Journal of Statistical Physics, Vol. 96, Nos. 1/2, 1999

Continuum Field Model of Driven Lattice Gases

F. de los Santos[1] and P. L. Garrido[1]

Received February 5, 1998; final November 30, 1998

We define a soft-spins approach to the driven lattice gas model (C-DLG) at the level of a master equation. As a result, we obtain a Langevin equation for the C-DLG which depends on the microscopic transition probabilities. We then show how this dependence affects the critical behavior of the the C-DLG, placing the finite- and the infinite-driving-field cases into different universality classes. In the same vein, we propose a continuum description of two other well-known anisotropic, conservative, nonequilibrium models: the two-temperature model (C-TT) and the randomly driven model (C-RDLG). We show that the C-RDLG with infinite averaged field and the C-TT with $T_{||} = \infty$ fall in the same universality class as the infinitely driven C-DLG. A Langevin equation for the driven bilayer lattice gas model is also presented.

KEY WORDS: Nonequilibrium systems; driven lattice gases; Langevin equations.

1. INTRODUCTION

Equilibrium statistical mechanics has succeeded in predicting the collective behaviour of a system in thermal equilibrium given the laws governing its microscopic behaviour. But most natural phenomena belong to the field of *non-equilibrium* statistical mechanics, a subject far less well understood and still in its developing stage. Being overwhelmed by the enormous complexity of non-equilibrium systems, it is expedient to focus on those which settle into non-equilibrium *steady states* and to trace its behaviour to specific model ingredients. Then, results pertaining to collective behaviour can be obtained by several methods. For instance, Monte Carlo simulations in which a great deal of our understanding of the large scale properties

[1] Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain.

0022-4715/99/0700-0303$16.00/0

of a system is based and also serve to test predictions based on other approaches. Mean field theories, the usual starting point of the analytic route, can provide us with some insight into the phase diagram, but they fail to describe the true collective behaviour properly, specially in systems with local interactions in low dimensionalities. Further analytical developments are seriously hampered by, among other things, the (commonly) discrete structure of the system, so to describe the physics at the macroscopic level a new avenue is required. To accomplish this goal one can benefit from the *mesoscopic* approach in terms of Langevin equations, which concentrates on the long-time, large-distance properties and tries to eliminate the lattice altogether, by making the order parameter into a continuous field. This mesoscopic picture can be derived, at least in principle, starting from the microscopic master equation and implementing a coarse graining procedure, but in practice this route proves to be an insurmountable task for most systems. This predicament is usually overcome by postulating phenomenological equations based on the choice of an order parameter and the underlying system symmetries. When equilibrium critical phenomena are considered, one can then appeal to the framework of the renormalization group, and *universality* then appears in the sense that the results are independent of the microscopic details, in particular the microscopic dynamic rules.[1] But non-equilibrium critical phenomena is still a challenging matter which displays striking features.[2] More specifically, in contrast to equilibrium, in non-equilibrium situations the transition rates are not a simply matter of convenience. The observable critical behaviour can depend on some details of the microscopic dynamics, a fact which is often underestimated in the literature. An analysis concluding about the relevant features that characterize the universal properties of a non-equilibrium system at criticallity is still lacking.

This paper is devoted to continuum description of the *driven lattice gas* (DLG henceforth) model and other related models. First devised by Katz, Lebowitz and Spohn,[3] the DLG is a kinetic lattice gas of interacting particles subject to an external uniform field. A more detailed description of the DLG will be presented in the second section of this paper, but for the time being let us remark that various generalizations of the DLG have been argued to be relevant to the understanding of a wealth of varietes of natural phenomena (see ref. 2 for a review). On the theoretical front, the DLG is one of the simplest non-equilibrium models that is hoped to serve as a paradigm for the behaviour of those systems which do not posses a thermodynamic equilibrium state. It also seems to capture the essence of strongly anisotropic systems. Despite the utterly simple specifications of the DLG, it displays far from trivial behaviour. To our belief open questions abound and new inroads on this subject are needed. So, seeking a new

framework for answers, we propose a new continuum approach whose preliminary results were presented in ref. 4.

The remainder of this paper is organized as follows: In the first part of Section 2 we briefly summarize the basic ingredients of the DLG. We then give our derivation of a Fokker–Planck equation for a soft-spins approach to the DLG and its stochastically equivalent Langevin equation. In Section 3 the continuous counterparts of the two-temperature model, the Randomly Driven Lattice Gas and the multilayer variant of the DLG are studied employing the formalism of Section 2. Our concluding remarks are in Section 4.

2. FROM DRIVEN LATTICE GASES TO DRIVEN DIFFUSIVE SYSTEMS

Consider a set of particles confined to a box, $\lambda \subset Z^d$, with periodic boundary conditions. A configuration of this system is specified by giving all the site occupation variables, $n_{\mathbf{x}} = 1, 0$, reflecting the fact that a particle may be present or not at site $\mathbf{x}$. Besides this hard-core constraint, the model also includes a nearest-neighbour (NN) interaction so, given a configuration $C = \{n_{\mathbf{x}}\}_{\mathbf{x} \in \lambda}$, the Hamiltonian reads

$$H(C) = -J \sum_{NN} n_{\mathbf{x}} n_{\mathbf{y}} \tag{1}$$

The configurations C evolve according to a stochastic hopping dynamics which conserves the number of particles, or equivalently the density ϱ. Up to the present, all we have is the familiar kinetic lattice gas[5] for which the following *master equation* for the time evolution of the probability distribution $P_t(C)$ applies

$$\partial_t P_t(C) = \sum_{C'} \{ W[C' \to C]\, P_t(C') - W[C \to C']\, P_t(C) \} \tag{2}$$

Here, $W[C \to C']$ stands for the rate at which the system makes a transition from C to C', and C and C' can differ by a single nearest-neighbour particle-hole exchange. The choice of $W[C \to C'] = D(\beta \Delta H)$, where D is any function satisfying $D(-x) = e^x D(x)$ and $\Delta H = H(C') - H(C)$, ensures that the stationary solution of (2) is the equilibrium one, i.e., $P(C) \propto e^{-\beta H(C)}$, and $\beta = 1/T$ is the inverse of the temperature of the thermal bath.

Next, let us introduce a uniform (in both space and time) external drive $\mathbf{E}$ pointing along one of the principal axis of the lattice. We refer to it as the *electric* field while imagining that the particles behave as positive

ions only in relation to it. The field biases the rates favouring jumps along its direction, suppressing jumps against it, and leaving unaffected those in the tranverse directions. For hard wall boundary conditions the only effect of the drive would be to add a gravitational potential energy to the Hamiltonian (1), and the resulting steady state would be an equilibrium one. But due to the periodic boundary conditions, the drive has a dramatical effect on the system static properties, preventing the system from achieving an equilibrium stationary state. In this case, the electric force is nonconservative so it is not derivable from a global potential. However, the local effect of the drive could be mirrored by adding to the Hamiltonian (1) the work done by the field during the jump, so we choose the rates in the form

$$W[C \to C'] = D(\beta \Delta H + \beta \ell E) \tag{3}$$

where $\ell = (1, 0, -1)$ for jumps (against, transverse to, along) $\mathbf{E}$, and E is the strength of the electric field ($E = |\mathbf{E}|$). When $E = 0$, one recovers the familiar equilibrium rates.

