

## Effect of Correlated Fluctuations of Synapses in the Performance of Neural Networks

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We consider stochastic neural networks in which synaptic intensities rapidly fluctuate—around means given by a learning rule—competing with neuron activity. Each snapshot of synaptic intensities contains the neuron-neuron correlations in one of the stored patterns chosen at random. The result is apparently noisy behavior which induces robustness, including improved associative and pattern recognition processes. The main result here might apply to biological systems that exhibit fluctuating patterns of synapses. [S0031-9007(98)07250-0]

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It has long since been noticed that, unlike for digital computers, both redundancy and robustness characterize the functioning of biological neural networks [1]. Recent experiments confirm that individual synapses are extremely unreliable which, however, seems to enhance the system processing power as a whole; see, for instance, Refs. [2–11] (for example, *stochastic resonance* or coupling of—apparently—noisy signals might be important to the processing of information in biological systems [12]). How redundancy and robustness develop in practice, what is the precise relation between them and the different types of “noise” that one observes, and how these influence emergent properties are not yet well enough understood, however. Neural networks defined by means of a master equation with competing kinetics [13] provide one of the simplest scenarios in which these issues may be investigated. Following this approach, we show in this Letter how a stochastic neural network may be adapted to behave as a sort of filter for the relevant information when synaptic variations are appropriately included. That is, in addition to thermal noise, we consider *correlated* fluctuations of synaptic intensities such that the characteristic time for the fluctuations is much smaller than the one for neuronal activity. We demonstrate that such noise may notably increase the efficiency in transmitting the signal. Although we expect this to hold more generally, we explicitly illustrate it here by a few analytical results and a series of computer simulations concerning a simple case, namely, a variation of Hopfield’s neural network [14–16]. The Hopfield model is a mathematically well-defined approach that contains some of the essentials of the processes of interest: cooperative transmission competing with thermal noise which exhibits associative memory under certain circumstances. In fact, we have already shown [17] that such oversimplified representation of a biological system allows for studying the consequences of various types of noise on the network design and performance. Somewhat related but showing a behavior which essentially differs from the one in our model are, for instance, the variations of the Hopfield model in Refs. [18–22] and a system [23] in

which neurons evolve fast as compared to synapses and a Langevin-type of dynamics is justified. Further analytical details of our model, including further comparison with related cases, will be published elsewhere [24].

Consider a set of  $N$  (binary) neurons,  $s_x = +1$  or  $-1$  (“firing” and “silent” states, respectively), coupled to each other by synapses of intensities  $J_{xy}$ ;  $x, y = 1, \dots, N$ . The configuration  $\mathbf{s} = \{s_x\}$  changes at each time assuming a value for  $\mathbf{J} = \{J_{xy}\}$  chosen at random from a given distribution,  $p(\mathbf{J})$ . In general, such a competition between variations of  $\mathbf{s}$  and  $\mathbf{J}$  induces asymptotically a *nonequilibrium* steady state. That is, a conflict exists whose effect is similar to the one that an external agent acting on  $\mathbf{s}$  would cause: Contrary to better understood cases, the stationary state resulting from the stochastic evolution implied by the agent together with the underlying heat bath at temperature  $T$  is not solely determined by the values for  $T$  and for the configurational energy that characterizes  $\mathbf{s}$ . One may represent such dynamic conflict by assuming that the transition probability per unit time (*rate*) for the change (*flip*)  $s_x \rightarrow -s_x$  is a superposition,  $\omega(s_x \rightarrow -s_x) = \int d\mathbf{J} p(\mathbf{J}) \varphi_{\mathbf{J}}(s_x)$ , where  $\varphi_{\mathbf{J}}(s_x)$  stands for the (elementary) rule when synapses are set to the value  $\mathbf{J}$ . In practice, we simulated this evolution in the computer starting from an initial configuration,  $\mathbf{s}_0$ , in which each variable  $s_x$  is given any of the two possible values at random, and then choosing a neuron at random, and attempting the flip  $s_x \rightarrow -s_x$  with one of the following rates

$$\begin{aligned} \omega_1 &= \min \left\{ 1, \exp \left[ \frac{2P}{T} \left( \frac{1}{N} - s_x \frac{1}{P} \sum_{\mu} \xi_x^{\mu} m^{\mu} \right) \right] \right\}, \\ \omega_2 &= \exp \left[ -\frac{P}{T} \left( 1 + s_x \frac{1}{P} \sum_{\mu} \xi_x^{\mu} m^{\mu} \right) \right], \\ \omega_3 &= \frac{1}{P} \sum_{\mu} \exp \left[ -\frac{P}{T} (1 + s_x \xi_x^{\mu} m^{\mu}) \right]. \end{aligned} \quad (1)$$

