\) solve the equations of motion if \(p', q'\) (i.e., \(\gamma, \delta\)) are positive. But the equations of motion are analytic, hence the formulas (4.1) together with \(t \rightarrow (p' e^{i\theta_0}, q' e^{-i\theta_0})\), with \(g_0 = g_0(p' q')\), give solutions of the pendulum equations independently of the sign of \(p', q'\), provided the series converge. The convergence requires \(|p' q'| < 1\): which represents many data, in particular, those in the vicinity of the separatrix.

Hence in the domain where \(|p' q'| < 1\) the motion is linearized in the sense developed in Ref. 5 but it is not yet in symplectic coordinates.

The coordinates can be called “hyperbolic” being suitable to describe motions near the separatrix (where \(p' q' = 0\)). We also see that time evolution preserves both volume elements \(dBd\beta\) and \(dp' dq'\); which means that the Jacobian determinant \(\partial(B, \beta)/\partial(p', q')\) must be a function constant over the trajectories, hence a function \(D(x')\) of \(x' = p' q'\). Note that \(D(x')\) has dimension of an action.

It is then possible to change coordinates setting \(p = a(x') p', q = a(x') q'\) and choose \(a(x)\) so that the Jacobian determinant for \((B, \beta) \leftrightarrow (p, q)\) is def \(\equiv 1\). A brief calculation shows that this is achieved by fixing
\[ a^2(x') = \frac{1}{x'} \int_0^{x'} D(y) dy, \]  

(4.2)

which is possible for \( x \) small because, from Eqs. (3.5) and (2.2), it is \( D(0)=32Ig>0 \). Therefore, the variables, which will have the dimension of \( a \), hence of a square root of an action,

\[ p = p'(a(x')), \quad q = q'(a(x')), \]  

(4.3)

have Jacobian determinant 1 with respect to \((B, \beta)\) and the map \((B, \beta) \leftrightarrow (p, q)\) is area preserving, hence canonical. The Hamiltonian, Eq. (1.2), becomes a function \( U(x) \) of \( x=pq \) and the derivative of the energy with respect to \( x \) has to be \( g_0(x') \) (because the \( p, q \) are canonically conjugated to \( B, \beta \)). Note that \( x \) has the dimension of an action, while \( p, q \) are, dimensionally, square roots of action.

This allows us to find \( D(x') \): by imposing that the equations of motion for the \((p, q)\) canonical variables have to be the Hamilton’s equations with Hamiltonian \( U(x) = U(x) = H(B, \beta) \) it follows that \( dU(x)/dx = g_0(x') \), i.e., \([dU(x')/dx'](dx'/dx) = g_0(x') \) or \( dU(x')/dx' = g_0(x')(d/dx') \times(x'(a(x')^2)) = g_0(x')D(x') \) by the above expression for \( a(x') \). The just obtained relation, together with Eq. (2.2), gives

\[ D(x') = g_0(x')^{-1} \frac{d}{dx} U(x'), \]  

(4.4)

which is an explicit expression for the Jacobian \( \partial(B, \beta)/\partial(p', q') = \partial(R, S)/\partial(p', q') = \partial(p, q)/\partial(p', q') \) [note that the Jacobian between \((B, \beta)\) and \((p, q)\) is identically 1 by construction]. Equation (4.4) is dimensionally correct because \( x' \) is dimensionless so that \( U(x') \) has the correct dimension (i.e., energy).

The function \( U(x') \) is in Eq. (2.7) where \( k^2 = h'^2/h^2 \), by Eq. (2.1), is related to \( x' = \xi(h) \) by Eq. (2.2), so that [Ref. 1, (8.197.3) and (8.197.4)]

\[ U(x') = 2g^2 \frac{1}{k^2} \frac{h^2}{h'^2} = 32Ig^2 x' \prod_{n=1}^{\infty} \left( 1 + x'^{2n} \right)^{\frac{8}{1 - x'^{2(2n-1)}}}. \]  

(4.5)

To complete the determination of the canonical hyperbolic coordinates it remains to find an expression for \( D(x') \), \( U(x) \) in terms of the elliptic functions to obtain the canonical variable and the Hamiltonian in closed form (rather than as power series as done so far).