Let us end this résumé of the main features of the DLG (see refs. 2 and 6 for fairly detailed reviews) with a few words on its collective behaviour. At half-filling ($\varrho = 0.5$) and $E = 0$, a second order phase transition at the Onsager critical temperature, T_c, is known to occur in two dimensions. For $E \neq 0$ and still at half-filled lattice, from the gleanings provided by Monte Carlo simulations, the DLG undergoes a second order phase transition at a higher critical temperature, $T_c^{(E)}$, saturating at about $T_c^{(\infty)} = 1.4 T_c$ for $E \to \infty$. The main observation is that for temperatures above $T_c^{(E)}$ the particles are distributed homogeneously while below $T_c^{(E)}$ the DLG segregates into a particle-poor phase and a particle-rich region, the latter having domain walls parallel to the field.

2.1. The C-DLG: Soft-Spins Approach to the DLG

In this paper we are mainly concerned with the continuum field description of the DLG, so firstly we aim at making the order parameter into a continuous field. To serve this purpose, let us define a coarse-grained excess particle density field, $\phi(\mathbf{r}, t)$, with $\mathbf{r} \in T^d$ and where T^d is a d-dimensional torus. We adopt the usual Landau–Ginzburg form as a suited Hamiltonian analogue to the internal energy (1), that is

$$H(C) = \Omega^d \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{\tau}{2} \phi^2 + \frac{g}{4!} \phi^4 \right] \tag{4}$$

where C here stands for a given ϕ configuration. This assumption is very much in the spirit of the *soft-spins* approach to the Ising model, where the field can fluctuate around two maxima. As for the occurrence of the factor Ω^d in $H(C)$, it can be argued that in a naive continuum limit the field $\phi(\mathbf{r})$ is defined as an average over the occupation variables in a region around $\mathbf{r}$ containing Ω^d lattice sites. The factor Ω^d is then a remnant of a coarse graining over the underlying lattice.

Now, we postulate the time evolution of our model system by associating a time dependent statistical weight with each configuration, $P_t(C)$, which evolves in time accordingly to the following master equation

$$\partial_t P_t(C) = \sum_a \int_R d\eta\, f(\eta) \int d\mathbf{r} \{ W[C^{\eta \mathbf{r} a} \to C]\, P_t(C^{\eta \mathbf{r} a}) - W[C \to C^{\eta \mathbf{r} a}]\, P_t(C) \} \tag{5}$$

Here, $C^{\eta \mathbf{r} a}$ is defined as

$$C^{\eta \mathbf{r} a} = \{ \phi(\mathbf{x}) + \eta \Omega^{-d} \nabla_{\mathbf{x}_a} \delta(\mathbf{x} - \mathbf{r}) \}_{\mathbf{x} \in T^d} \tag{6}$$

the label a stands for the $\mathbf{a}$ direction and $f(\eta)$ is an even function of η. Before continuing to prescribe the exchange rates $W[C \to C']$, let us digress briefly to mention that, in general, there does not exist a rigorous connection between the microscopic and the mesoscopic description of a system. For instance, the continuous equivalent to the discrete exchange evolution mechanism would be some kind of infinitesimal-exchange term whose precise form is unknown. Thus, seeking a wider context we introduce the variable η and its probability distribution $f(\eta)$ in an attemp to cover a wider range of situations.

Lastly, we choose the transition rates $W[C \to C']$ such that

$$W[C \to C^{\eta \mathbf{r} a}] = D(H(C^{\eta \mathbf{r} a}) - H(C) + H_E(C \to C^{\eta \mathbf{r} a})) \tag{7}$$

with

$$H_E(C \to C^{\eta \mathbf{r} a}) = \eta\, \mathbf{a} \cdot \mathbf{E}(1 - \phi(\mathbf{r})^2) + O(\varepsilon) \tag{8}$$

where $\varepsilon \equiv \Omega^{-d}$. This means that the transition rates depend on the energy difference between configurations plus a term whose dominant part in ε is the natural choice to mirror the effects of the drive as far as it accounts for the local increment of energy due to the driving field. Correcting terms of higher order in ε will be fixed later. Again, as in the DLG case described above, in absence of the drive the system stationary state is the equilibrium one characterized by the ϕ^4 Hamiltonian (4).

We shall name the model we have just defined C-DLG, for it is intended to be a continuum counterpart of the DLG.

2.2. From Fokker–Planck to Langevin

The next step we take is to get a Fokker–Planck equation by expanding the Master equation in ε up to ε^2 order. To keep matters simple, we will consider $E=0$ for the moment. We will recover the $E\neq 0$ case later on.

We avail ourselves of the results collected in Appendix A to expand $H(C^{\eta \mathbf{r}a})$ and $P_t(C^{\eta \mathbf{r}a})$ around C. The Master equation transforms into

$$\partial_t P_t(C)=\sum_a \int d\mathbf{r}\, d\eta\, f(\eta)\left\{(D(-\Delta H)-D(\Delta H))\, P_t(C) + \sum_{n=1}^{\infty}\frac{(-\eta\varepsilon)^n}{n!}\left(\nabla_{\mathbf{r}_a}\frac{\delta}{\delta\phi(\mathbf{r})}\right)^n P_t(C)\, D(-\Delta H)\right\} \tag{9}$$

where

$$\Delta H=\eta\lambda_a-\frac{\eta^2\varepsilon}{2}\left(\nabla_{\mathbf{r}_a}\frac{\delta}{\delta\phi}\right)\lambda_a+O(\varepsilon^2) \tag{10}$$

with

$$\lambda_a=-\nabla_{\mathbf{r}_a}\frac{\delta H}{\delta\phi(\mathbf{r})} \tag{11}$$

We would like to stress that λ_a is of order one rather than order ε. This is due to the factor Ω^d in (4) which ensues that our expansion of D is not around zero. This assurance is of the most importance because it bestows dependence upon the dynamics on Eq. (10). Otherwise we would have found a very different story: our expansion would have resulted in a simple Model B[7] where any dependence on the dynamics would have vanished. Recalling again the results of Appendix A and noticing that much simplification is obtained because of the integration of odd terms in η, one is led to

$$\partial_t P_t(C)=\sum_a \int d\mathbf{r}\left(\nabla_{\mathbf{r}_a}\frac{\delta}{\phi(\mathbf{r})}\right)\left\{\varepsilon h(\lambda_a)\, P_t(C)+\frac{\varepsilon^2}{2}e(\lambda_a)\left(\nabla_{\mathbf{r}_a}\frac{\delta}{\phi(\mathbf{r})}\right)P_t(C)\right\} \tag{12}$$

where

$$h(\lambda_a) = \int_R d\eta\, f(\eta)\, \eta D(\eta\lambda_a)$$
$$e(\lambda_a) = \int_R d\eta\, f(\eta)\, \eta^2 D(\eta\lambda_a) \tag{13}$$