It is assumed here that the system stores  $P$  patterns,  $\xi^{\mu} \equiv \{\xi_x^{\mu} = \pm 1\}$  with  $\mu = 1, \dots, P$ , and that  $m^{\mu} = N^{-1} \sum_x \xi_x^{\mu} s_x$  measures the overlap of the current neuron

state with the  $\mu$ th “memorized” pattern. (It is likely that the condition of sequential updating which is implied here is not essential to our results.)

The choices  $\omega_1$  and  $\omega_2$  correspond to the Hopfield case with  $p(\mathbf{J}) = \delta(\mathbf{J} - \mathbf{J}_0)$ , where  $\mathbf{J}_0 = \{J_{xy}^0\}$  corresponds to the Hebbian rule,  $J_{xy}^0 = N^{-1} \sum_{\mu} \xi_x^{\mu} \xi_y^{\mu}$ . One obtains  $\omega_1$  after using the Metropolis rule, namely,  $\varphi_{\mathbf{J}} = \min\{1, \exp(-\Delta/T)\}$ , where  $\Delta$  is the change of energy—as given by the Hopfield-Hebb Hamiltonian  $\mathcal{H}_{\mathbf{J}}(\mathbf{s}) = -\frac{1}{2} \sum_{x \neq y} J_{xy} s_x s_y$  for given  $\mathbf{J}$ —brought about by the attempted flip. One obtains  $\omega_2$  after using instead the rule  $\varphi_{\mathbf{J}} = \exp(-\Delta/2T)$ , introduced before in another problem [13], and normalizing to unity. The case  $\omega_3$  characterizes our model. This corresponds to a simplified version of the following, more general stochastic process: Assume that one performs the change  $s_x \rightarrow -s_x$  with probability  $q\varphi_{\mathbf{J}}$  and, with probability  $(1 - q)\phi$ , where—for simplicity— $\phi$  is independent of  $\mathbf{s}$ , one performs the change  $J_{xy} \rightarrow J'_{xy}$ ;  $q \in (0, 1)$ . For  $q = 1$ , this process leads asymptotically to the equilibrium state for  $T$  and  $\mathcal{H}_{\mathbf{J}}(\mathbf{s})$ ; both  $\omega_1$  and  $\omega_2$  drive any  $\mathbf{s}_0$  to this state (the two cases exhibit a different relaxation, however). The simplest nontrivial situation that involves competition between different tendencies, each for a different value of  $\mathbf{J}$  in  $\mathcal{H}_{\mathbf{J}}$ , occurs for  $q \rightarrow 0$ . In this limit, two relevant time scales exist (once the previous plasticity learning process is completed): a fine time scale,  $\tau$ , in which neurons do not appreciably evolve but synapse intensities fluctuate locally; and a coarse scale,  $t = q\tau$  for  $q \rightarrow 0$  and  $\tau \rightarrow \infty$ , in which neurons evolve as in the presence of a steady distribution for the synapses,  $p(\mathbf{J})$ . Such distinction of well-separated time scales has been reported to occur in some biological systems [25–27]. Under this limiting condition (but not for  $1 > q > 0$ , an interesting case that we do not consider explicitly here), assuming that

$$p(\mathbf{J}) = \frac{1}{P} \sum_{\mu} \prod_{x,y} \delta\left(J_{xy} - \frac{P}{N} \xi_x^{\mu} \xi_y^{\mu}\right), \quad (2)$$