V. DETERMINATION OF THE JACOBIAN: REMARKS

It is remarkable that the function \( a^2 \) defined above, hence such that \( D(x') = d/dx'(x' a^2(x')) \), seems to be simply

\[ a^2(z) = 8I \frac{d}{dz} g_0(z), \]  

(5.1)

in a common holomorphy domain, for both sides, around \( z=0 \). This is suggested by the agreement of the first 200 coefficients of the expansion of the two sides in powers of \( z \); however, this is not a proof and the relation, Eq. (5.1), holds because it can be seen to be equivalent to an identity on elliptic functions, as discussed in Appendix B below, or it can be independently derived from symplectic geometry, as discussed in Appendix A below.

Remarks:

1. The expansion of \( D(x') \) in powers of \( x' \) can be derived from Eqs. (4.5) and (2.6), while that of \( a^2(x') \) is obtained from Eq. (5.1) and, again, Eq. (2.6).
(2) It is perhaps natural to guess that the function \( a(x')^2 \) should be closely related to \( g_0(x') \); this is a guide to its determination as it becomes, then, natural to look for it among the derivatives of \( g_0 \) with respect to \( x' \). By dimensional analysis all \( x' \)-derivatives of \( I_0 \) are of the same dimension as \( x'^2 \).

Looking also at the derivatives of \( g_0 \) as candidates for \( x' \) is an idea due to one of us (P.G.).

This follows a similar line of thought which led to a conjecture on the canonical integrability of the “Calogero lattice,” whose proof was discovered in two subsequent works (Refs. 7 and 8).

Other peculiarities are, setting \( 2! = 1 \), \( g = 1 \), the following.

(1) The function \( g_0(x') \), \( U(x') \), hence \( (d/dx')g_0(x') \), \( D(x') \), have Taylor coefficients in powers of \( x' \) which are all positive integers as it follows from the relations (4.5) and (2.6), while \( U(x) = x \) seems to have alternating sign Taylor coefficients,

\[
U(x) - x = 2x^2 - 4x^3 + 20x^4 - 132x^5 + 1008x^6 + \ldots, \tag{5.2}
\]

where \( U(x) \) is obtained by power series inversion of \( x' = a(x')^2 \) and from \( U(x) = U(x') \) together with Eq. (4.5).

(2) The function \( U(x') \), energy of the pendulum expressed as a function of \( x' \), has also the form

\[
U(x') = 32I_0 [f' U_{xx}(p') + q' V_{xx}(q')] \int [f' V_{xx}(p') + q' U_{xx}(q')] = x'f(x'), \tag{5.3}
\]

which, remarkably, has by Eq. (4.5) to depend only on \( x' \), and has the form \( x'f(x') \) for some \( f \). This is not a priori evident, unless the mechanical interpretation is kept in mind, from the expressions found for \( U, V \), namely,

\[
U_{xx}(z) = \sum_{\ell=0}^{\infty} \frac{x^{'2\ell}}{1 + (x^{'2\ell + 1}z^2)}, \quad V_{xx}(z) = \sum_{\ell=0}^{\infty} \frac{x^{'2\ell + 1}}{1 + (x^{'2\ell + 1}z^2)}. \tag{5.4}
\]

(3) The existence of an analytic canonical map integrating, near the hyperbolic point, the system with energy Eq. (1.1) into one with Hamiltonian \( U(pq) = gpq + O((pq)^2) \), is well known; it can be established without an explicit calculation by perturbation analysis, see Ref. 3, Appendix A3, for instance.

**APPENDIX A: A DEDUCTIVE PROOF OF EQ. (5.1)**

A first proof for this formula can be based directly on the equalities on symplectic forms,

\[
\text{d}B \wedge \text{d}B = D(\xi) \text{d}p' \wedge \text{d}q' = \frac{d}{d\xi} \text{d}(\xi^2) \text{d}p' \wedge \text{d}q'. \tag{A1}
\]

In principle, the computation of \( \text{d}S = (p', q') \wedge \text{d}R = (p', q') \) should provide the value of the function \( D \). But the computation gets too involved. The idea is to compute only the class of the volume forms in the relative cohomology of the function \( \xi = p'q' \). This type of computation is well known in the theory of limit cycles of plane vector fields, but is perhaps more novel in the context of Hamiltonian dynamics.