Turning our focus to the $E \neq 0$ case, let us choose H_E in (8) as

$$H_E(C \to C^{\eta \mathbf{r}a}) = \eta\lambda_a^{(E)} + \sum_{l=1}^{\infty} \frac{\eta^{l+1}(-\varepsilon)^l}{(l+1)!} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta\phi}\right)^l \lambda_a^{(E)} \tag{14}$$

with $\lambda_a^{(E)} = \mathbf{a} \cdot \mathbf{E}(1 - \phi(\mathbf{r})^2)$. We stress that $H_E(C \to C^{\eta \mathbf{r}a}) = -H_E(C^{\eta \mathbf{r}a} \to C)$, i.e., H_E can be considered as a local increment of energy (although, of course, a global potential energy cannot be defined) and local detailed balance in (7) holds. The point of these manouevres is that the Kramers Moyal expansion is now trivial. That is, one should only substitute in (12) and (13) λ_a by $\Lambda_a \equiv \lambda_a + \lambda_a^{(E)}$. So, this election allows us to write

$$\partial_t P_t(C) = \sum_a \int d\mathbf{r} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta\phi(\mathbf{r})}\right) \left\{\varepsilon h(\Lambda_a)\, P_t(C) + \frac{\varepsilon^2}{2} e(\Lambda_a) \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta\phi(\mathbf{r})}\right) P_t(C)\right\} \tag{15}$$

We have just derived a Fokker–Planck for the C-DLG. Now, we proceed to find out its stochastically equivalent Langevin equation. We shall invoke the main result of Appendix B in virtue of which the Fokker–Planck equation

$$\partial_t P_t(C) = \sum_a \int d\mathbf{r} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta\phi(\mathbf{r})}\right) \times \left\{f_a(\phi; \mathbf{r})\, P_t(C) + \frac{1}{2} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta\phi(\mathbf{r})}\right) g_a(\phi; \mathbf{r})^2\, P_t(C)\right\} \tag{16}$$

is equivalent to the Langevin equation

$$\partial_t \phi(\mathbf{r}, t) = \sum_{a=1}^{d} \nabla_{\mathbf{r}_a} [f_a(\phi; \mathbf{r}) + g_a(\phi; \mathbf{r})\, \zeta_a(\mathbf{r}, t)] \tag{17}$$

using the Ito prescription and with $\zeta_a(\mathbf{r}, t)$ being a gaussian white noise, i.e., $\langle \zeta_a(\mathbf{r}, t) \rangle = 0$ and $\langle \zeta_a(\mathbf{r}, t)\, \zeta_{a'}(\mathbf{r}', t') \rangle = \delta_{a, a'}\, \delta(t-t')\, \delta(\mathbf{r}-\mathbf{r}')$. Then, relating to our case, it is straightforward to get

$$\partial_t \phi(\mathbf{r}, t) = \sum_a \nabla_{\mathbf{r}_a} [h(\Lambda_a) + e(\Lambda_a)^{1/2}\, \zeta_a(\mathbf{r}, t)] \tag{18}$$

where, time has been rescaled by a factor ε, and finally ε has been set to 1 since no more perturbative expansions in ε are going to be considered.

Before we proceed further, several comments are in order. First, the basic symmetries of the DLG are present in the Langevin Eq. (18): it is invariant under the simultaneous change $E \to -E$ and $\phi \to -\phi$, and it is also invariant under translations in space and time. But the central hallmark is that it depends strongly upon the dynamics. We believe this is a real step forward, if only because it goes beyond phenomenological approaches. Gratifyingly, we shall see shortly how to exploit this new state of affairs.

2.3. Power Counting

We focus on the critical region where large fluctuations on all length scales dominate. Further simplification in (18) is possible in this regimen by dropping the irrelevants terms in the renormalization group sense. Following the standard field theoretic methods let us introduce an external momentum scale μ and make the following anisotropic scale transformations: $t \to \mu^{-z} t$, $r_\perp \to \mu^{-1} r_\perp$, $r_{||} \to \mu^{-\sigma} r_{||}$, and $\phi \to \mu^{\delta} \phi$, where $\|$ stands for the direction parallel to the driving field $\mathbf{E}$, and $\perp$ for those perpendicular to it. As usual, the noise scales as $\zeta_a \to \mu^{(z+d+\sigma-1)/2} \zeta_a$. Next, we expand the Langevin equation in powers of μ around $\mu = 0$, keeping only the leading terms. The time scale, the transverse noise, and the transverse spatial interaction are forced to remain invariant under the transformation. With this understood the values of z and δ can be determined. One gets $z = 4$ and $\delta = (\sigma + d - 3)/2$. Different scenarios are now possible depending on the value of σ. Demanding that the coefficients of $\nabla_\perp^4$ and $\nabla_{||}^2$ scale in the same way, as in the standard analysis of the critical behaviour of the DLG,[8] would lead to the choice $\sigma = 2$. We show the most representative terms

$$\begin{aligned} \partial_t \phi = & -\frac{e(0)}{2} \left[\Lambda_\perp^2 \phi + \mu^{2\sigma-2} \Lambda_\perp \Lambda_{||} \phi - \mu^{-2} \tau \Lambda_\perp \phi - \mu^{\sigma+d-5}\, \frac{g}{6}\, \Lambda_\perp \phi^3 \right] \\ & + h'(E) [\mu^{2\sigma+2} \Lambda_\perp \Lambda_{||} \phi + \mu^{4\sigma-4} \Lambda_{||}^2 \phi - \mu^{2\sigma-4} \tau \Lambda_{||} \phi \\ & - \mu^{3\sigma+d-7} \Lambda_{||} \phi^3 - \mu^{(3\sigma+d-11)/2} E \nabla_{||} \phi^2] \\ & + e(0)^{1/2} \sum_\perp \nabla_\perp \zeta_\perp + \mu^{\sigma-1} e(E)^{1/2}\, \nabla_{||} \zeta_{||} \end{aligned} \tag{19}$$

which after setting $\sigma = 2$ and taking the limit $\mu \to 0$, assuming $d > 3$, gives

$$\partial_t \phi(\mathbf{r}) = \frac{e(0)}{2}\left[-\Delta_\perp^2 \phi + \tau \Delta_\perp \phi + \frac{g}{6} \Delta_\perp \phi^3 \right]$$

$$- \tau h'(E)\, \nabla_{||}^2 \phi - E h'(E)\, \nabla_{||} \phi^2 + \sqrt{e(0)} \sum_\perp \nabla_\perp \zeta_\perp(\mathbf{r}) \qquad (20)$$

h' is the first derivative of the function $h(\Delta_a)$, closely related to the first derivative of the transition rate D.