the effective rate is  $\omega(s_x \rightarrow -s_x) = P^{-1} \sum_{\mu} \varphi_{\mathbf{J}_{\mu}}(s_x)$ , where  $\mathbf{J}_{\mu} = \{\xi_x^{\mu} \xi_y^{\mu}\}$ . This, which equals  $\omega_3$  for  $\varphi_{\mathbf{J}} = \exp(-\Delta/2T)$  after proper normalization, implies that synaptic intensities randomly fluctuate about means given by the Hebbian values, i.e.,  $\overline{J_{xy}} \equiv \int d\mathbf{J} p(\mathbf{J}) J_{xy} = J_{xy}^0$ ; this feature is convenient in order to have a well-known reference. The choice (2) induces two more features that are essential to the result below, namely, (i) that memories are held in the system as fluctuating patterns of synaptic connections, such that each pattern  $\xi^{\mu}$  contributes to dynamics with certain probability, and, even more important, (ii) that local fluctuations are the result of  $\mathbf{J}$  taking at random one of the  $P$  possible synaptic configurations,  $\mathbf{J}_{\mu}$ , so that the spatial correlations within the memorized  $\xi^{\mu}$  are influencing dynamics. In fact, the analytical results in [17], where different rates corresponding to a factorized function  $p(\mathbf{J})$  were studied, indicate that lacking such correlations does not significantly enhance

robustness—though some interesting behavior is still induced. Of course, one may think of other choices for  $p(\mathbf{J})$  having convenient properties. For example, assuming that  $p(\mathbf{J})$  is a sum of appropriate Gaussians would generalize (2) while allowing for more synaptic noise, which—in light of our results—is worth being investigated explicitly (one may also avoid in this way the assumption  $\overline{J_{xy}} = J_{xy}^0$  above).

By standard methods, in the limit  $N \rightarrow \infty$  [13,17], some explicit analytical results may be obtained for  $\omega_3$ , assuming that one may replace  $s_x$  by its ensemble average—which is expected to be a realistic hypothesis for this highly connected system. For example, one can demonstrate the unusual property that mixture states are not locally stable for  $\omega_3$ ; instead, a transition occurs at  $\tilde{T}(P)$  towards pure ( Mattis) states, with

$$\tilde{T}(P) = \begin{cases} 1 & \text{for } P < 3 \\ \frac{1}{\theta} (\cosh \theta + P - 1) \sinh \theta & \text{for } P > 3 \end{cases},$$

where  $\theta + (P - 1)(\theta \cosh \theta - \sinh \theta) - \sinh \theta \cosh \theta = 0$ ; the values  $(T = 1, P = 3)$  mark a *tricritical point*. That is, the overlap  $m^{\mu}$  behaves discontinuously for  $P > 3$  at  $\tilde{T}(P)$ , as in a phase transition of first order, so that the process of associative memory is then very efficient. In other words, the system with  $\omega_3$ , unlike the corresponding Hopfield case, is characterized by large overlaps below  $\tilde{T}$  for any  $P > 3$ , the error associated with the retrieval process being negligible, except for very small  $P$  near  $T = 1$ . (It is to be remarked that the observed trend for large  $P$  indicates that our results, obtained for finite  $P$ , hold for  $P \rightarrow \infty$  as well [24].) In order to further investigate such interesting behavior, which can only be demonstrated exactly for orthogonal memorized patterns, we performed computer simulations of a network with (up to)  $N = 2500$  and  $P = 10$ , assuming  $J_{xy} = J_{yx}$ —though the model admits asymmetric couplings, which would require using a different choice for the effective rate  $\omega$ .

A principal conclusion is that our analytical results are fully confirmed by the Monte Carlo study, e.g., steady mixture states were never observed. Some main features of the behavior revealed by the simulations are depicted in Figs. 1–3. Figure 1 shows the time variation of the overlaps  $m^{\mu}$ ,  $\mu = 1, \dots, P$ , at a given—relatively low—temperature. The noisy horizontal behavior shown by most evolutions in the inset of Fig. 1 is the familiar (Hopfield) behavior implied by  $\omega_1$ . That is, starting from any initial state, there is a rapid evolution (not visible on the scale of this figure) in which the overlap with one of the  $P$  memorized patterns becomes relatively large (about 0.7 in this simulation) while the others decay practically to zero. The situation remains stationary, except for thermal noise, which is added to the signal in this case; increasing temperature would finally impede associative memory. For comparison purposes, we also