Recall that any holomorphic 2-form \( \phi(p', q') \text{d}p' \wedge \text{d}q' \) has a unique decomposition,

\[
\phi(p', q') \text{d}p' \wedge \text{d}q' = \psi(\xi) \text{d}p' \wedge \text{d}q' + d\xi \wedge d\eta. \tag{A2}
\]

Remark that the function \( \psi(\xi) = \sum_{n=0}^{\infty} \phi_n(p'q')^n \) is obtained from \( \phi(p', q') = \sum_{m,n=0}^{\infty} \phi_{mn}p'^m q'^n \) by collecting all terms in the series of equal exponents for both variable \( p' \) and \( q' \) and \( \eta = \sum_{n=0}^{\infty} \phi_{nm}p'^m q'^n / (m-n) \). This is a consequence of the identity

\[
\psi(\xi) = \sum_{n=0}^{\infty} \phi_{nm}p'^m q'^n / (m-n). \tag{A3}
\]
\[ p^n q^m dp' \wedge dq' = d(p' q') \wedge d\left(\frac{p^n q^m}{m-n}\right) \]  

(A3)

by linearity.

The class of cohomology of the 2-form \( \phi(p', q') dp' \wedge dq' \) relative to the function \( \xi \) is defined as the quotient of the holomorphic 2-forms modulo the 2-forms of type \( d\xi \wedge d\eta \) for \( \eta \) holomorphic. It is conveniently represented by \( \psi(\xi) dp' \wedge dq' \) [or equivalently by the function \( \psi(\xi) \)]. In the local version near an isolated singularity, this is a very special case of a general theory due to Brieskorn and Sebastiani.\(^{11} \)

There is another equivalent representation of the cohomology class of a 2-form \( \phi(p', q') dp' \wedge dq' \). Write

\[ \phi(p', q') dp' \wedge dq' = d[f(p', q') p' q' \omega], \]  

(A4)

with

\[ \omega = \frac{1}{2} d \log \left(\frac{q'}{p'}\right) = \frac{1}{2} \left[ \frac{dq'}{q'} - \frac{dp'}{p'} \right]. \]  

(A5)

Then this yields

\[ \phi(p', q') = f(p', q') + \frac{1}{2} \left( p' \frac{\partial f}{\partial p'} + q' \frac{\partial f}{\partial q'} \right), \]  

(A6)

which defines a 1-1 linear correspondence between \( \Sigma_{m,n} \phi_{mn} p'^m q'^n \) and \( \Sigma_{m,n} f_{mn} p'^m q'^n \) by

\[ \phi_{mn} = f_{mn} \left[ 1 + \frac{1}{2} (m + n) \right]. \]  

(A7)

Remark that this 1-1 transformation defines by restriction a 1-1 correspondence between \( \psi(\xi) = \Sigma \phi_{mn}(p', q')^n \) and \( F(\xi) = \Sigma f_{mn}(p', q')^n \). In other words, the cohomology class of the 2-form \( \phi(p', q') dp' \wedge dq' \) is uniquely defined by the function \( F(\xi) \) or equivalently by \( \psi(\xi) \).

The application to our problem of finding an expression for \( a(\xi) \) can be implemented by remarking that

\[ D(\xi) dp' \wedge dq' = \frac{d}{d\xi} \left[ \xi a(\xi)^2 \right] dp' \wedge dq' = \frac{d}{d\xi} \left[ \xi a(\xi)^2 \right] d\xi \wedge \omega = d[\xi a^2(\xi) \omega]. \]  

(A8)

So that, by Eq. (A1), the cohomology class of the symplectic form \( dB \wedge d\beta \) with respect to \( \xi \) [i.e., \( D(\xi) \)] corresponds to the function \( a(\xi)^2 \) via Eq. (A7), and we can use the formulas (3.1) to compute it.

