Langevin equation for driven diffusive systems

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An open controversy exists about the nature of the second-order nonequilibrium phase transition exhibited by a lattice gas in which particles are driven along one of the lattice directions by an external agent. Field theoretical predictions and Monte Carlo estimates for the critical exponent values do not seem to agree with each other. In this paper we introduce a Langevin equation in which the effects of the microscopic dynamics are carefully taken into account. We show that the order parameter critical exponent when the drive is *infinite* (no backwards jumps) is not mean-field-like, in contrast with the prediction for finite values of the drive. This finding seems to reconcile field theoretical and numerical results. [S1063-651X(97)10012-5]

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Much attention has been devoted in the past decade to the study of nonequilibrium stationary states (NSS) [1,2], which exhibit instabilities, phase transitions, and critical phenomena much more varied than the equilibrium states. The absence of a general theory for NSS similar to the powerful equilibrium ensemble theory makes the understanding of nonequilibrium phase transitions difficult and challenging. *Universality* is often invoked in order to disregard the influence of most microscopic details on critical behavior. Nonequilibrium critical phenomena are often studied by postulating a Langevin equation (or field theory) which contains the features that are assumed to be relevant near a critical point [3,4]. However, the situation concerning universality is less clear-cut here than in equilibrium, and there is at least one case in the literature in which a significant disagreement between Monte Carlo results and field theoretical predictions seems to occur: the driven lattice gas (DLG).

The DLG is one of the simplest nonequilibrium models that illustrates some of the basic features of systems which do not possess a thermodynamic equilibrium state. The DLG, which models some situations of practical interest [1], is a stochastic particle system on a lattice in contact with a thermal bath at some given temperature. The particles move according to a probabilistic rule that tries to minimize a local energy function. Such a function has two parts: a configurational (typically Ising-like) energy and one corresponding to the action of an external constant electric field that drives the particles (say, positive ions) along a given lattice direction. This can be implemented as a Markovian master equation that characterizes the model by transition probabilities which depend on the local energy increments. Periodic boundary conditions are considered, hence the electric field is not derivable from a global potential. The system can reach, however, a (nonequilibrium) stationary state in which there is a net current of particles in the direction of the field. On the other hand, if the field is turned off, the system relaxes to an equilibrium state characterized by a Gibbsian distribution.

Since the DLG was first proposed [5], mean field analysis [6], Monte Carlo simulations [7], and field theoretical approaches [8,9] have been performed, and most of its qualitative physics is now quite well understood (see [1] and [10] for comprehensive reviews). In particular, it is well known that, for a half-filled lattice in the presence of a nonzero driving field, the DLG undergoes at some temperature a second-order phase transition above dimension 1. The critical point separates the high temperature phase, in which the system is disordered, from the low temperature, highly anisotropic phase, in which a strip of particles parallel to the field direction is exhibited. An interesting question is the role of universality in this simple nonequilibrium problem and, in particular, the values for the critical exponents that characterize the phase transition.

Vallés and Marro [7] concluded from Monte Carlo data for an infinite field (i.e., particles are not allowed to jump against the field direction) that the order parameter critical exponent, β, seemed to be close to 1/3 for dimension 2. In contrast, Leung and Cardy [8] and Janssen and Schmittmann [9] showed that β is obtained from a mesoscopic Langevin equation, which is often known as a driven-diffusive system (DDS), which is assumed to be the coarse grained version of the DLG, takes the value 1/2, that is, the DDS has classical or mean-field critical behavior. Further computer simulations and data analysis did not solve the problem. It was claimed that if Monte Carlo data are analyzed, taking into account the intrinsic anisotropy of the system [12], then the classical value 1/2, for β ensues. However, an analysis of the data in [12] was presented in [11] that seems to invalidate the previous conclusion and confirms that β is close to 1/3, in accordance with the original prediction and with the behavior of other anisotropic lattice gases [1].

