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On the influence of microscopic dynamics in non-equilibrium stationary states: a mean field example*

M A Muñoz† and P L Garrido‡

† Departamento de Física Aplicada, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain
‡ Departamento de Física Moderna, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain

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Abstract. We discuss the role of the microscopic stochastic dynamics in the macroscopic properties of a simple spin-flip non-equilibrium mean field model where ferromagnetic and antiferromagnetic interactions are competing. We have found that different analytical forms for the dynamic mechanism give rise to different stationary states at low temperature, in contrast to what happens at equilibrium. We also find an effective free-energy functional which allows us to classify the stable and metastable stationary solutions.

1. Introduction

In the statistical physics context, this century has been characterized and dominated by a deep understanding of systems in equilibrium states. In particular, the connection between the microscopic and macroscopic properties of these systems by means of the Gibbs ensemble theory has been one of the essential tools responsible of this success. In contrast, the study of systems under the action of external agents driving them to non-equilibrium stationary states still lacks a well-defined theory equivalent in generality and in its practical applications to the equilibrium Gibbsian one.

In this context, our general strategy is clear: by using simple models describing non-equilibrium situations we would like to understand general features of these systems to obtain new ideas in order to build such a hypothetical theory.

A useful model in the development of equilibrium statistical mechanics has been the Ising model [1]. This has been so because the model is sufficiently simple to be treated with analytical tools, but is also complex enough to have a rich collective behaviour, such as, for example, a phase transition. Moreover, in the study of non-equilibrium stationary states, we think that it is interesting to build general models which include the equilibrium as a particular case for some values of their external parameters. Therefore, a generalization of the Ising model may be the most appropiate model to use as a well known reference system for comparison purposes. In this work we use two reference equilibrium systems: the mean field versions of the ferromagnetic and antiferromagnetic Ising model. In particular, the non-equilibrium model treated in this paper is built as the competition of two Markov processes, each one, when

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acting apart, driving the system to a pure Gibbs ferromagnetic and antiferromagnetic Gibbsian state, respectively. Then, by varying the frequency of the two processes we can move smoothly away from the equilibrium behaviour.

The paper is structured as follows. Section 2 is dedicated to defining the model system. In section 3 we describe the N-expansion of the model, the solutions of the deterministic and Fokker–Planck equations are derived and a theory to discriminate between stable and metastable states is introduced. In section 4 we remark on the main conclusions of this work.

2. The model

In each node, x, of a finite d-dimensional simple cubic lattice, $\Lambda \subset Z^d$, we define a spin variable, s_x, which can take $+1$ or -1 values. A system configuration s is given by a set of values s_x in Λ, i.e. $s \equiv \{s_x; x \in \Lambda\}$, and it evolves in time according to a stochastic Markovian spin-flip dynamics. The system time evolution is characterized by the probability to find the system in a configuration s at a given time t, i.e. $P(s; t)$, which obeys a Markovian master equation

$$\frac{\partial}{\partial t} P(s; t) = \sum_{x\in\Lambda} [w(s^x; x)P(s^x; t) - w(s; x)P(s; t)] \tag{2.1}$$

where s^x is the s-configuration with the spin at x flipped, i.e. $s_x \to -s_x$, and $w(s; x)$ is the transition probability per unit time that the spin at x flips [2]. Once the rate $w(s; x)$ is specified, the model is totally defined. In particular, if we want to describe a system evolving from an arbitrary initial state towards a final equilibrium one, which is characterized by a Hamiltonian, $H(s)$, we may use

$$w(s; x) = \phi(H(s^x) - H(s)) \qquad \phi(\lambda) = e^{-\lambda}\phi(-\lambda) \tag{2.2}$$

which guarantees that the stationary distribution is $P_{st}(s) \equiv \lim_{t\to\infty} P(s; t) = Z^{-1}\exp(-H(s))$. We will assume the normalization $\phi(0) = 1$.