The symplectic form can be written as

\[ dB \wedge d\beta = dR'(p', q') \wedge dS'(p', q') = d[R'(p', q')dS'(p', q')], \]  

(A9)

and this yields

\[ R'(p', q') dS'(p', q') = 16f_0(\xi) \left\{ \sum_{m=0}^{\infty} \left[ \frac{(p' q')^m p'}{1 + ((p' q')^m p')^2} + \frac{(p' q')^m q'}{1 + ((p' q')^m q')^2} \right] \right\} \]  

\[ \cdot \left\{ \sum_{l=0}^{\infty} \left[ \frac{(p' q')^l dq'}{1 + ((p' q')^l q')^2} - \frac{(p' q')^l dp'}{1 + ((p' q')^l p')^2} \right] \} \]  

(A10)

modulo \( d(p' q') \).

In this double sum, it is convenient to first isolate the terms corresponding to both \( l=m=0 \).
relative cohomology. The term
\[
16l_g(\xi) \left[ \frac{dp'}{1 + p'^2} - \frac{dq'}{1 + q'^2} \right] = 16l_g(\xi) \frac{p'dp' - q'dq'}{(1 + p'^2)^2}.
\]
(A11)
The term
\[
16l_g(\xi) \left( \frac{p'dp' - q'dq'}{(1 + p'^2)^2} \right)
\]
gives 0 in the relative cohomology. The term
\[
16l_g(\xi) \frac{p'dq' - q'dp'}{(1 + q'^2)(1 + p'^2)} = 32l_g(\xi) \frac{p'q'}{(1 + q'^2)(1 + p'^2)^2}.
\]
(A13)
contributes with
\[
32l_g(\xi) p'q' \sum_{i=0}^{+\infty} (p'q')^{2i} \omega,
\]
(A14)
and hence
\[
32l_g(\xi) \frac{1}{1 - (p'q')^{2} \omega}
\]
in calculating the relative cohomology by Eqs. (A4) and (A7).
The other terms of the double sum can be written as
\[
\sum_{l,m; l+m \geq 1} \left( \frac{(p'q')^{l+m}}{1 + [(p'q')^m p'^2]} \frac{(p'q')^{l+m-1} q'^2}{1 + [(p'q')^m q'^2]} \right) p'dq'
\]
\[- \sum_{l,m; l+m \geq 1} \left( \frac{(p'q')^{l+m}}{1 + [(p'q')^m p'^2]} \frac{(p'q')^{l+m-1} p'^2}{1 + [(p'q')^m q'^2]} \right) q'dp',
\]
(A16)
hence they have the form
\[
P(q',p')p'dq' + Q(q',p')q'dp' = \alpha(q',p')(p'dq' + q'dp') + \beta(q',p')(p'dq' - q'dp'),
\]
(A17)
with \(\alpha=(P+Q)/2, \beta=(P-Q)/2\). Notice then that the term \(\alpha(q',p')(p'dq' + q'dp')\) gives 0 in relative cohomology. The term \(\beta(q',p')(p'dq' - q'dp')\) yields
\[
8l_g(\xi) \sum_{l,m; l+m \geq 1} \left( (p'q')^{l+m-1} q'^2 \right) \left( \frac{2(p'q')^{l+m}}{1 + [(p'q')^m p'^2]} \frac{(p'q')^{l+m-1} p'^2}{1 + [(p'q')^m q'^2]} \right)
\]
\[\quad \quad \quad + \frac{(p'q')^{l+m-1} q'^2}{1 + [(p'q')^m p'^2]} \frac{(p'q')^{l+m-1} p'^2}{1 + [(p'q')^m q'^2]} \right).
\]
(A18)
The series formed by the addends
This can be used to obtain an expression for \( \frac{d}{dq} \) leading to the equality, recalling Eq. (A8),

\[
d^2 \xi = 8I \frac{dg_0(\xi)}{d\xi}.
\]

