



# On the effect of synaptic fluctuations during retrieval processes in neural network models

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## Abstract

We studied stochastic neural network models in which, in addition to long term learning, fast fluctuations of the synaptic intensities occur so that the neuron activity update takes place assuming a set of synaptic intensities taken at random from a given distribution. The influence of this distribution on the performance of the retrieval process and on the general behavior of the network is discussed. © 1999 Elsevier Science B.V. All rights reserved.

The mechanisms underlying synaptic dynamics and the influence of synaptic changes on the behavior of real neural networks is not fully understood. Experiments in the cortex and the hippocampus indicate a high variability of synaptic response to spike trains, and it seems that it is the dynamics of synaptic efficacy on short time-scales – rather than random noise – what causes this variability [1]. Experiments also indicate that individual synapses are extremely unreliable in spite of the high efficiency of the whole system [2]. It seems that the apparently noisy behavior is, rather than a drawback for the efficiency of the system, responsible for the observed high processing power and robustness of biological neural networks [3,4]. This is also suggested by the fact that different kinds of noise seem to couple each other to produce *stochastic resonance* under certain circumstances in biological systems [5]. We report here on our studies of the consequences of certain synaptic variations. Motivated by recent observations, we consider a neural-network model which, in addition to (slow) learning plastic-

ity, involves fast fluctuations of the synapse intensities around mean values corresponding to the given learning rule.

The model is a stochastic neural network of  $N$  binary, fully connected *neurons* whose configuration,  $\mathbf{s} = \{s_x = \pm 1; \mathbf{x} = 1, \dots, N\}$ , evolves in time competing with synaptic variations as given by the master equation

$$\begin{aligned} \partial_t P_t(\mathbf{s}, \mathbf{J}) &= p \sum_{\mathbf{x}} [\omega_{\mathbf{J}}(\mathbf{s}^{\mathbf{x}}; \mathbf{x}) P_t(\mathbf{s}^{\mathbf{x}}, \mathbf{J}) - \omega_{\mathbf{J}}(\mathbf{s}; \mathbf{x}) P_t(\mathbf{s}, \mathbf{J})] \\ &+ (1-p) \sum_{\substack{\mathbf{x}, \mathbf{y} \\ \mathbf{x} \neq \mathbf{y}}} \sum_{J'_{xy}} [\omega(J'_{xy} \rightarrow J_{xy}) P_t(\mathbf{s}, \mathbf{J}^{xy}) \\ &- \omega(J_{xy} \rightarrow J'_{xy}) P_t(\mathbf{s}, \mathbf{J})]. \end{aligned} \quad (1)$$

$\mathbf{J} = \{J_{xy} \in \mathfrak{R}\}$  is the synapse configuration, and  $\mathbf{s}^{\mathbf{x}}$  ( $\mathbf{J}^{xy}$ ) stand for  $\mathbf{s}$  ( $\mathbf{J}$ ) after the change  $s_x \rightarrow -s_x$  ( $J_{xy} \rightarrow J'_{xy}$ ). For simplicity, we assume

$$\omega_{\mathbf{J}}(\mathbf{s}; \mathbf{x}) = \varphi \left[ \frac{2}{T} h_{\mathbf{x}}(\mathbf{s}, \mathbf{J}) \right],$$

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where  $h_{\mathbf{x}}(\mathbf{s}, \mathbf{J}) = \sum_{y \neq x} J_{xy} s_y$  is a local field. We consider here the choices

$$\varphi(X) = \exp(-X/2) \quad (\text{rule } V),$$

$$\varphi(X) = 2(1 + \exp X)^{-1} \quad (\text{rule } K),$$

and

$$\varphi(X) = \min\{1, \exp(-X)\} \quad (\text{rule } M).$$

This has a familiar limit case. That is, the stationary solution of (1) for  $p = 1$  (quenched synaptic intensities) and  $J_{xy} = J_{yx}$ , is the canonical equilibrium state for temperature  $T$  and energy

$$H_{\mathbf{J}}(\mathbf{s}) = -\frac{1}{2} \sum_{\mathbf{x}} h_{\mathbf{x}} s_{\mathbf{x}}.$$