As we said above, we are interested in the study of simple systems with non-equilibrium stationary states. As we explained in the introduction, a way to build this kind of system is by introducing the competititon of two dynamic mechanisms, each driving the system to a different final equilibrium state (for their physical relevance, see [3–5]), that is,

$$w(s; x) = p\phi(H_1(s^x) - H_1(s)) + (1-p)\phi(H_2(s^x) - H_2(s)) \tag{2.3}$$

where $p \in [0, 1]$ gives the relative frequency between the two mechanisms. Obviously, when $p = 1(0)$ the final stationary state is an equilibrium one with Hamiltonian $H_{1(2)}(s)$. In this paper we consider for simplicity H_1 and H_2 being the ferromagnetic and antiferromagnetic mean field Hamiltonians for the Ising model, i.e.

$$H_{1(2)}(s) = -2NKm_{F(A)}(s)^2 \tag{2.4}$$

where $K \equiv J/k_B T$, T being the system temperature,

$$m_{F(A)}(s) = \frac{1}{2N}\sum_{x\in\Lambda_1} s_x + (-)\frac{1}{2N}\sum_{x\in\Lambda_2} s_x \tag{2.5}$$

are the ferromagnetic and antiferromagnetic order parameters, respectively, and $\Lambda_{1,2}$ are two disjoint sets of N points, $N \equiv |\Lambda|/2$. One can imagine the lattice as composed by two equal interpenetrating sublattices. When the initial condition for $P(s; t)$ is such that all spins in each sublattice, Λ_i, are equivalent, we can perform in the master equation (2.1) a partial sum over configurations with fixed magnetizations at each sublattice or, equivalently, with fixed m_F and m_A. That is, we can define

$$Q(m_F, m_A; t) = \sum_s \delta(m_F - m_F(s))\delta(m_A - m_A(s))P(s; t). \tag{2.6}$$

Taking a partial derivative with respect to time in equation (2.6) and using the master equation (2.1) and equations (2.3)-(2.5) we get

$$\begin{aligned}\frac{\partial}{\partial t} Q(m_F, m_A; t) &\\ = \sum_{\mu=\pm 1} \sum_{\nu=\pm 1} \bigg[&c\left(\mu, \nu; \frac{2}{N}\right) Q\left(m_F + \frac{\mu}{N}, m_A + \frac{\nu}{N}; t\right) \\ &- c(\mu, \nu; 0) Q(m_F, m_A; t)\bigg]\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}c(\mu, \nu; m_F, m_A, \Delta) &\\ = \frac{N}{2}(1 + \mu m_F + \nu m_A + \Delta) &\\ \times \left\{ p\phi\left[2K\left(2m_F\mu + \Delta - \frac{1}{N}\right)\right] + (1-p)\phi\left[2K\left(2m_A\nu + \Delta - \frac{1}{N}\right)\right]\right\}.\end{aligned} \tag{2.8}$$

In the limiting case $p = 1(0)$ the master equation (2.7) has, by construction, the following equilibrium stationary solution:

$$Q_e(m_F, m_A) = \binom{N}{\frac{1}{2}N(1 + m_F + m_A)} \binom{N}{\frac{1}{2}N(1 + m_F - m_A)} \exp(2NKm^2_{F(A)}). \tag{2.9}$$

For these cases and in the thermodynamic limit, $N \to \infty$, the model exhibits a phase transition [1]: $v_{F(A)} \equiv \langle m_{F(A)} \rangle \neq 0$ when $K > K_c = \frac{1}{2}$ and $v_{F(A)} = 0$ when $K < K_c$, being all the stationary properties independent of the functional form, ϕ, considered for the dynamics. Some dynamic properties, such as the decay time for metastable states and the tunnel effect, have been studied by computer simulation methods [6] and, as expected, they depend on the used rate. The goal of this paper is to study not the dynamic but the stationary behaviour of the system and the rate-type influence on its properties.