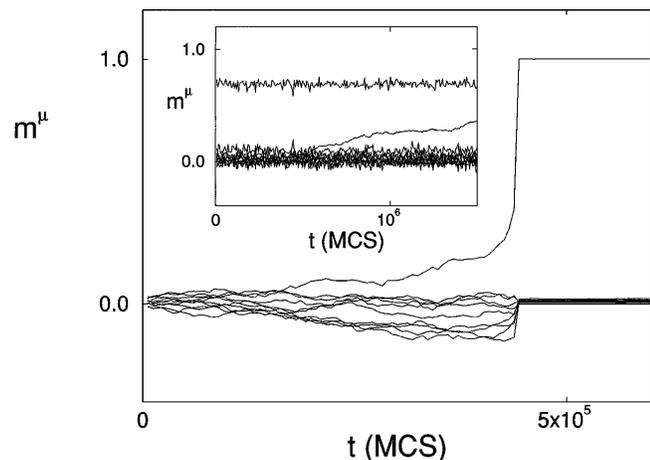


FIG. 1. Evolution with time at temperature  $T = 0.8$  of the normalized overlap  $m^\mu = N^{-1} \sum_x \xi_x^\mu s_x$  for  $N = 2500$ ; different lines are for  $\mu = 1, \dots, 10$ . (Note: the unit of time is the number of “flips” attempted per site, i.e., Monte Carlo steps, and we set Boltzmann’s constant to unity.) The main graph illustrates the case with correlated synaptic fluctuations,  $\omega_3$ . The inset contains the corresponding Hopfield result; all of the noisy horizontal behaviors are for  $\omega_1$  (Metropolis rule); the line increasing with time is for  $\omega_2$  and  $\mu$  such that  $m^\mu \neq 0$ . This illustrates how correlated fluctuations of synapsis can notably enhance both the stability and efficacy of the retrieval process in a neural network; the convergence time is also reduced.

illustrate the behavior for  $\omega_2$ , i.e., a different choice for the rule  $\varphi_J$ . This is represented by the line increasing with time in the inset of Fig. 1, which corresponds to the only nonzero overlap. The retrieval process in this version of the Hopfield model is slower but more robust against thermal noise; one should have expected the latter fact given that  $\exp(-\Delta/2T)$  favors low “energy” states

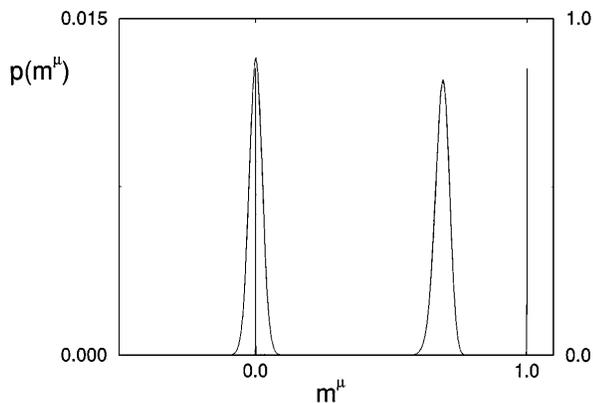


FIG. 2. The distribution of the fluctuations of the overlap with time as exhibited by the system in Fig. 1, during the stationary regime, for two typical values of  $\mu$ , normalized to unity. The scale on the left side corresponds to the bell-shaped cases (centered about 0.0 and 0.7, respectively) that occur for  $\omega_1$ . The scale on the right side corresponds to distributions (centered around 0.0 and very near 1, respectively) for  $\omega_3$  that are hardly indistinguishable (on the scale of this figure) from Dirac delta functions.

more than the Metropolis rule. As illustrated in the main graph in Fig. 1, the performance of the retrieval process is further improved if the evolution proceeds according to  $\omega_3$ , namely, in the presence of fluctuations as described by (2). That is, although the convergence to the desired result is slower for  $\omega_3$  than for  $\omega_1$ —but not for  $\omega_2$ —, both the evolution and the stationary state are very robust, and the retrieval error is negligible for the parameter values in the figure. We observed the same qualitative behavior varying  $N$ ,  $P$ , and  $T$  within wide ranges (which is confirmed by analytical results mentioned above, as will be reported elsewhere [24]). It should be remarked that the additive nature of (2), which makes dynamics respect the spatial correlations that characterize the stored patterns  $\xi^\mu$ , is essential for this result. In particular, as mentioned above, assuming that the effects from different patterns are multiplicative in  $p(\mathbf{J})$  induces a synaptic noise which, in a sense, simply adds to the thermal one, and differences ensue which are not so dramatic, though also interesting [17]. Figure 2, for the distribution of  $m^\mu$  values, confirms the result in Fig. 1. This graph clearly shows that  $\omega_3$ , unlike  $\omega_1$ , leads to saturation and induces a very high level of robustness (note two different scales for the vertical axis here, as described in the figure caption).