This is the Hopfield Hamiltonian if intensities  $J_{xy}$  are given by the Hebb's rule [6]. However, our interest is in the limit  $p \rightarrow 0$ , i.e., when synapses undergo very fast variations independently of the much more slowly-varying neurons, which we believe may correspond to some actual conditions in Nature. As discussed in detail in [7,8], one can invoke – once the learning process is completed – two time scales which are relevant in relation with (1): There is a fine time scale,  $\tau$ , in which neurons do not appreciably evolve while synapse intensities fluctuate locally and, on the other hand, neurons evolve in the presence of stationary synapses on the coarse scale  $t = p\tau$  for  $p \rightarrow 0$  and  $\tau \rightarrow \infty$ . Assuming  $\omega(J_{xy} \rightarrow J'_{xy})$  independent of the current  $\mathbf{s}$ , one may represent this situation by means of

$$\partial_t P_t(\mathbf{s}) = \sum_{\mathbf{x}} [\varpi(\mathbf{s}^{\mathbf{x}}; \mathbf{x}) P_t(\mathbf{s}^{\mathbf{x}}) - \varpi(\mathbf{s}; \mathbf{x}) P_t(\mathbf{s})], \quad (2)$$

where  $\varpi(\mathbf{s}; \mathbf{x}) = \int d\mathbf{J} f(\mathbf{J}) \omega_{\mathbf{J}}$  is the (effective) rate for transitions  $s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}}$  and  $f(\mathbf{J})$  is the distribution of synaptic fluctuations. The evolution of  $\mathbf{s}$  is thus by superposition of canonical mechanisms, each for a different local field  $h_{\mathbf{x}}(\mathbf{s}, \mathbf{J})$ , such that  $\varpi(\mathbf{s}; \mathbf{x})$  will not satisfy detailed balance in general. Therefore, the system evolves asymptotically towards a *nonequilibrium* steady state which strongly depends on both  $f(\mathbf{J})$  and  $\varphi(X)$ .

Let us consider in the following only functions  $f(\mathbf{J})$  such that

$$\langle J_{xy} \rangle \equiv \int d\mathbf{J} f(\mathbf{J}) J_{xy} = \frac{1}{N} \sum_{\mu} \xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu},$$

where  $\xi^{\mu} \equiv \{\xi_{\mathbf{x}}^{\mu} = \pm 1\}$  are  $\mu = 1, \dots, P$  memory patterns, so that fluctuations are around values corresponding to the Hebb's learning rule in this particular case of our model. The simplest behavior occurs for

$$f(\mathbf{J}) = f_1(\mathbf{J}) \equiv \prod_{\mathbf{x} \neq \mathbf{y}} \delta\left(J_{xy} - \frac{1}{N} \sum_{\mu=1}^P \xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu}\right),$$

i.e., each individual synapse has a fixed Hebb value. In practice, this avoids any synaptic fluctuations, in particular,  $\sigma_{xy}^2(\xi) \equiv \langle (J_{xy} - \langle J_{xy} \rangle)^2 \rangle = 0$ , and the model simply reduces to the Hopfield one. More interesting are the cases:

$$\begin{aligned} f_2(\mathbf{J}) &= \prod_{\substack{\mathbf{x}, \mathbf{y} \\ \mathbf{x} \neq \mathbf{y}}} \sum_{\mu=1}^P a_{\mu} \delta\left(J_{xy} - \frac{\xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu}}{N a_{\mu}}\right), \\ f_3(\mathbf{J}) &= \sum_{\mu=1}^P a_{\mu} \prod_{\substack{\mathbf{x}, \mathbf{y} \\ \mathbf{x} \neq \mathbf{y}}} \delta\left(J_{xy} - \frac{\xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu}}{N a_{\mu}}\right), \end{aligned} \quad (3)$$

$\sum_{\mu} a_{\mu} = 1$ . The variance is then  $\sigma_{xy}^2(\xi) = (P/N)^2 - N^{-2} \sum_{\mu, \nu} \xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{y}}^{\mu} \xi_{\mathbf{x}}^{\nu} \xi_{\mathbf{y}}^{\nu}$ . The choice  $f_2$  corresponds to synapses evolving in time independently of the others, taking at each time the corresponding synaptic intensity from one of the stored patterns, with probability  $a_{\mu}$ . The factorization in this function implies that fluctuations in different synapses are not correlated with each other. This case admits a 'quasi-canonical' representation in terms of an effective Hamiltonian [8]. This has the Hopfield structure in certain limits, and some exact results may then be obtained [7]. A main result for  $f_2$  is that such synaptic fluctuations add as an extra noise to the thermal one, which reflects the lack of correlations implied by  $f_2$ . More interesting is the superposition of products, each with probability  $a_{\mu}$ , in  $f_3$ . Each product corresponds to a spatial map of synapse intensities associated to the  $\mu$ th stored pattern. The synaptic system globally visits then, with some probability, the set of synapses characterizing each of the stored patterns. Therefore, the fluctuations described by  $f_3$  contain the spatial correlations in the stored patterns. This feature turns out to be essential for the interesting behavior we illustrate next.