Let us take a glance at the structure of Eq. (20). It can be easily checked that all we have is the Langevin equation postulated by Leung and Cardy for the DLG and which is often known as *driven diffusive system*. But in stark contrast to ref. 9, Eq. (20) displays the precise form in which the microscopic field enters the mesoscopic picture of the DLG. More precisely, the two different critical temperatures introduced by Leung and Cardy for longitudinal and transverse ordering are indentified here as $\tau h'(E)$ and τ respectively, while the mesoscopic version of the field **E** finds its counterpart in $Eh'(E)$. Originally, it-was assumed that an infinite microscopic electric field implied a finite non-zero coarse grained driving field.[9] In Eq. (20) we show explicitely that this is not the case. In fact, it happens there that when $E = \infty$ the driving term disappears. We believe that this is the reason of the mismatch between the simulational results and the analysis of the Langevin Eq. (20). The latter is not suited for comparison with computer simulations because these are always performed with infinite drive and in Eq. (20) a finite non-zero driving term is present. Let us take the calculation a stage further by setting E to infinity in (20). Then, all the terms depending on the electric field E become identically zero. This fact can be easily checked, irrespective of the equation considered, (18) before power counting or (20) after rescaling. So, (20) simplifies to

$$\partial_t \phi(\mathbf{r}) = \frac{1}{2} e(0) \left[-\Delta_\perp^2 \phi + \tau \Delta_\perp \phi + \frac{g}{6} \Delta_\perp \phi^3 \right] + \sqrt{e(0)} \sum_\perp \nabla_\perp \zeta_\perp(\mathbf{r}, t) \qquad (21)$$

The latter equation constitutes a simple model B[7] in the transverse directions and no structure in the parallel one. Thus, the scaling $\sigma = 2$ for $E = \infty$ leads to a trivial behaviour. Furthermore, one realizes that imposing $\nabla_\perp^4$ and $\nabla_{||}^2$ to scale in the same way is meaningless because there is no parallel gradient term appearing in the Langevin equation. Nevertheless, we have the freedom to choose σ to look for different critical theories.

A natural choice is $\sigma = 1$. With this election, an equation analogue to (19) can be written down, however much more involved:

$$\partial_t \phi = -\frac{e(0)}{2}\left[\Delta_\perp^2 \phi + \Delta_\perp \Delta_{||} \phi - \mu^{-2}\tau \Delta_\perp \phi - \mu^{d-4}\,\frac{g}{6}\,\Delta_\perp \phi^3 \right]$$

$$\cdot h'(E)\left[\Delta_\perp \Delta_{||} \phi + \Delta_{||}^2 \phi - \mu^{-2}\tau \Delta_{||} \phi - \mu^{d-4}\,\frac{g}{6}\,\Delta_{||} \phi^3 - \mu^{(d-8)/2} E \nabla_{||} \phi^2 \right]$$

$$+ h''(E)\left[\mu^{(d-4)/2}\tau^2 \nabla_{||}(\nabla_{||}\phi)^2 + \mu^{d-4}\,\frac{2}{3}\,\tau E \Delta_{||} \phi^3 + \mu^{3(d-4)/2} E^2 \nabla_{||} \phi^4 \right]$$

$$- \mu^{(5d-16)/2} E^3\,\frac{h'''(E)}{6}\,\nabla_{||} \phi^6 + e(0)^{1/2} \sum_\perp \nabla_\perp \zeta_\perp + e(E)^{1/2}\,\nabla_{||} \zeta_{||} \qquad (22)$$

In particular, for $E = 0$ one recovers the equilibrium theory (model B). For finite E, the critical dimension is $d_c = 8$ but some of the irrelevant terms might be taken into account below the critical temperature. When $E = \infty$ we get a much more simple equation:

$$\partial_t \phi = \frac{1}{2}\,e(0)\left[-\Delta_\perp \Delta_{||} \phi - \Delta_\perp^2 \phi + \tau \Delta_\perp \phi + \frac{g}{6}\,\Delta_\perp \phi^3 \right]$$

$$+ \sqrt{e(0)} \sum_\perp \nabla_\perp \zeta_\perp(\mathbf{r}, t) + \sqrt{\frac{e(0)}{2}}\,\nabla_{||} \zeta_{||}(\mathbf{r}, t) \qquad (23)$$

This equation is the central result of this work. We now proceed to discuss its physical implications. To begin with, Eq. (23) is structureless in the parallel direction. So, it corresponds to a lattice gas in which particles are exchanged at random in the direction of the field while jumps in the transverse directions are subject to energetics. The most obvious distinction between the finite E case and the infinite one lies in the steady-state current term. This does not appear in (23). Of course, such a current exists but it has no bearing on critical properties. Thus, Eq. (23) does not gather it.

It is quite surprising to find that, for infinite driving field, the Langevin equation changes dramatically when compared to the one that characterizes the finite E case. This would have been difficult to work out only on symmetry grounds. Also remarkable is the lack of Galilean invariance and the emergence of a single correlation length (cf. ref. 8). Anticipating renormalization (the computation of the critical exponents will be presented elsewhere), we should remark that the upper critical dimension is now

$d_c = 4$ and it yields a distinct universality class from that obtained for finite fields. Therefore, a value of β different from 1/2 is expected.

We wish to remark that Eq. (23) is renormalizable when $E = \infty$, while for finite values of the field it can be seen that it is not easily renormalizable due to a plethora of dangerous irrelevant operators. Presumably, a strong crossover from our theory to the finite E case occurs for very long impressed fields which obscures the interpretation of Monte Carlo data. This is borne out by the following mean field analysis of the steady-state structure factor which for $d > 8$ and any E is equal to

$$S(\mathbf{k}_\perp, k_{||}) = \frac{k_\perp^2 + (e(E)/e(0))\, k_{||}^2}{k_\perp^2 - (2h'(E)/e(0))\, k_{||}^2} \frac{1}{k^2 + \tau} \tag{24}$$

Of course, for $E = 0$ one recovers the usual form $(k^2 + \tau)^{-1}$. Since $h'(E)$ is an exponentially decreasing function of E, $S(\mathbf{k}_\perp, \mathbf{k}_{||})$ will rapidly adopt the form of the $E = \infty$ structure factor, namely

$$S(\mathbf{k}_\perp, k_{||}) = \frac{k_\perp^2 + k_{||}^2/2}{k_\perp^2 (k^2 + \tau)} \tag{25}$$

This leads us to the conclusion that correlations behave effectively as in the infinitely driven case even for not very large values of the electric field. A similar, but smoother, crossover occurs between the equilibrium and E finite cases.

Finally, we comment that other choices for H_E, the term responible for the drift effects in the transition rates (8), are also possible. For instance, $H_E = \eta \mathbf{a} \cdot \mathbf{E}(1 - \phi^2) + O(\phi^4)$, but the corrections will turn out to be irrelevant for universal properties. The election $H_E = \eta \mathbf{a} \cdot \mathbf{E}$ would lead to (23) again for $E = \infty$, while for finite values of the electric field a new Langevin equation emerges. However, qualitative differences always exist between the infinite and finite E cases.