**APPENDIX B: ALTERNATIVE PROOF OF EQ. (5.1) VIA AN IDENTITY BETWEEN ELLIPTIC FUNCTIONS**

Alternatively the formula can be reduced to a well known identity between elliptic functions. Calling \( E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \alpha)^{1/2} d\alpha \), it is \( E(h) = h h' (dK(h)/dh) + h^2 K(h) \) see Ref. 1, (8.123.2), and \( E(h) K(h') + E(h') K(h) - K(h) K(h') = \pi/2 \) see Ref. 1, (8.122); the latter “Legendre’s relation,” (Ref. 4, p. 520), combined with \( dh'/dh = -h'/h'' \), yields the identity

\[
h'' h' \left( K(h) \frac{dK(h')}{dh'} - K(h') \frac{dK(h)}{dh} \right) = \frac{\pi}{2}.
\]

This can be used to obtain an expression for \( d \log x'/dh' \): keeping in mind \( x' = e^{-\pi K(h')/K(h)} \), it is

\[
( d \log x'/dh') = -\pi ((1/K(h))(dK(h')/dh') - (K(h')/K(h))^2 (dK(h)/dh'))
\]

which is transformed
into \((d \log x'/dh')= \log x'(1/IK(h'))(dK(h')/dh')-(1/IK(h))(dK(h)/dh')\).

Form Eq. (B1) it follows, therefore,

\[
\frac{d}{dh} \log x' = \frac{\log x'}{2h' h'^2 K(h) K(h')}, \quad \frac{d}{dh} \log x' = -\frac{\log x'}{2h' h'^2 K(h) K(h')},
\]

and the corresponding derivatives with respect to \(h\) are obtained by multiplying both sides by \(-h'/h'\).

To establish Eq. (5.1) consider the relation

\[
\frac{d}{dh} \left( h h'^2 K(h) \right) - h K(h) = 0,
\]

see Ref. 1, (8.124.1). This implies by simple algebra, and keeping in mind that \(h/h'=-dh'/dh\), the following identity:

\[
h K(h) = h'^1 \frac{d}{dh} \left( h h'^2 \left( -\frac{1}{h'} \frac{d}{dh} K(h) + \frac{h}{h'^3} K(h) \right) \right),
\]

which is a known linear equation, solved by \(K(h)\). This can be rewritten, since \(h/h'=-dh'/dh\), as

\[
\frac{h}{h'^2} K(h) = \frac{d}{dh} \left( h h'^2 \left( -\frac{1}{h'} \frac{d}{dh} K(h) - \frac{1}{h'^2} \frac{d}{dh} K(h) \right) \right) = \frac{d}{dh} \left( h h'^2 K(h) \frac{1}{h'^2} \frac{d}{dh} K(h) \right).
\]

Remarking that \(2h/h'^4=(d/dh)(h^2/h'^2)\), Eq. (B5) implies, multiplying both sides by \(2/h'K(h)\),

\[
\frac{d}{dh} \left( \frac{h}{h'} \right)^2 = \frac{2}{h'K(h)} \frac{d}{dh} \left( h h'^2 K(h)^2 \left( \frac{d}{dh} \frac{1}{h'^2} K(h) \right) \right) = \frac{2 \pi}{h'K(h)} \frac{d}{dh} \left( h h'^2 K(h) \frac{1}{h'} K(h) \left( \frac{d}{dh} \frac{1}{h'^2} K(h) \right) \right)
\]

and by the first of Eq. (B2) multiplied by \(dh'/dh=-h'/h\) this is, using \(k^2=h^2/h'^2\),

\[
\frac{d}{dh} \frac{1}{h'^2} = \frac{\pi^2}{h'K(h)} \frac{d}{dh} \log x' \left( \frac{d}{dh} \frac{1}{h'^2} K(h) \right) = \frac{\pi^2}{h'K(h)} \frac{d}{dh} \left( x' \frac{d}{dx'} \frac{1}{h'} K(h) \right),
\]

and multiplying by \(2 \pi g^2 (dh/dx')\) it follows

\[
2 \pi g^2 \frac{d}{dx'} \frac{1}{k'_2} = 8I \frac{\pi g}{2h'K(h)} \frac{1}{dx'} \left( \frac{d}{dx'} \frac{1}{h'} K(h) \right),
\]

and setting \(a(x')^2=8I(d/dx')(\pi g/2)(1/h'K(h))=8I(d/dx')g_0(x')\) the last relation is \((d/dx')U(x')=g_0(x')(d/dx')(x'a(x')^2)\) so that Eqs. (4.4) and (4.2) imply Eq. (5.1).