3. The master equation N-expansion

The general solution of the master equation (2.7) is highly non-trivial for an arbitrary N-value. It is natural in the statistical mechanics context to study the microscopic systems from a macroscopic point of view, i.e. when $N \to \infty$. In this case, and assuming that the central limit theorem holds for the spin stochastic variables, it is natural to write

$$m_F = v_F(t) + \frac{1}{\sqrt{N}}\eta \qquad m_A = v_A(t) + \frac{1}{\sqrt{N}}\xi \tag{3.1}$$

where $v_{F(A)}(t)$ are functions to be determined and η and ξ are stochastic variables describing the fluctuations around v_F, v_A, respectively. Introducing the expressions of equations (3.1) in equations (2.7) and (2.8), expanding these in powers of N (this is the well known Van Kampen N-expansion [7]) and identifying the leading orders, we get

$O(\sqrt{N})$:

$$\begin{aligned}\frac{dv_F}{dt}&=-\sum_{\mu=\pm1}\sum_{\nu=\pm1}c_0(\mu,\nu;v_F,v_A)\mu\\ \frac{dv_A}{dt}&=-\sum_{\mu=\pm1}\sum_{\nu=\pm1}c_0(\mu,\nu;v_F,v_A)\nu\end{aligned}\tag{3.2}$$

$O(N^0)$:

$$\begin{aligned}\frac{\partial}{\partial t}\Pi(\eta,\xi;t)=\bigg\{&b_{00}^{(2)}(\eta,\xi,2)-b_{00}^{(2)}(\eta,\xi,0)\\ &+b_{10}^{(1)}(\eta,\xi,0)\frac{\partial}{\partial\eta}+b_{01}^{(1)}(\eta,\xi,0)\frac{\partial}{\partial\xi}\\ &+\frac{b_{00}^{(0)}(\eta,\xi,0)}{2}\left[\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right]+b_{11}^{(0)}(\eta,\xi,0)\frac{\partial^2}{\partial\eta\partial\xi}\bigg\}\Pi(\eta,\xi;t)\end{aligned}\tag{3.3}$$

where $\Pi(\eta,\xi;t)$ is the probability distribution at time t for the random variables η and ξ; we have used

$$\begin{aligned}c&\left(\mu,\nu;\eta,\xi,v_F,v_A,\frac{a}{N}\right)\\ &=Nc_0(\mu,\nu;v_F,v_A)+\sqrt{N}\,c_1(\mu,\nu;\eta,\xi,v_F,v_A)\\ &\quad+c_2(\mu,\nu;\eta,\xi,v_F,v_A,a)+O(N^{-1/2})\end{aligned}\tag{3.4}$$

with

$$\begin{aligned}c_0(\mu,\nu;v_F,v_A)&=\tfrac{1}{2}(1+\mu v_F+\nu v_A)[p\phi(4K\mu v_F)+(1-p)\phi(4K\nu v_A)]\\ c_1(\mu,\nu;\eta,\xi,v_F,v_A)&\\ &=2(1+\mu v_F+\nu v_A)[K\mu\eta p\phi'(4K\mu v_F)+K(1-p)\nu\xi\phi'(4K\nu v_A)]\\ &\quad+\tfrac{1}{2}(\mu\eta+\nu\xi)[p\phi(4K\mu v_F)+(1-p)\phi(4K\nu v_A)]\\ c_2(\mu,\nu;\eta,\xi,v_F,v_A,a)&\\ &=(1+\mu v_F+\nu v_A)[pK(a-1)\phi'(4K\mu v_F)+(1-p)K(a-1)\phi'(4K\nu v_A)\\ &\quad+4pK^2\eta^2\phi''(4K\mu v_F)+4(1-p)K^2\xi^2\phi''(4K\nu v_A)]\\ &\quad+(\mu\eta+\nu\xi)[2pK\mu\eta\phi'(4K\mu v_F)+2(1-p)K\nu\xi\phi'(4K\nu v_A)]\\ &\quad+\tfrac{1}{2}a[p\phi(4K\mu v_F)+(1-p)\phi(4K\nu v_A)]\end{aligned}\tag{3.5}$$

and

$$b_{nm}^{(i)}(\eta,\xi,a)\equiv\sum_{\mu,\nu}c_i(\mu,\nu;\eta,\xi,v_F,v_A,a)\mu^n\nu^m.\tag{3.6}$$