3. RELATED MODELS

3.1. Two-Temperature Model

In this subsection we consider a model closely related to the DLG. The *two-temperature Ising lattice gas*[10] consists of a set of particles endowed with an Ising Hamiltonian and lying on a hypercubical d-dimensional lattice. In contrast to the usual kinetic lattice gas, particle-hole exchanges are controlled by rates with different temperatures according to the following rules: if the vector $\mathbf{a}$ pointing from the particle to the hole lies

in an n-dimensional subspace, the "parallel" subspace to say, then a transition rate of the form $D(\Delta H/T_{||})$ is at play. On the other hand if **a** belongs to the "transverse" space then the exchanges are coupled to a bath with temperature $T_{\perp}$.

We now proceed to build a continuum theory for the two-temperature model. Let us start with the master equation in continuous space (5). Now $W[C \to C'] = D(H(C') - H(C))$ where $H(C)$ is the usual ϕ-four hamiltonian (4) but it depends on **a** that the coefficient τ of $\phi^2/2$ will adopt different values. That is, $\tau = \tau_{\perp}$ if **a** lies in the transverse subspace and $\tau = \tau_{||}$ otherwise. Being initially interested in the $\tau_{||} \to \infty$ case, we shall assume that particle-hole exchanges in the parallel subspace are subject to

$$H = \Omega^d \frac{\tau_{||}}{2} \int d\mathbf{x}\, \phi^2(\mathbf{x}) \tag{26}$$

We shall call the model so furnished continuous two-temperature model, from now on C-TT model. So, it is expedient to separate (5) into two parts, the first being a sum over the transverse subspace and the second over the parallel one. Next, we follow the same steps as in Section 2.2, i.e., an expansion of ΔH and $P_t(C^{\eta \mathbf{r}a})$ around C. The calculation is straigtforward, for the (simbollicaly) $a \in \perp$ case leads to the same result as in Section 2.2 and when $a \in \parallel$ one simply gets $\Delta H = -\eta \tau_{||} \nabla_{\mathbf{r}_a} \phi$. Then the following Fokker–Planck equation can be easily computed

$$\begin{aligned} \partial_t P_t(C) = & \sum_{a \in \perp} \int d\mathbf{r} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta \phi(\mathbf{r})} \right) \left\{ \varepsilon h(\lambda_a)\, P_t(C) + \frac{\varepsilon^2}{2} e(\lambda_a) \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta \phi(\mathbf{r})} \right) P_t(C) \right\} \\ & + \sum_{a \in \parallel} \int \frac{\varepsilon^2}{2} e(\tau_{||} \Delta_{\mathbf{r}_a} \phi) \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta \phi(\mathbf{r})} \right)^2 P_t(C) \end{aligned} \tag{27}$$

and in the limiting case $\tau_{||} \to \infty$ its equivalent Langevin equation reads

$$\partial_t \phi(\mathbf{r}, t) = \sum_{a \in \perp} \nabla_{\mathbf{r}_a} [h(\lambda_a) + e(\lambda_a)^{1/2} \zeta_a(\mathbf{r}, t)] + \frac{\sqrt{e(0)}}{2} \sum_{a \in \parallel} \nabla_{\mathbf{r}_a} \zeta_a \tag{28}$$

Finally, we restrict ourselves to a 1-dimensional parallel subspace. A naive dimensional analysis yields, after dropping irrelevant terms and assuming $\sigma = 1$, the same equation we found for the C-DLG at criticallity with $E = \infty$. That is, the C-DLG with $E = \infty$ and the C-TT model with $T_{||} = \infty$ are members of the same universality class. To illustrate this connection further, let us consider our continuum version of the DLG with infinite drift. As we saw in the previous section this corresponds to completely

random particle hops in the field direction. Then we note that the same picture arises if we choose $T_{||} = \infty$.

No adiditional effort is necessary to study the finite $T_{||}$ case. We again seperate (5) into two parts, but now we take the full expression for the Hamiltonian (4) rather than (26). A derivation running along the same lines of the previous subsection is possible. We only remind that the scaling $z = 4$, $\sigma = 1$ and $\delta = (d-2)/2$ is used in the power counting procedure which entails

$$\partial_t \phi = \frac{e(0)}{2} \left[\Delta^2 \phi + (\tau_\perp \Delta_\perp + \tau_{||} \Delta_{||}) \phi + \frac{g}{3!} \Delta \phi^3 \right] + \sqrt{e(0)} \sum_a \nabla_{\mathbf{r}_a} \zeta_a(\mathbf{r}, t) \qquad (29)$$

Despite of the fact that anisotropy is only present as far as the masses are concerned, we should remark that more general anisotropies could be expected. Then, the collective behaviour of the C-TT model with finite $T_{||}$ could be more adequately predicted by means of an extension of (29) to an equation with full anisotropy in the coefficients.

3.2. Randomly Driven Lattice Gases

Here we take up an extension of the DLG, the *randomly driven lattice gas* (RDLG hereafter).[12] Let us consider a DLG in which the driving field fluctuates accordingly to an even distribution $p[E(\mathbf{x}, t)]$ which is δ-correlated in space and time. The RDLG is easier to realize in the laboratory than the DLG and one can also benefit from a higher analytical simplicity because a random drive induces no steady-state current and the particle-hole symmetry is preserved. Like the DLG, it exhibits a second order phase transition at half filling from a disordered state to striplike order. There is hardly any difference between typical ordered configurations associated with the DLG with an infinite drive imposed, the two-temperature model and the randomly driven model. Now, to provide a better comparison we put forward a continuum equation for the RDLG. Following our criterion for notation, we shall name the resulting model C-RDLG. Our starting point is again equation (5) with the prescriptions (7) and (8). We should then average over a random E, but we can defer such an average to a later stage. First, we repeat the steps we performed to arrive at a Langevin equation for the DLG. All results carry over without change until we face to averages over E in (22) of the type $E^m h^{(n)}(E)$ and $\sqrt{e(E)}$. As a consequence of a symmetric $p[E(\mathbf{x}, t)]$ and the integrations over η, the averages with $m+n$ being an even integer vanish whilst the remainder terms will yield the following finite values:

$$\gamma_1 \equiv \int dE\, p[E]\, h'(E)$$

$$\gamma_2 \equiv \int dE\, p[E]\, Eh''(E) \tag{30}$$

$$\gamma_3 \equiv \int dE\, p[E] \sqrt{e(E)}$$

The Langevin equation can then be written down in the form

$$\partial_t \phi = -\frac{e(0)}{2} \Delta_\perp^2 \phi + \gamma_1 \Delta_{||}^2 \phi + \left(\gamma_1 - \frac{e(0)}{2}\right) \Delta_\perp \Delta_{||} \phi + \frac{e(0)}{2} \tau \Delta_\perp \phi - \tau \gamma_1 \Delta_{||} \phi$$
$$+ \frac{e(0)}{2} \frac{g}{3!} \Delta_\perp \phi^3 + \frac{2}{3} \tau(\gamma_2 - \gamma_1) \Delta_{||} \phi^3 + \sqrt{e(0)} \sum_\perp \nabla_\perp \zeta_\perp + \gamma_3 \nabla_{||} \zeta_{||} \tag{31}$$

We have arrived at an entirely anisotropic equation, i.e., all the gradient operators have been split into components parallel and transverse to E. In accordance with our earlier discussion of the C-TT model, we conclude that the C-RDLG and the C-TT model with finite $T_{||}$ share the same critical behaviour. But we caution that, as in the C-DLG, in the C-RDLG two cases have to be distinguished. We can think of the simplest distribution $p[E]$, namely the bimodal $\frac{1}{2}[\delta(E+E_o)+\delta(E-E_o)]$, and then take the limit $E_o \to \infty$. Not surprisingly, in this limit $\gamma_i = 0$, $i = 1, 2, 3$, and the Langevin Eq. (23) associated with the infinitely driven C-DLG emerges again. We therefore can conclude that the generical critical properties of the C-TT model with $T_{||} = \infty$, the C-DLG with E set to infinity and the C-RDLG driven with an effectively infinite field are indistinguishable.