In our opinion, the controversy may be due to the fact that the microscopic DLG master equation and the mesoscopic DDS Langevin equation are not describing the same physical situation. In order to support our point, we have derived a Langevin equation from the DLG master equation. Our study casts serious doubts on the validity of the DDS Langevin equation as a continuous representation of the DLG, or at least on the range of validity of such assumption.

We proceed as follows: First we introduce a master equation, which defines the DLG. From this equation, after using standard techniques in the theory of stochastic processes [13], we derive a Fokker-Planck equation, and determine its

stochastically equivalent Langevin equation. By following this strategy, we ensure that the details of the microscopic dynamics are properly taken into account. We then show, in particular, that the analytic form of the microscopic transition probability and the precise value of the field may affect the critical behavior of the DLG. Our study thus clarifies somewhat the concept of universality in nonequilibrium phenomena.

The original model [5] consists of a d-dimensional lattice, whose sites have occupation variables $n_i=1$ or 0, for site i. These variables evolve following a particle-hole exchange dynamics. Let us define at each point $\mathbf{r}\in\mathbb{Z}^d$ a density variable, $\Phi_{\mathbf{r}}\in\mathbb{R}$, which is the averaged value of the occupation variables in a region of volume ε^{-1} around $\mathbf{r}$. Let us introduce a dynamics for the density variables which is inspired in the original lattice dynamics. We postulate that the system evolves from a given configuration $\mathbf{\Phi}$ to another $\mathbf{\Phi}'$ by choosing at random a particle at point $\mathbf{r}$ and exchanging it with its nearest neighbor in the $\mathbf{a}$ direction, namely

$$\Phi'_{\mathbf{x}}=\Phi_{\mathbf{x}}+\varepsilon(\delta_{\mathbf{x},\mathbf{r}}-\delta_{\mathbf{x},\mathbf{r}+\mathbf{a}})\equiv\Phi^{\mathbf{r}a}_{\mathbf{x}}. \tag{1}$$

When ε^{-1} is large enough, $\Phi_{\mathbf{x}}$ is assumed to be a continuous function of $\mathbf{x}$, say $\Phi(\mathbf{x})$, so that we have $\mathbf{\Phi}^{\mathbf{r}a}=\{\Phi(\mathbf{x})+\varepsilon\nabla_{x_a}\delta(\mathbf{x}-\mathbf{r}),\Phi(\mathbf{x})\in\mathbf{\Phi}\}_{\mathbf{x}\in\mathbb{R}^d}$. We can also generalize this dynamics to consider exchanges of magnitude $\eta\varepsilon$ with probability amplitude $f(\eta)$, the latter being an even function of η, e.g.,

$$f(\eta)=\tfrac{1}{2}[\delta(\eta+1)+\delta(\eta-1)]. \tag{2}$$

See [14] for a more detailed explanation of this procedure.

The probability of finding the system at some given configuration, $P_t(\mathbf{\Phi})$, evolves according to the following Markovian master equation:

$$\partial_t P_t(\mathbf{\Phi})=\sum_a\int_R d\eta\, f(\eta)\int_{R^d}d\mathbf{r}[c^E(\mathbf{\Phi}^{\eta\mathbf{r}a}\to\mathbf{\Phi})P_t(\mathbf{\Phi}^{\eta\mathbf{r}a})-c^E(\mathbf{\Phi}\to\mathbf{\Phi}^{\eta\mathbf{r}a})P_t(\mathbf{\Phi})]. \tag{3}$$