3.1. Macroscopic stationary properties and fluctuations

Let us study first the stationary solutions of the deterministic equations (3.2). Substituting equations (3.5) in equations (3.2) we get

$$\frac{\mathrm{d}v_{\mathrm{F}}}{\mathrm{d}t}=2p\phi(4Kv_{\mathrm{F}})\exp(2Kv_{\mathrm{F}})[\sinh(2Kv_{\mathrm{F}})-v_{\mathrm{F}}\cosh(2Kv_{\mathrm{F}})]$$
$$-2(1-p)v_{\mathrm{F}}\cosh(2Kv_{\mathrm{A}})\phi(4Kv_{\mathrm{A}})\exp(2Kv_{\mathrm{A}}) \tag{3.7}$$

$$\frac{\mathrm{d}v_{\mathrm{A}}}{\mathrm{d}t}=2(1-p)\phi(4Kv_{\mathrm{A}})\exp(2Kv_{\mathrm{A}})[\sinh(2Kv_{\mathrm{A}})-v_{\mathrm{A}}\cosh(2Kv_{\mathrm{A}})]$$
$$-2pv_{\mathrm{A}}\cosh(2Kv_{\mathrm{F}})\phi(4Kv_{\mathrm{F}})\exp(2Kv_{\mathrm{F}}). \tag{3.8}$$

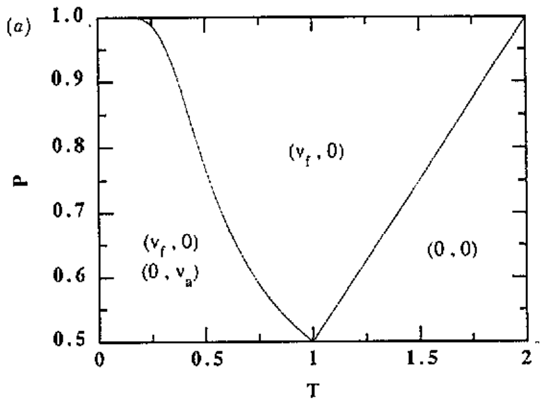
The stationary state, $\boldsymbol{v}_{\mathrm{st}}\equiv(v_{\mathrm{F,st}}, v_{\mathrm{A,st}})$, is defined as the solution of the latter system of equations with $\mathrm{d}v_{\mathrm{F,st}}/\mathrm{d}t=\mathrm{d}v_{\mathrm{A,st}}/\mathrm{d}t=0$.

Some general properties for any rate can be easily worked out from equations (3.7) and (3.8) (i) for the infinite temperature limit, $T\to\infty$, there is a unique solution $v_{\mathrm{st}}=(0,0)$ which is dynamically stable under small perturbations when $T>T_{\mathrm{c}}^{(1)}(p)\equiv 2\max\{p,1-p\}$, (ii) $T_{\mathrm{c}}^{(1)}$ is the critical temperature of a second-order phase transition with mean field critical exponents $v_{\mathrm{F}}\approx A(T_{\mathrm{c}}^{(1)}-T)^{1/2}$ where $A^{-2}=4p(24p\partial_{\lambda\lambda}\phi(\lambda)|_{\lambda=0}-1)$ and (iii) $v_{\mathrm{F}}+v_{\mathrm{A}}=1$ when $T=0$ and $v_{\mathrm{F}}, v_{\mathrm{A}}\neq 0$.

The structure of $\boldsymbol{v}_{\mathrm{st}}$ below $T_{\mathrm{c}}^{(1)}(p)$ is not trivial and it depends strongly on the rate ϕ we use. We have studied in particular two rates: $\phi_1(\lambda)=\mathrm{e}^{-\lambda/2}$ and $\phi_2(\lambda)=1-\tanh(\lambda/2)$ and, because equations (3.7) and (3.8) are invariant under the exchange of $p\leftrightarrow(1-p)$ and $v_{\mathrm{F}}\leftrightarrow v_{\mathrm{A}}$, we have studied only the case $p>\frac{1}{2}$.