3.3. Layered Driven Lattices Gases

We turn our attention to a generalization of the DLG.[13] Let us consider a pair of identical square lattices placed *back to back*. Each plane is a copy of a two dimensional DLG. No inter-layer coupling is allowed, but particles can hop from one plane to their "nearest neighbour" site in the other one. This process is controlled by in-plane energetics alone and it is not affected by the drive. The overall particle density is fixed at 1/2. Concerning the nature of the phase transition, Monte Carlo data ($E = \infty$) have revealed that intriguinly two transitions appear.[13] Here, we just give the gist of references[14, 15] where the phase diagram in the (E, T)-plane has been mapped out employing Monte Carlo simulations and dynamic mean-field theory. As T is lowered the system first orders from a homogeneous

state into a state with strips in both layers. This transition is observed to be characterized by the same critical indexes as the DLG. Decreasing T further, we reach the second transition where homogeneous layers with different density appear. This transition belongs to the Ising universality class for any $E < E_c \approx 2$, while values of E beyond the threshold field E_c lead to a first order phase transition.[14]

Although physical motivations for this model come from various directions,[2] we shall focus on the theoretical side. As far as comparison with simulations is intended, we shall provide a mesoscopic picture to place into a coherent analytical context these two phase transitions. Next, we propose a continuum mesoscopic theory in the spirit of Section 2. We first discuss the equilibrium case $E = 0$. Jumps in this model can be naturally divided into two types: *in*-layer jumps and *inter*-layer jumps. So, the following notation will prove to be convenient: we shall refer to $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ as the coarse grained density field in plane one and two respectively, and an arbitrary global configuration will be termed $C \equiv \{\phi_1(\mathbf{r}), \phi_2(\mathbf{r})\}$. We shall denote $C_i^{\eta \mathbf{r}a}$ the configuration after an exchange of density $\varepsilon\eta$ is performed in the $\mathbf{a}$ direction with an infinitesimal neighbour of $\mathbf{r}$ in plane i. Exchanges between planes will lead to configurations named $C^{\eta \mathbf{r}}$. More specifically,

$$\begin{aligned} C_1^{\eta \mathbf{r}a} &= \{\phi_1(\mathbf{x}) + \varepsilon\eta \nabla_{\mathbf{x}_a} \delta(\mathbf{x} - \mathbf{r}), \phi_2(\mathbf{x})\} \\ C_2^{\eta \mathbf{r}a} &= \{\phi_1(\mathbf{x}), \phi_2(\mathbf{x}) + \varepsilon\eta \nabla_{\mathbf{x}_a} \delta(\mathbf{x} - \mathbf{r})\} \\ C^{\eta \mathbf{r}} &= \{\phi_1(\mathbf{x}) + \varepsilon\eta \delta(\mathbf{x} - \mathbf{r}), \phi_2(\mathbf{x}) - \varepsilon\eta \delta(\mathbf{x} - \mathbf{r})\} \end{aligned} \tag{32}$$

With this understood, then the following master equation can be written down,

$$\begin{aligned} \partial_t P_t(C) = \sum_a \int d\mathbf{r}\, d\eta\, f(\eta) \{ & W[C_1^{\eta \mathbf{r}a} \to C]\, P_t(C_1^{\eta \mathbf{r}a}) - W[C \to C_1^{\eta \mathbf{r}a}]\, P_t(C) \\ & + W[C_2^{\eta \mathbf{r}a} \to C]\, P_t(C_2^{\eta \mathbf{r}a}) - W[C \to C_2^{\eta \mathbf{r}a}]\, P_t(C) \} \\ & + \int d\mathbf{r}\, d\eta\, f(\eta) \{ W[C^{\eta, \mathbf{r}} \to C]\, P_t(C^{\eta, \mathbf{r}}) - W[C \to C^{\eta, \mathbf{r}}]\, P_t(C) \} \end{aligned} \tag{33}$$

$W[C \to C']$ has its usual meaning, i.e., $W[C \to C'] = D(H(C') - H(C))$ and $H(C)$ is again the Hamiltonian (4). The calculus towards a Fokker–Planck equation can be carried out as we did in Section 2, the only difference being the inter-layer current term. One can easily get

$$\partial_t P_t(C) = \sum_{i=1}^{2} \sum_a \int d\mathbf{r} \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta \phi_i(\mathbf{r})} \right) \left\{ \varepsilon h(\lambda_a^{(i)})\, P_t(C) + \frac{\varepsilon^2}{2} e(\lambda_a^{(i)}) \left(\nabla_{\mathbf{r}_a} \frac{\delta}{\delta \phi_i(\mathbf{r})} \right) P_t(C) \right\} + \int d\mathbf{r} \left(\nabla_{12} \frac{\delta}{\phi(\mathbf{r})} \right) \left\{ \varepsilon h(\lambda_{12})\, P_t(C) + \frac{\varepsilon^2}{2} e(\lambda_{12}) \left(\nabla_{12} \frac{\delta}{\phi(\mathbf{r})} \right) P_t(C) \right\} \tag{34}$$

We now explain our notation. The first term in this equation models two decoupled Ising lattice gases with appropiate constraints. As for the second one, it is simply a "discrete" version of the former, as long as it has to be with exchanges across the layers. Thus, $\lambda_a^{(i)}$ has the same meaning as in Section 2, but restricted to plane i. The functions h and e are also defined as in the previous section. With $\nabla_{12}(\delta/\delta\phi)$ we simply denote the operator

$$\frac{\delta}{\delta \phi_1(\mathbf{r})} - \frac{\delta}{\delta \phi_2(\mathbf{r})}$$

and λ_{12} stands for $(\nabla_{12}(\delta/\delta\phi))\, H(C)$. Turning the drive on, one only has to move $\lambda_a^{(i)}$ into $\Lambda_a^{(i)}$ thereby taking into account the effect of the electric field. A Langevin equation can then be derived following the same lines of Subsection 2.2. One gets

$$\partial_t \phi_1(\mathbf{r}) = -\Gamma \lambda_{12} - \Gamma e(\lambda_{12})^{1/2}\, \zeta(\mathbf{r}, t) + \sum_a \nabla_{\mathbf{r}_a} [h(\Lambda_a^{(1)}) + e(\Lambda_a^{(1)})^{1/2}\, \zeta^{(1)}{}_a(\mathbf{r}, t)] \tag{35}$$

$$\partial_t \phi_2(\mathbf{r}) = \Gamma \lambda_{12} + \Gamma e(\lambda_{12})^{1/2}\, \zeta(\mathbf{r}, t) + \sum_a \nabla_{\mathbf{r}_a} [h(\Lambda_a^{(2)}) + e(\Lambda_a^{(2)})^{1/2}\, \zeta^{(2)}{}_a(\mathbf{r}, t)]$$

ζ and $\zeta_a^{(i)}$ are gaussian white noises while Γ is a hand introduced transport coefficient that measures the rate at which the system changes due to the inter-layer exchange mechanism.