Figure 3 illustrates another interesting feature of  $\omega_3$ . The evolutions here start with a state obtained by perturbing substantially any of the memorized (random) patterns. The main graph illustrates how  $\omega_3$  always drives the system to the “right” pattern for  $T = 0.6$ . Further decreasing of  $T$  increases the relaxation time, but no

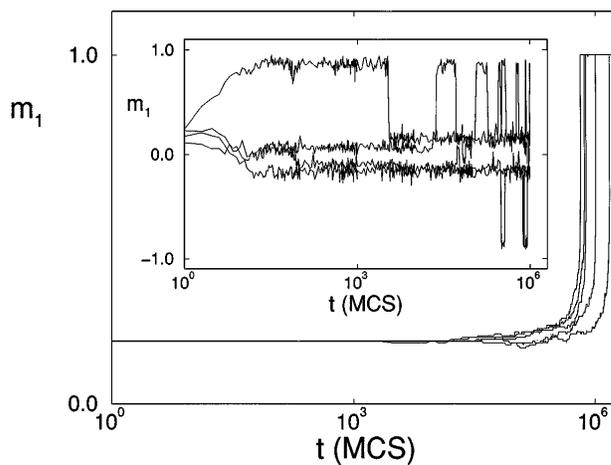


FIG. 3. Semilogarithmic plot showing independent evolutions at  $T = 0.6$  for  $N = 400$  and  $P = 10$  (random) memorized patterns starting with the same initial state. The latter is obtained by perturbing one of the given patterns in such a way that 236 (out of 400) randomly placed neurons remain at the same state. The main graph shows six independent typical evolutions with  $\omega_3$ . The inset shows four independent typical evolutions with  $\omega_1$ . This illustrates that pattern recognition processes in a neural network may be notably improved in the presence of correlated fluctuations of synapses as described by  $\omega_3$ .

other qualitative effects are evident. The inset shows the corresponding result for the Hopfield case  $\omega_1$ . The system becomes then much less efficient in recognizing the right pattern, and eventually destabilizes after some relatively short time (note that time in Fig. 3 is in logarithmic scale in order to show the details of the early evolution). The situation depicted in Fig. 3 is confirmed in much larger evolutions; that is, contrary to the situation shown by the inset, one never observes escape from the “saturated” state for  $\omega_3$  (in fact, this involves a phase transition of first order for which fluctuations are relatively much weaker).

In summary, we have described the basic features of a dynamic neural network as obtained from some analytical results and a series of computer simulations. The model assumes time evolution of the neuron-synapse configuration,  $(\mathbf{s}, \mathbf{J})$ , according to a Markov process, with  $q(1 - q)^{-1}$  as the rate of variation of  $\mathbf{s}$  relative to that of  $\mathbf{J}$ . We studied the effective transition rates in (1), namely, Hopfield’s cases ( $\omega_1$  and  $\omega_2$ ) corresponding to  $q \rightarrow 1$ , and a case ( $\omega_3$ ) for  $q \rightarrow 0$  which describes synapse intensities fluctuating rapidly—as compared to neuron changes—and randomly with distribution  $p(\mathbf{J})$ , [Eq. (2)]. A principal conclusion is that competition between synaptic and neuron activities induces a nonequilibrium condition and, consequently, it influences essentially the emergent behavior. Our model explains memory as a dynamic stochastic, fluctuating property of the *whole* network, instead of being stored as a sort of average (e.g., Hebbian) value. As a consequence, the process of learning consists of modifying the dynamical mechanism rather than a kind of topological plasticity. We considered explicitly here the synapses oscillating synchronously in such a way that each connection fluctuates around a mean value corresponding to a learning, e.g., Hebbian value, and the correlations characterizing the stored patterns are present in the fluctuating mechanism. That is, the set of synaptic couplings keeps (very quickly) visiting, at random, each of the  $P$  available patterns  $\mathbf{J}_\mu$ . It follows, which is likely to hold under more general conditions, that, as compared with the situation in which such types of fluctuations do not occur, the evolution and the stationary state are much more robust, and the retrieval error and the convergence time are smaller in our model. This is consistent with, and gives a simple interpretation to, recent observations in biological systems. That is, some of the synaptic noise which is reported to occur in Nature could be the most relevant part of the (dynamic) retrieval mechanism. Confirming this would require specific experimental studies.

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