Solving analytically and/or numerically equations (3.7) and (3.8) we show in figure 1 the solution structure for the rates ϕ_1 and ϕ_2. We see in both cases that there exists a new phase transition with critical temperature $T_{\mathrm{c}}^{(2)}<T_{\mathrm{c}}^{(1)}$ which is a function of the rate. In particular: $T_{\mathrm{c}}^{(2)}(p;\phi_1)=2[1+(1-p)p^{-1}\cosh(x)]^{-1}$, where x is solution of the implicit equation $px=(1-p)\sinh(x)$, and $T_{\mathrm{c}}^{(2)}(p;\phi_2)=2(1-p)$. The locally stable solutions when $T\in[T_{\mathrm{c}}^{(2)}(p;\phi_{1,2}), T_{\mathrm{c}}^{(1)}(p)]$ are of the form $\boldsymbol{v}_{\mathrm{st}}=\pm(v_{\mathrm{F}},0)$. Below $T_{\mathrm{c}}^{(2)}(p;\phi)$ the solution form depends also on the rate used. For the rate ϕ_1 there is a coexistence of two locally stable solutions, $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{I}}=(v_{\mathrm{F}}^{\mathrm{I}},0)$ and $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{II}}=(0,v_{\mathrm{A}}^{\mathrm{II}})$, while for ϕ_2 there is a single solution, $\boldsymbol{v}_{\mathrm{st}}=(v_{\mathrm{F}}, v_{\mathrm{A}})$.

In figures 2 and 3 the explicit behaviour of v_{F} and v_{A} for both rates as a function of temperature for fixed $p=0.8$, and as a function of p for different temperatures are shown, respectively. Some interesting properties are reflected in these figures. When the rate is ϕ_1, $v_{\mathrm{st}}^{\mathrm{II}}$ appears in a discontinuous form which is typical for and characteristic of a first-order phase transition and, *a priori* more surprisingly, $|v_{\mathrm{F}}^{\mathrm{I}}|$ and $|v_{\mathrm{A}}^{\mathrm{II}}|$ tend to one when $T\to 0$ for any $p\in[\frac{1}{2},1)$. That is, it may be possible to find a pure antiferromagnetic state for $p\approx 1$ (an almost pure ferromagnetic dynamic mechanism) at low enough temperatures. This solution has to be unphysical because one expects that, in these mean field system types, small perturbations around the equilibrium state will give equally small corrections for the physical obervables. The latter argument implies that $v_{\mathrm{st}}^{\mathrm{II}}$ is not a real macroscopic stationary state but a metastable one. In contrast to the latter case, for the rate ϕ_2 there is a true second-order phase transition below $T_{\mathrm{c}}^{(2)}$. The behaviour of v_{F} and v_{A} is regular and mutually independent because the evolution equations (3.7) and (3.8) are decoupled for this particular rate and they reach their maximum values, $v_{\mathrm{F}}=p$ and $v_{\mathrm{A}}=1-p$, at $T=0$. It is interesting to remark on the sublattice magnetization behaviour, $v_1\equiv v_{\mathrm{F}}+v_{\mathrm{A}}$ and $v_2\equiv v_{\mathrm{F}}-v_{\mathrm{A}}$, in this case: when



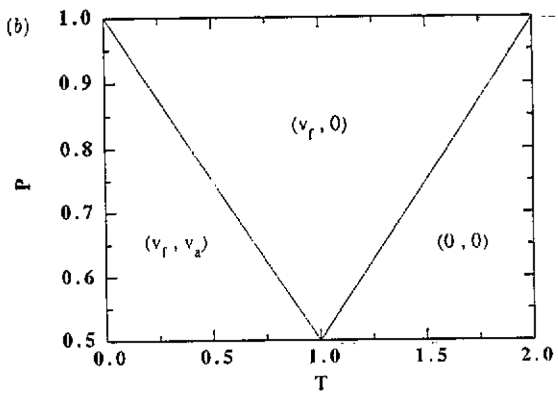


Figure 1. The stationary solutions structure arising from equations (3.7) and (3.8) for the rates (*a*) $\phi_1(\lambda) = e^{-\lambda/2}$ and (*b*) $\phi_2(\lambda) = 1 - \tanh(\lambda/2)$.