Bearing density conservation in mind, we introduce two new fields:

$$m(\mathbf{x}) \equiv (\phi_1 + \phi_2)/2, \qquad \varphi(\mathbf{x}) \equiv (\phi_1 - \phi_2)/2 \tag{36}$$

Equations (35) are easily expressed in terms of the new fields $m(\mathbf{x})$ and $\varphi(\mathbf{x})$. Now, simulation results hint at which field will be treated as an order parameter. We shall take $m(\mathbf{x})$ as the ordering field for the DLG type transition, the one that occurs at a higher temperature. Then, in a naive dimensional analysis there is much freedom to choose the scale of observation. We perform the following scale transformation: $t \to \mu^{-z} t$, $r \to \mu^{-\sigma} r$,

$\varphi \to \mu^{\delta}\varphi$, and $m \to \mu^{\gamma} m$. In particular, if we fix the exponents $z=4$, $\sigma=1$, $\delta=d/2$ and $\gamma=(d-2)/2$, it can be easily checked that, after neglecting terms that are irrelevant in the renormalization group sense, we are left with nothing but Eq. (22) for the C-DLG. Hence, the critical properties belong to the C-DLG universality class, a picture consistent with simulations.

Turning next to the second transition, we consider φ as a non-conserved order parameter. We propose a critical theory naively consistent with $z=2$, $\sigma=1$, $\delta=(d-2)/2$ and $\gamma=d/2$. This scaling leaves us with a couple of equations that take the form:

$$\begin{aligned}
\partial_t \varphi &= \left(1-\frac{\tau h'(0)}{2\Gamma}\right) \Delta_{\perp}\varphi + \left(1-\frac{\tau h'(E)}{2\Gamma}\right) \Delta_{||}\varphi - \mu^{-2}\tau\varphi \\
&\quad - \mu^{d-4}\,\frac{g}{6}\,\varphi^3 - \mu^{d-2}\,\frac{g}{2}\,\varphi m^2 - \sqrt{e(0)}\,\zeta \\
\partial_t m &= \tau(h'(0)\,\Delta_{\perp} + h'(E)\,\Delta_{||})\,m - \mu^{d-2}\,\frac{g}{2}\,(h'(0)\,\Delta_{\perp} + h'(E)\,\Delta_{||})(\varphi^2 m) \\
&\quad - \mu^{(d-2)/2} E h'(E)\,\nabla_{||} m^2 - \mu^{(d-6)/2} E h'(E)\,\nabla_{||}\varphi^2 \\
&\quad + \sqrt{e(0)} \sum_{\perp} \nabla_{\perp}(\zeta_{\perp}^{(1)} + \zeta_{\perp}^{(2)}) + \sqrt{e(E)}\,\nabla_{||}(\zeta^{(1)}{}_{||} + \zeta_{||}^{(2)})
\end{aligned} \tag{37}$$

We have dropped all terms that give a negligible contribution in the limit $\mu \to 0$. Due to the electric field, which singles out a lattice axis, all gradient terms have become anisotropic. The situation is then very much reminiscent of the driven lattice gases with repulsive interactions.[16] We have an electric field $\mathbf{E}$ that has an effect on the phase transition only through an auxiliary non-ordering field $m(\mathbf{x})$. The naive dimension of E turns out to be $(d-2)/2$, in contrast to $(d-8)/2$ (Section 2.3), so it is highly irrelevant compared to $g\varphi^3$. We note that essentially the same set of equations results in ref. 16, so our analysis of (37) will follow the same lines of this reference. That is, $\mathbf{E}$ is naively irrelevant for the Gaussian fixed point until d is lower than two. We conjecture that E is not relevant to the Wilson–Fisher fixed point for $d<4$, so the unique effect of the drive consists in generating anisotropies. It should be noted that the field $m(\mathbf{x})$ does not order, so we do not need to keep track of it as far as critical behaviour is concerned. Thus, the critical properties for the two-layer driven lattice gas are given by the Langevin equation for $\varphi(\mathbf{x})$ and they fall into the Ising universality class. However, corrections to order $O(E^2)$ show that g and τ decrease to an amount that depends on E. Eventually, both of them may vanish simultaneously, a mechanism that would be liable for a tricritical point.

Then, in qualitative agreement with simulations, the transition would be discontinous for E beyond a critical field value. Interestingly enough, it could be worked out the dependence of the transition temperature on the dynamics.

4. SUMMARY AND CONCLUSIONS

Here, we take stock of what we have done. Section 1 outlines the task undertaken in this article: to arrive at a field theoretic description of the DLG model and to show how this description explains why the DLG, and three models intimately related to it, have the various properties that they exhibit. Inspired by the dynamics at the microscopic level, we were able to postulate a master equation in continuous space which characterizes a model we have called C-DLG. The goal of the avenue we have pursued was to take into account the microscopic details, an effort towards disentangling the role of dynamics in non-equilibrium critical behaviour. An expansion of the continuous master equation was then possible due to the factor Ω^d, a remnant of a coarse graining over the underlying lattice. The role of the factor Ω^d can only be fully appreciated noticing that all dependence on the dynamics relies on it. Then, the full Langevin equation for the C-DLG was derived. The richness of Eq. (18) was brought to fruition when viewed on different length scales, leading to the emergence of distinct critical theories. In particular, we have recovered the Langevin equation of reference[9] after considering an anisotropic scale transformation ($\sigma = 2$). This result, interesting though it is, is unsatisfactory because the case $E = \infty$, which is with few exceptions the most studied one, was carried out explicitely the result being a trivial equation. The situation became more transparent after we resorted to the detailed dependence of the coefficients on the microscopic dynamics: the choice $\sigma = 2$ makes no sense, as it was seen in Section 2. Following well honed arguments[17] that invoke a single effective correlation length, we have turned to the choice $\sigma = 1$. Possibly against intuition, different critical behaviour has been found for the finite E and $E = \infty$ cases. The upper critical dimension associated with the former is $d_c = 8$ whereas the latter is characterized by $d_c = 4$, thereby in either case yielding an universality class other than that obtained in ref. 9.