Our analytical results here concern the limit  $N \rightarrow \infty$  for finite  $P$ ; however, both analytical and numerical results indicate that our conclusions hold well for arbitrarily large  $P$ . Consider the mean field approxima-

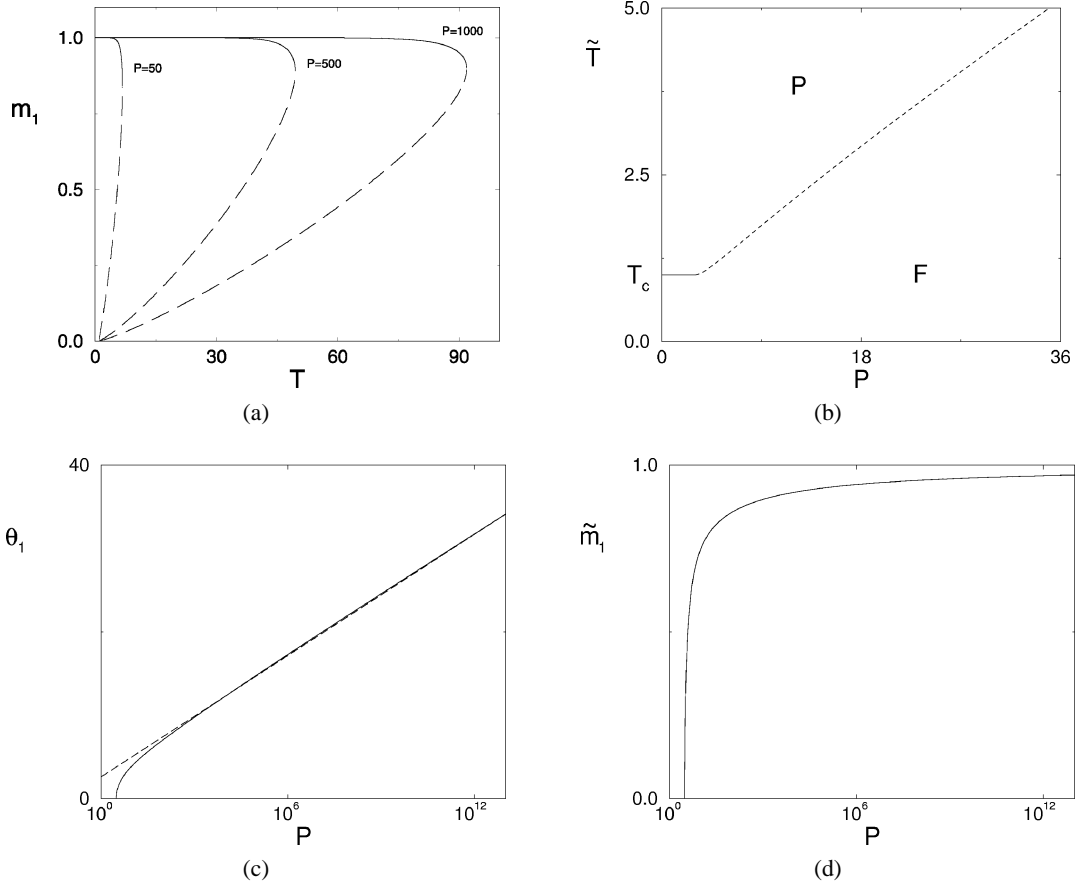


Fig. 1. Mean-field behavior of the stochastic network (2) for fluctuations  $f_3$  and rule  $V$ : (a) Pure states  $m_1(T)$  for  $P = 50, 500$  and  $1000$ ; solid and dashed lines correspond to stable and unstable solutions, respectively. (b) The phase diagram for  $n = 1$ ; the line indicates nonequilibrium phase transitions of first (dashed) and second (solid) order, separating a region in which the system exhibits associative memory ( $F$ ) from a region in which such property is absent ( $P$ ). Graphs (c) and (d): asymptotic behavior of  $\theta_1$  and  $\tilde{m}_1$ ; this illustrates a linear behavior of  $\theta_1$  with  $\ln P$  for  $P \in [10^4, 10^{13}]$  and that  $\tilde{m}_1 \rightarrow 1$  for  $P \rightarrow \infty$ .