$T > T_c^{(2)}$, both sublattices have the same magnetization, $v_1 = v_2$, which presents a second-order phase transition at $T_c^{(1)}$; when $T < T_c^{(2)}$ one of the sublattices tends to saturation at low temperatures, i.e. $v_1 \to 1$ when $T \to 0$, and, in compensation, the other sublattice loses net magnetization and reaches its minimum value, $v_2 \to 2p - 1$, when $T \to 0$.

The probability distribution of the system fluctuations, η and ξ, around the deterministic solution, v, which was computed in the last section, can be obtained by solving the Fokker–Planck equation (3.3). This equation can be solved explicitly in this model by using standard techniques (see [7]), and it can be shown that the general solution is a Gaussian distribution. In particular, when $T > T_c^{(1)}(p)$ we get for any rate ϕ that the fluctuations decay exponentially fast towards its stationary Gaussian distribution:

$$\Pi_{st}(\eta, \zeta) = \frac{1}{\pi} \sqrt{1 - 2pK}\, \sqrt{1 - 2(1-p)K}\, \exp\{-(1 - 2pK)\eta^2 - [1 - 2(1-p)K]\xi^2\}. \qquad (3.9)$$

When $T \to T_c^{(1)}$, the Gaussian width goes to infinity and its normalization constant goes to zero, as the distribution is singular at the critical point. It is interesting to remark that the latter is a consequence of the fact that the critical behaviour appears at all observational scales. For temperatures below $T_c^{(1)}$, the mean and the standard deviation of the distribution depend on the rate we use.

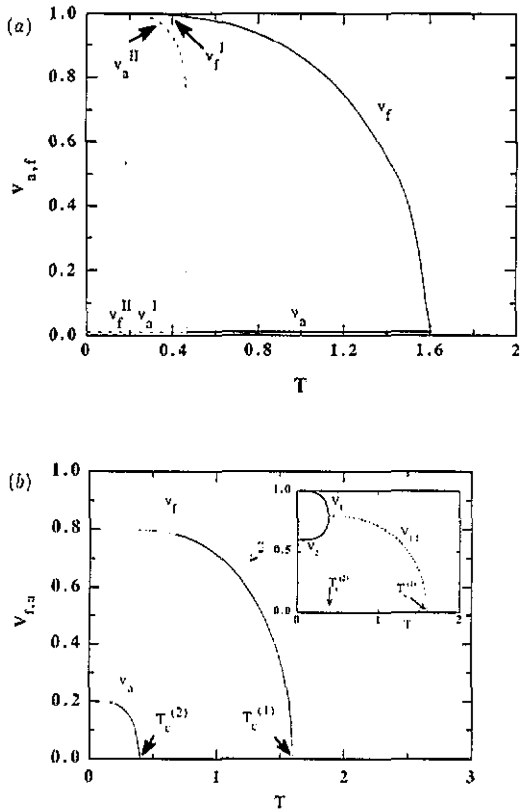


Figure 2. Phase diagrams for $v_{F,A}$ versus T for fixed $p=0.8$ and for the rates (a) ϕ_1 and (b) ϕ_2. The inset in (b) shows the behaviour of $v_{1(2)}=v_F+(-)v_A$.