In a subsequent section we have provided three examples of the applicability of our methods in the shape of the two-temperature model, the random driven lattice gas and the driven bi-layer lattice gas. A Langevin equation identical to the one associated with the infinitely driven C-DLG has resulted for our continuum version of the two-temperature model (named C-TT) with $T_{||} = \infty$ in a derivation that runs along the same lines of Section 2. The finite $T_{||}$ case has also been studied. Different critical

behaviour from the $T_{||} = \infty$ case has been found. Remarkably, the C-DRLG model with finite averaged external field has resulted in a Langevin equation identical to that associated with the two-temperature model with $T_{||}$ finite, whilst the C-RDLG with an infinite averaged driving field was proved to belong to the same universality class of the infinitely driven C-DLG. Summing up, we conclude that the infinitely driven C-DLG, the C-TT model with $T_{||} = \infty$ and the C-RDLG with infinite drive, are described by the same Langevin equation. On the other hand, the two-temperature model with finite $T_{||}$ and the C-RDLG with finite averaged field belong into the same universality class.

In the last part of Section 3 we have tackled with the two-layer driven lattice gas. A viable explanation for the two transitions exhibited in this model has been provided in the frame of field theory. Again, the election $\sigma = 1$ has proved to suffice our purposes, i. e. the understanding of collective behaviour in these systems.

Despite the Langevin equations of refs. 9 and 12, for example, yield a sizable number of analytical results, in our opinion the new ones exhibited in this paper deserve study in its own right. It is not only that we dissent from the standard approach, but the complex behaviour displayed in this far from equilibrium scenario persuades and compels us to explore new directions. We believe that the models under the prefix C- that we have introduced capture the essential properties of the discrete ones in the large-scale long-time limit. No doubt, much work is still to be done as more detailed studies on these subjects would be highly desirable.

APPENDIX A. FORMAL DEVELOPEMENTS

Supose that F is a functional of a function $\phi(x)$. If ϕ changes to $\phi + \delta\phi$, then a Taylor series expansion can formally be written down

$$F(\phi + \delta\phi) = F(\phi) + \int dx_1 \frac{\delta F}{\delta\phi(x_1)} \delta\phi(x_1) + \frac{1}{2} \int dx_1\, dx_2 \frac{\delta^2 F}{\delta\phi(x_1)\, \delta\phi(x_2)} \delta\phi(x_1)\, \delta\phi(x_2) + \cdots \tag{38}$$

where $\delta F/\delta\phi$ means the functional derivative of $F(\phi)$ with respect to $\phi(x)$. Now, we explicitely treat the case we are concerned with, namely

$$\delta\phi = \varepsilon \nabla_x \delta(x - r) \tag{39}$$

In such a case it inmediately follows that

$$\begin{aligned} F(\phi+\delta\phi) &= F(\phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \, \frac{\delta^n F(\phi)}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \\ &\quad \times \varepsilon^n \nabla_{x_1} \delta(x_1 - r) \cdots \nabla_{x_n} \delta(x_n - r) \\ &= F(\phi) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \left\{ \prod_{k=1}^{n} \int dx_k \nabla_{x_k} \delta(x_k - r) \frac{\delta}{\delta\phi(x_k)} \right\} F(\phi) \\ &= F(\phi) + \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \left(\nabla_r \frac{\delta}{\delta\phi(r)} \right)^n F(\phi) \end{aligned} \tag{40}$$

The operator $(\nabla_r(\delta/\delta\phi))$ satisfies

$$\left(\nabla_r \frac{\delta}{\delta\phi} \right) F(\phi) = \nabla_r \left(\frac{\delta F(\phi)}{\delta\phi(r)} \right) - \frac{\delta}{\delta\phi(r)} (\nabla_r F(\phi)) \tag{41}$$

which can be better proved by putting the last expression in a lattice. Finally, the usual properties of functional derivatives can be applied, v.g.

$$\left(\nabla_r \frac{\delta}{\delta\phi} \right) F_1(\phi)\, F_2(\phi) = F_1(\phi) \left(\nabla_r \frac{\delta}{\delta\phi(r)} \right) F_2(\phi) + F_2(\phi) \left(\nabla_r \frac{\delta}{\delta\phi(r)} \right) F_1(\phi) \tag{42}$$

APPENDIX B. FOKKER–PLANCK AND LANGEVIN EQUATIONS FOR SYSTEMS WITH CONSERVED ORDER PARAMETER

A Langevin equation with conserved order parameter has the general form

$$\partial_t \phi(x, t) = \sum_a \nabla_{x_a} [f_a(\phi, x) + g_a(\phi, x)\, \zeta_a(x, t)] \tag{43}$$

ζ_a being a Gaussian white noise: $\langle \zeta_a(x, t)\, \zeta'_a(x', t') \rangle = \delta_{a,a'}\, \delta(x - x') \times \delta(t - t')$; $\langle \zeta_a(x, t) \rangle = 0$. Let us introduce a time discretization. Then

$$\phi_{n+1} = \phi_n + \varepsilon \sum_a \nabla_{x_a} [f_a(\phi_n, x) + g_a(\phi_n, x)\, \zeta_{a,n}] \tag{44}$$

where $\phi_n \to \phi_t$ and $t_n = n\varepsilon \to t$ when $\varepsilon \to 0$ and $n \to \infty$. A factor ε has been absorbed into the noise and we are using the Ito prescription.

The probability that the system is in the configuration $\phi_{n+1}(r)$ at time t_{n+1} is given by

$$P_{n+1}(\phi_{n+1}) = \left\langle \int d\phi_n \, P_n(\phi_n) \, \delta(S_n) \right\rangle_\zeta \tag{45}$$

$$\begin{aligned}\delta(S_n) \approx \delta(\phi_{n+1} - \phi_n) + \int dx_1 \, \frac{\delta\delta(\phi_{n+1} - \phi_n)}{\delta\phi_n} \left[-\varepsilon \sum_a \nabla_{x_a} [f_a + g_a \zeta_a] \right] \\ + \frac{1}{2} \int dx_1 \, dx_2 \, \frac{\delta^2 \delta(\phi_{n+1} - \phi_n)}{\delta\phi_n(x_1) \, \delta\phi_n(x_2)} \\ \times \left[-\varepsilon \sum_a \nabla_{x_{1,a}} \cdots \right] \left[-\varepsilon \sum_a \nabla_{x_{2,a}} \cdots \right] + \cdots\end{aligned} \tag{46}$$

The calculus now reduces to noise averages. An intermediate step is

$$\begin{aligned}\varepsilon^{-1}(P_{n+1} - P_n) = -\sum_a \int dx_1 \, \frac{\delta}{\delta\phi} [P_n \nabla_{x_a} f_a] \\ + \frac{1}{2} \int dx_1 \, dx_2 \, \frac{\delta^2}{\delta\phi(x_1) \, \delta\phi(x_2)} \\ \times \left[P_n \sum_a \nabla_{x_{1,a}} \nabla_{x_{2,a}} (g_a^2 \, \delta(x_1 - x_2)) \right]\end{aligned} \tag{47}$$

and after a bit of algebra, applying the results of Appendix A, our final result reads

$$\partial_t P_t(\phi) = \sum_a \int dx \left(\nabla_{x_a} \frac{\delta}{\delta\phi(x)} \right) \left[P_t f_a + \frac{1}{2} \left(\nabla_{x_a} \frac{\delta}{\delta\phi(x)} \right) (P_t g_a^2) \right] \tag{48}$$

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