tion  $s_{\mathbf{x}} = \langle s_{\mathbf{x}} \rangle$ , which is compatible with the fact that the local field  $h_{\mathbf{x}}$  involves long-range interactions, and either orthogonal patterns, namely,  $N^{-1} \sum_{\mathbf{x}} \xi_{\mathbf{x}}^{\mu} \xi_{\mathbf{x}}^{\nu} = \delta_{\mu\nu}$ , or *quasi-orthogonal*, as when  $\{\xi_{\mathbf{x}}^{\mu}\}$  is a set of  $P \times N$  statistically-independent random variables. Under these assumptions, one obtains from (2) for  $f_3$  that

$$\partial_t m_{\mu} = -2m_{\mu} \sum_{v=1}^P a_v B_v^+ - 2a_{\mu} B_{\mu}^-, \quad (4)$$

where  $m_{\mu} \equiv \frac{1}{N} \sum_{\mathbf{x}} \xi_{\mathbf{x}}^{\nu} \langle s_{\mathbf{x}} \rangle$  is the overlap of the mean network activity with the  $\mu$  pattern, and

$$B_{\mu}^{\pm} = \frac{1}{2} \left\{ \varphi \left( \frac{2}{a_{\mu} T} m_{\mu} \right) \pm \varphi \left( -\frac{2}{a_{\mu} T} m_{\mu} \right) \right\}.$$

The stationary solution of (4) is  $\mathbf{m} = (m_1, \dots, m_P)$ , with  $m_{\mu} = -a_{\mu} B_{\mu} / (\sum_{v=1}^P a_v B_v^+)^{-1}$ , and non-trivial solutions  $(m_1, \dots, m_n, 0, \dots, 0)$  with  $n \leq P$  and  $1 \geq |m_{\mu}| > 0$  appear continuously for  $T < T_c = 1$ ; pure states are for  $n = 1$ , and mixture states are for  $n > 1$  and  $\mathbf{m} = m_n(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$  for  $a_{\mu}$  independent of  $\mu$ . The behavior of the solutions for  $T < T_c$  and near  $T_c$  depends on the elementary rule  $\varphi$ . It ensues a second order phase transition for any  $\varphi$ ; for rule  $V$  this occurs only for  $3n > P$  while the solutions

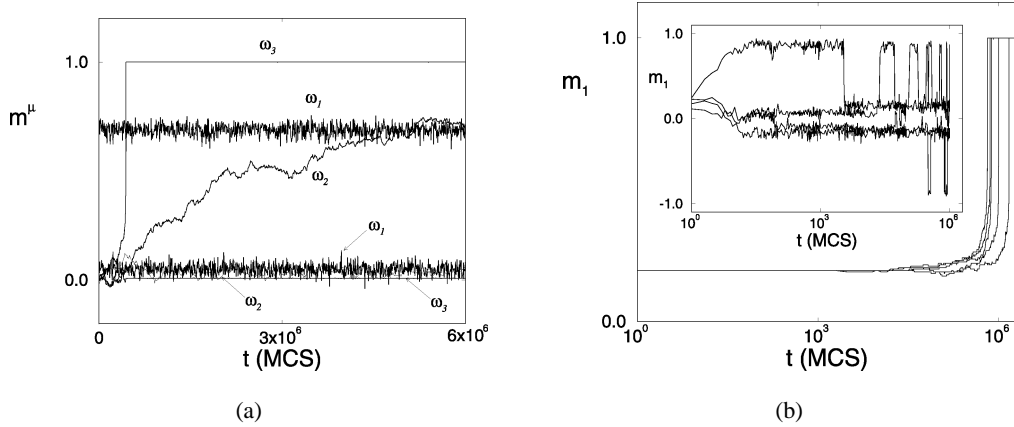


Fig. 2. Behavior from computer simulations of (2). This illustrates a dramatic improvement of pattern recognition processes for the fluctuations  $f_3$  and rule  $V$ : (a) Evolution with time of two overlaps  $m_\mu(s)$ ,  $\mu = 1, 2$ , for  $N = 2500$ ,  $P = 10$ , and  $T = 0.8$ ; the two noisiest trajectories correspond to the standard Hopfield model for rates  $\omega_1$  (horizontal line) and  $\omega_2$  (slowly increasing line), respectively; the more regular behavior is for  $\omega_3$ . (b) Independent evolutions at  $T = 0.6$  for  $N = 400$  and  $P = 10$ , starting with the same initial condition, for  $\omega_1$  (inset) and  $\omega_3$  (main graph).