3.2. Effective free energy functional: metastable state analysis

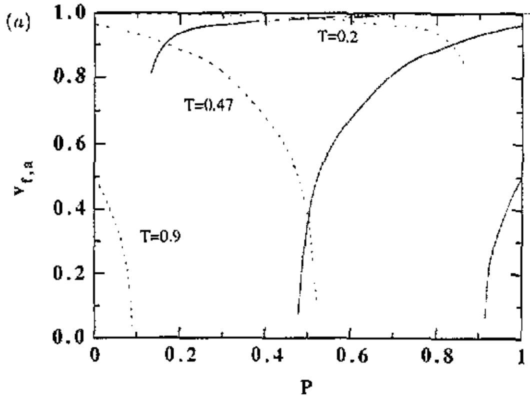
In general, whether or not a stationary state has a metastable or stable character is not a simple question to be answered in the context of a general non-equilibrium model system. This is because there is a lack of a well-defined functional potential, similar to the functional free energy for equilibrium problems, such that their absolute minima correspond to the stable states while the other relative minima are transient or metastable states. Some attempts to build a theory have been developed (see [8] and references therein) but, in practice, the resulting ones are too complicated to be applied. In this section we are going to develop a simple scheme which can be useful in order to discriminate between those types of locally stable states.

Let us define a new dynamic process in which, superimposed on the original microscopic dynamic mechanism, there is an external one which is able to fix at any time the macroscopic field to a given arbitrary value, $v=(v_F, v_A)$. This new process is described by the master equation

$$\frac{\partial}{\partial t} Q(s|v;t)=\sum_x (w(s^x|v;x)Q(s^x|v;t)-w(s|v;x)Q(s|v;t)) \tag{3.10}$$

where

$$w(s|v;x)=p\phi(4Kv_F s_x)+(1-p)\phi(4\rho K v_A s_x) \tag{3.11}$$



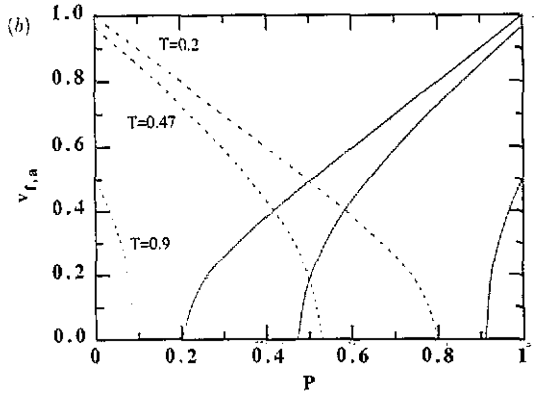


Figure 3. Phase diagrams for $v_{F,A}$ versus p for different temperatures and for the rates (a) ϕ_1 and (b) ϕ_2.

and $\rho = 1(-1)$ when $x \in \Lambda_{1(2)}$. The stationary distribution, $Q_{st}(s\,|\,v)$, for the dynamic process defined by equations (3.10) and (3.11) is given by

$$Q_{st}(s\,|\,v) = \frac{1}{Z(v)} \exp\left(- \sum_{i=1,2} h_i(v) \sum_{x\in\Lambda_i} s_x \right) \tag{3.12}$$

where

$$h_i(v) = \frac{1}{2}\log\left(\frac{p\phi(4Kv_F) + (1-p)\phi[4(3-2i)Kv_A]}{p\phi(-4Kv_F) + (1-p)\phi[4(2i-3)Kv_A]} \right) \qquad i = 1,2 \tag{3.13}$$

and

$$\begin{aligned} Z(v) &= \sum_s \exp\left(- \sum_{i=1,2} h_i(v) \sum_{x\in\Lambda_i} s_x \right) \\ &= \sum_{m_F, m_A} \exp[-N(h_1(v) + h_2(v))m_F - N(h_1(v) - h_2(v))m_A] \\ &\quad \times \sum_s \delta(m_F - m_F(s))\delta(m_A - m_A(s)) \\ &\equiv \sum_m \exp(-Nf(m\,|\,v)). \end{aligned} \tag{3.14}$$

The free energy functional, $f(\boldsymbol{m}|\boldsymbol{v})$, defined in equation (3.14), characterizes the fluctuation distribution of the $\boldsymbol{m}$-field for a given fixed external macroscopic field $\boldsymbol{v}$. In the thermodynamic limit, $N\to\infty$, the most probable states are the minima values of the free energy functional, $(\partial f(\boldsymbol{m}|\boldsymbol{v})/\partial \boldsymbol{m})_{\boldsymbol{m}=\boldsymbol{m}^*}=0$, i.e.