Here, $c^E(\mathbf{\Phi}\to\mathbf{\Phi}')$ is the probability of transition per unit time (or transition rate) from $\mathbf{\Phi}$ to $\mathbf{\Phi}'$. It is given for the DLG as

$$c^E(\mathbf{\Phi}\to\mathbf{\Phi}^{\eta\mathbf{r}a})=D(H[\mathbf{\Phi}^{\eta\mathbf{r}a}]-H[\Phi]+H_E[\mathbf{\Phi}\to\mathbf{\Phi}^{\eta\mathbf{r}a}]), \tag{4}$$

where $H[\Phi]$ is the equilibrium Φ^4 Hamiltonian,

$$H(\mathbf{\Phi})=\varepsilon^{-1}\int_{\mathbb{R}^d}d\mathbf{r}\left[\tfrac{1}{2}(\boldsymbol{\nabla}\Phi(\mathbf{r}))^2+\frac{\rho}{2}\Phi(\mathbf{r})^2+\frac{g}{4!}\Phi(\mathbf{r})^4\right], \tag{5}$$

and

$$H_E[\mathbf{\Phi}\to\mathbf{\Phi}^{\eta\mathbf{r}a}]=\eta\mathbf{a}\cdot\mathbf{E}[1-\Phi(\mathbf{r})^2]+O(\varepsilon). \tag{6}$$

We have discarded here terms of order ε which only have some influence at a technical level.

The first term on the right-hand side of Eq. (6) is the local increment of energy due to the action of the driving field when a density $\varepsilon\eta$ is exchanged in the $\mathbf{a}$ direction with an infinitesimal neighbor of $\mathbf{r}$. The function $D(z)$ is any function satisfying the property $D(z)=e^{-z}D(-z)\geqslant 0$ (for instance, $D(z)=[1-\tanh(z/2)]/2$); this constraint ensures that the master equation (3) satisfies in the limiting case $E=0$, the detailed balance (DB) property [1], and, consequently, the stationary solution to Eq. (3) for $E=0$ is the equilibrium state corresponding to $H[\mathbf{\Phi}]$, i.e., $P_{\mathrm{st}}(\mathbf{\Phi})\propto e^{-H[\mathbf{\Phi}]}$.

Once the model has been defined, the next step consists in deriving a Langevin equation for it. With this aim, we perform a Kramers-Moyal expansion of the Master equation (3), i.e., a series expansion in powers of the small parameter ε to be truncated at the second order [13]. After some algebra, using the Ito prescription, one is led to the following Langevin equation:

$$\partial_\tau\Phi_\tau(\mathbf{r})=\sum_a\nabla_{r_a}[h(\Lambda_a)+e(\Lambda_a)^{1/2}\psi_{a,\tau}(\mathbf{r})], \tag{7}$$

where

$$h(\Lambda_a)=\int_{\mathbb{R}}d\eta\, f(\eta)\,\eta D(\eta\Lambda_a),$$

$$e(\Lambda_a)=\int_{\mathbb{R}}d\eta\, f(\eta)\,\eta^2 D(\eta\Lambda_a), \tag{8}$$

with $\Lambda_a(\mathbf{\Phi},\mathbf{r},\mathbf{E})\equiv\mathbf{a}\cdot\mathbf{E}[1-\Phi(\mathbf{r})^2]-\nabla_{\eta_a}(\delta H[\mathbf{\Phi}]/\delta\Phi(\mathbf{r}))$. ψ_a is a Gaussian white noise, i.e., $\langle\psi_{a,\tau}(\mathbf{r})\rangle=0$ and $\langle\psi_{a,\tau}(\mathbf{r})\psi_{b,\tau'}(\mathbf{r}')\rangle=\varepsilon\delta_{a,b}\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')$ and $\tau=\varepsilon t$. We then set $\varepsilon=1$ as no more perturbative expansions in ε are considered.

Let us point out that the basic symmetries of the DLG are indeed present in the Langevin equation (7): it is invariant under translations in space and time, and it is also invariant under the simultaneous change $E\to-E$ and $\Phi\to-\Phi$.