are discontinuous for  $T < \tilde{T}(n, P) \equiv \tilde{m}_n P \theta_n^{-1}$ , where  $\tilde{T}(n, P < 3n) = T_c$ , with

$$\tilde{m}_n = \sinh \theta_n (n \cosh \theta_n + P - n)^{-1}$$

and

$$n \theta_n + (P - n)(\theta_n \cosh \theta_n - \sinh \theta_n)$$

$$-n \sinh \theta_n \cosh \theta_n = 0.$$

After linearizing the system (4), one obtains  $\partial_t n^\mu = \sum_{\nu=1}^P Q_{\mu\nu} n^\nu + O(n^2)$ , with  $\mathbf{n} = (n^1, \dots, n^P)$  and  $n^\mu \equiv m_\mu - m_{\mu,0} \forall \mu$ , where  $\mathbf{m}_0$  is a stationary solution of (4). Any stable solution requires negative eigenvalues of  $Q_{\mu\nu}$ . Stability turns out to strongly depend on  $\varphi(X)$  and  $n$ : for rules  $K$  and  $M$  only mixture states with  $n = P$  are locally stable; for rule  $V$  the only locally stable states are the pure ( $n = 1$ ) ones – see Fig. 1(a) – that appear, for large enough values of  $P$ , for  $T < \tilde{T}(1, P)$ , with large overlap with a particular pattern,  $m_1 > \tilde{m}_1$ . In general, this will make very effective the retrieval of a pattern. Therefore, for  $f_3$  and rule  $V$ , the system exhibits associative memory which is not affected by the presence of mixture states that are locally unstable. In Figs. 1 (a) and (b) we show, respectively, the nature of pure states for three values of  $P$ , and the resulting phase diagram for  $n = 1$ , both for rule  $V$ . Note that good, monotonic behavior is suggested as  $P$  is further increased even up to very

large values, as shown in Figs. 1 (c) and (d) for the behavior of  $\theta_1$  and  $\tilde{m}_1$  with  $P$ , respectively.

We also performed a series of computer simulations of Eq. (2) in order to check and extend our analytical results. For this purpose we considered  $N \in [400, 3600]$ ,  $P = 10$  and the effective rates

$$\omega_1 = \min \left\{ 1, \exp \left[ \frac{2P}{T} \left\{ N^{-1} - s_x \frac{1}{P} \sum_{\mu} \xi_x^\mu m_\mu(\mathbf{s}) \right\} \right] \right\},$$

$$\omega_2 = \exp \left\{ - \frac{P}{T} \left[ 1 + s_x \frac{1}{P} \sum_{\mu} \xi_x^\mu m_\mu(\mathbf{s}) \right] \right\},$$

and

$$\omega_3 = \frac{1}{P} \sum_{\mu} \exp \left\{ - \frac{P}{T} \left[ 1 + s_x \xi_x^\mu m_\mu(\mathbf{s}) \right] \right\},$$

where  $m_\mu(\mathbf{s}) = \frac{1}{N} \sum_{\mathbf{x}} \xi_x^\mu s_{\mathbf{x}}$ .  $\omega_1$  and  $\omega_2$  correspond to the Hopfield case, namely,  $f_1$  and, respectively, rules  $M$  and  $V$ ;  $\omega_3$  corresponds to synaptic fluctuations as given by  $f_3$  and rule  $V$ . Our main results are summarized in Fig. 2: The retrieval of a particular pattern is much more efficient and robust for  $\omega_3$ ; see Fig. 2(a). Contrary to the case  $\omega_3$ , the retrieval of information for  $\omega_1$  and  $\omega_2$  is importantly affected by the noise; see Fig. 2(b).

Summing up, fluctuating synapses during neuronal activity determine the behavior of a neural network, in general. Our analytical and numerical studies of

a simple model that allows for a systematic study of such fluctuations support this idea. After the system has stored an arbitrary number of patterns in a (previous) learning process, fast synaptic fluctuations may strongly influence the retrieval of information. The form of the distribution characterizing such fluctuations and other details of the microscopic dynamics have a crucial role in the emergent behavior; in fact, the system is in a *nonequilibrium* steady state in our case. Our analysis can be extended to asymmetric synapses; see [9] for further details.

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