**APPENDIX C: PENDULUM AT THE STABLE EQUILIBRIUM: ACTION-ANGLE COORDINATES**

From the above results it is straightforward to find the canonical transformation that converts the pendulum Hamiltonian in its normal form around the stable equilibrium point. The Hamiltonian is now given by Eq. (1.1) with the substitution: \(g=ig\). It is natural to define \(k_s=ik\) in order to use the same set of equations from the unstable case. The system energy is then \(U_s=2g^2/k_s^2\) and large values of \(k_s\) correspond now to small oscillations around the equilibrium point.

Finally, it is convenient to define

\[
k'_s = \sqrt{1-k_s^2}, \quad h'_s(k_s) = \frac{k_s}{\sqrt{k_s^2-1}}, \quad h_s = \sqrt{1-h'_s^2},
\]
by looking for a function

\[ R_s = \frac{1}{h_s'(k_s)}, \quad h(k) = \frac{i h_s(k_s)}{h_s'(k_s)}, \quad (C1) \]

and one finds

\[ g_0^{(i)}(h_s) = -i g_0(h) = \frac{\pi}{2} \frac{g_s}{K(h_s)}, \quad (C2) \]

\[ x_s'(h_s) = e^{-\frac{\pi}{2}K(h_s)^2K(h_s)} = -x'(h), \]

where we have used Ref. 1, (8.128).

With these conventions, and going through computations similar to the ones performed to study the unstable point, the relations found for the latter can be converted into the corresponding ones for the equilibrium point. In particular, by choosing \( p' = \sqrt{x_s'} \cos(g_0^{(i)}) \) and \( q' = \sqrt{x_s'} \sin(g_0^{(i)}) \) the transformation given by Eq. (4.1) is now

\[ S_s' = 4i \sum_{m=0}^{\infty} (-1)^m \left(\frac{\pi}{2} \frac{g_s}{K(h_s)}(p'+iq') - \arctan((p'^2 + q'^2m)(p'-iq'))\right), \quad (C3) \]

\[ R_s' = -4i g_0^{(i)} \sum_{m=0}^{\infty} (-1)^m \left(1 - ((p'^2 + q'^2m(p'+iq'))^2 + \frac{(p'^2 + q'^2m(p'-iq'))^2}{1 - ((p'^2 + q'^2m(p'-iq'))^2)}\right), \]

where the relation \( R_s' = g_0^{(i)}(p' \partial_q - q' \partial_p)S_s' \) holds. Moreover, the energy can be written [see Eq. (4.5)]

\[ U_s(x'_s) = 2 g_0^2 I_k^2 = 32 I g^2 x'_s \prod_{n=1}^{\infty} \left(\frac{1 + x'^{2n}_{2s}}{1 + x'^{2n-1}_{2s}}\right)^8. \quad (C4) \]

The transformation \((B, \beta) \rightarrow (p', q')\) is not canonical. The canonical variables \((p, q)\) can be found by looking for a function \( a_s(x'_s) \) (which depends on the constant of motion \( x'_s \)), such that \((p, q) = (a_s(x'_s)p', a_s(x'_s)q')\) and the Jacobian of the transformation is one. It is, as in the hyperbolic case,

\[ a_s^2(z) = -16l \frac{d}{dz} g_0^{(s)}(z), \quad (C5) \]

Finally, the normal form of the Hamiltonian now reads

\[ U_s(x) = 32 I g^2 W\left(\frac{x}{64 I g^2}\right), \quad (C6) \]

where

\[ W(z) = z(1 - 2z - 4z^2 - 20z^3 - 132z^4 - 1008z^5 \ldots), \quad (C7) \]

which can be compared with the hyperbolic case expression, Eq. (5.2),

\[ U(x) = -32 I g^2 W\left(-\frac{x}{32 I g}\right). \quad (C8) \]

Action-angle coordinates \((A, \alpha) \in \mathbb{R} \times \mathbb{T}\) around equilibrium are related to \((p, q)\) by \( p = \sqrt{2A} \cos \alpha, q = \sqrt{2A} \sin \alpha. \)