In order to find the relevant part of the Langevin equation, we perform the following anisotropic scale transformations [8,9]: $\tau\to\mu^{-z}\tau$, $x_\perp\to\mu^{-1}x_\perp$, $x_\parallel\to\mu^{-\sigma}x_\parallel$, and $\Phi\to\mu^{\delta}\Phi$, where $\perp$ represents the $d-1$ orthogonal directions to the driving field $\mathbf{E}$ whose direction is represented by $\parallel$. Next we expand the Langevin equation (7) in powers of μ around $\mu=0$, and we keep only the leading terms. The values of z and δ can be determined by requiring that the coefficients of the transverse noise and of the transverse spatial interaction are invariants of the transformation. This implies that $z=4$ and $\delta=(\sigma+d-3)/2$. We still have the freedom to choose σ to look for different critical theories. A way to obtain a renormalizable theory is to require that the coefficients of the gradient terms $\nabla_\perp^4\Phi$ and $\nabla_\parallel^2\Phi$ both scale in the same way. Under this assumption, we get $\sigma=2$, and we obtain the critical theory for $d>3$:

$$\partial_\tau \Phi(\mathbf{r}) = \tfrac{1}{2} e(0) \left[-\sum_{\perp} \nabla_\perp^4 \Phi + \rho \Delta_\perp \Phi + \frac{g}{6} \Delta_\perp \Phi^3 \right] - \rho h'(E) \nabla_\parallel^2 \Phi - E h'(E) \nabla_\parallel \Phi^2 + \sqrt{e(0)} \sum_{\perp} \nabla_\perp \psi_{\perp,\tau}(\mathbf{r}), \tag{9}$$

where h' is the first derivative of the function $h(\Lambda_a)$, closely related to the first derivative of the transition rate D. $\Delta_\perp$ indicates the Laplacian associated with the $d-1$ directions orthogonal to the driving field. There are two effective temperatures: one is associated with the transverse directions to the field, ρ, and the other is associated with the field direction, $\rho h'(E)$. Under the renormalization group transformations they flow independently in the parameter space. Let us remark that Eq. (9) has precisely the structure of the Langevin equation postulated by Leung and Cardy as a mesoscopic description for the DLG [8,9], the only difference is that Eq. (9) exhibits an explicit dependence on the microscopic field parameter. That is, Leung and Cardy introduced a coarse-grained driven field that is identified here as $Eh'(E)$. The field theoretical analysis of Eq. (9) was performed by Janssen and Schmittmann [9]. The upper critical dimension, above which mean field results are exact, is $d=5$. For dimensions $d=5-\varepsilon$, an ε expansion can be performed. It is important to notice that the term proportional to g is naively irrelevant at the critical dimension, while it is necessary to take it into account to ensure stability below the critical temperature (that is, this term is a dangerous irrelevant operator [9,15]). The field theoretical analysis of the Langevin equation (9) shows that, due to a Galilean invariance present in the theory, the order parameter critical exponent is mean-field-like, i.e., $\beta=1/2$ even below the critical dimension.

Contrary to what happens for the Langevin equation postulated in [8,9], it is possible to set $E=\infty$ in Eq. (9), which corresponds to the case usually analyzed in Monte Carlo simulations. In this limit, all the terms depending on the electric field *E become identically zero*. This fact can be checked in the Langevin equation (7) before rescaling, and also in Eq. (9), which involves rescaling. The resulting critical theory is the equilibrium one, i.e., model B [16] for the transverse directions, and no structure ensues in the parallel direction. More explicitly, one obtains

$$\partial_\tau \Phi(\mathbf{r}) = \tfrac{1}{2} e(0) \left[-\Delta_\perp^2 \Phi + \rho \Delta_\perp \Phi + \frac{g}{6} \Delta_\perp \Phi^3 \right] + \sqrt{e(0)} \sum_{\perp} \nabla_\perp \psi_{\perp,\tau}(\mathbf{r}). \tag{10}$$

Physically this is due to the fact that, in the case of infinite field, at a microscopic level, jumps in the direction of the field occur with probability 1, independently of the field value, while jumps against the field direction are forbidden. Therefore, any dependence of configuration $\mathbf{\Phi}$ on the drive disappears in the case of infinite field. This is not to be associated with any of the approximations or truncations considered; in fact, it may be shown explicitly that the same stands when perturbation corrections are considered. Summing up, the scaling $\sigma=2$ for $E=\infty$ leads to a trivial behavior, so that such substitution is meaningless. In fact, $\sigma=2$ was obtained by imposing that the gradient terms $\nabla_\perp^4 \Phi$ and $\nabla_\parallel^2 \Phi$ scale in the same way. However, in the $E=\infty$ case, there is no parallel gradient term appearing in the Langevin equation. In this case, the theory can be renormalized in a consistent way using $\sigma=1$. This leads to

$$\partial_t \Phi = \tfrac{1}{2} e(0) \left[-\Delta_\perp \Delta_\parallel \Phi - \Delta_\perp^2 \Phi + \rho \Delta_\perp \Phi + \frac{g}{6} \Delta_\perp \Phi^3 \right] + \sqrt{e(0)} \sum_{\perp} \nabla_{r_\perp} \psi_{\perp,\tau}(\mathbf{r}) + \left(\frac{e(0)}{2} \right)^{1/2} \nabla_{r_\parallel} \psi_{\parallel,\tau}(\mathbf{r}). \tag{11}$$

This Langevin equation has a simple physical interpretation: it corresponds to a lattice gas in which particles are exchanged at random in the direction of the field, and they follow a local increment of the configurational energy in the transverse directions. Interestingly, such a situation would have been difficult to guess using only heuristic arguments and considering symmetries, conservation laws, or other general properties. Possibly against intuition, it follows *a different relevant Langevin equation in the infinite field limit*, which characterizes a critical behavior different from that for finite values of E.

Let us make some remarks concerning Eq. (11).

(i) The fact that no steady particle current is exhibited explicitly by Eq. (11) does not imply that there is no such current in our model. It only implies that the current does not depend on the field values and, consequently, does not affect the critical behavior.

(ii) The naive power counting leading to $\sigma=1$ does not imply that the scaling is isotropic. The anisotropy shows up when the diagrammatic corrections are taken into account.

(iii) In the scale $\sigma=1$, Eq. (11) is renormalizable when E is set to ∞, while for finite values of the field it can be seen that it is not. In this sense the point $E=\infty$ can be considered as a singular one. Similar remarks could be made on the case $E=0$, the reason being that they both separate a renormalizable case, $E=0$ or $E=\infty$, and the highly nontrivial one E finite.

(iv) The critical dimension in the infinite field limit associated with Eq. (11) is $d=4$, and it yields a universality class other than that obtained from Eq. (9) for finite fields. The detailed renormalization and computation of the critical exponents associated with Eq. (11) will be presented elsewhere. In any case, it is clear already that there is no reason to expect that the value of β is 1/2.

In conclusion, we have shown that, in the limit of infinite driving field, which is the case studied in Monte Carlo simulations of the DLG, the correct field theory that captures the relevant features of the underlying phase transition differs from that obtained for finite field values. Consequently, the order parameter exponent β is more likely to be different from 1/2, in contrast with the prediction from previous work. It is likely also that, for large field values, a strong crossover

from our theory to the fixed point computed in [9] occurs, and that this crossover makes difficult the interpretation of Monte Carlo data. Our results shed some light on the physics of a familiar nonequilibrium phase transition, and the discrepancies between the existing field theory and various Monte Carlo simulations are clarified. Finally, we mention that even though symmetries, dimensionality, and other general features of the system are very relevant to characterize its universality class, it ensues from above that one has to be extremely careful when constructing a continuous description for nonequilibrium systems. Some of the details of the microscopic dynamics can be crucial to determine the observable behavior, a fact which is often underestimated in the literature.

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