$$m^*_{\mathrm{F(A)}}=-\tfrac{1}{2}\tanh(h_1(\boldsymbol{v}))-(+)\tfrac{1}{2}\tanh(h_2(\boldsymbol{v})). \tag{3.15}$$

Up to this point we have only defined a new simple model distantly related to the one we have studied above from the master equation (2.7). Nevertheless, we can build a close relationship between both models by assuming that the original model defined by the spin-flip dynamics given by equations (2.3), evolves 'effectively' by using the following scheme: let us assume that the system at some given time t is in a macroscopic state characterized by the macroscopic field $\boldsymbol{v}_t$; due to microscopic fluctuations acting on a faster timescale in which $\boldsymbol{v}_t$ seems frozen, the system tries to evolve to the most probable fluctuation given by the minima of the free energy functional $f(\boldsymbol{v}_{t+\Delta t}|\boldsymbol{v}_t)$ where Δt is the natural timescale for the process. Namely, using equation (3.15) we can write a set of effective evolution equations for the macroscopic field:

$$v_{\mathrm{F}}(t+\Delta t)=\frac{-p\phi(4Kv_{\mathrm{F}}(t))[1-\exp(4Kv_{\mathrm{F}}(t))]}{p\phi(4Kv_{\mathrm{F}}(t))[1+\exp(4Kv_{\mathrm{F}}(t))]+(1-p)\phi(4Kv_{\mathrm{A}}(t))[1+\exp(4Kv_{\mathrm{A}}(t))]} \tag{3.16}$$

$$v_{\mathrm{A}}(t+\Delta t)=\frac{-(1-p)\phi(4Kv_{\mathrm{A}}(t))[1-\exp(4Kv_{\mathrm{A}}(t))]}{p\phi(4Kv_{\mathrm{F}}(t))[1+\exp(4Kv_{\mathrm{F}}(t))]+(1-p)\phi(4Kv_{\mathrm{A}}(t))[1+\exp(4Kv_{\mathrm{A}}(t))]}. \tag{3.17}$$

It is trivial to check that the fixed point for the latter dynamic system, i.e. $\boldsymbol{v}_{t+\Delta t}=\boldsymbol{v}_t=\boldsymbol{v}^*$, is the stationary state that we found by solving directly equations (3.7) and (3.8) in section 3.1.

In particular, it is interesting to study the case with rate $\phi_1(\lambda)=\mathrm{e}^{-\lambda/2}$ at some temperature $T<T_c^{(2)}$. In this case we see in figure 4 the dynamic flux described by equations (3.16) and (3.17) for different initial conditions. The local dynamic stability of both solutions described above, $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{I}}$ and $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{II}}$, is clear from the figure. It is interesting to remark that the dynamic flows obtained from equations (3.16) and (3.17) are very similar to those we got by solving equations (3.7) and (3.8). The main advantage in this context comes from the fact that, with our effective dynamic system, we have a

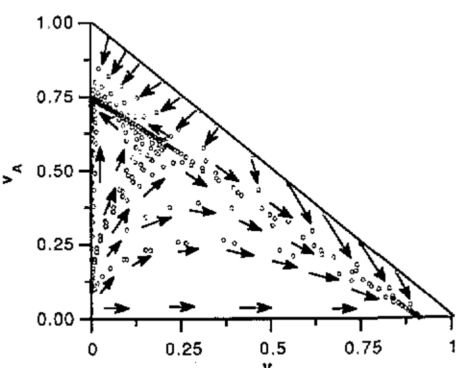


Figure 4. Dynamic flux arising from iteration of equations (3.19) and (3.20) for the rate ϕ_1, $p=0.6$ and $T<T_c^{(2)}$.

function which measures the probability of going from one point to another in the phase space:

$$P(\boldsymbol{m} \to \boldsymbol{m}') = \frac{\exp(-Nf(\boldsymbol{m}'|\boldsymbol{m}))}{\int \mathrm{d}\boldsymbol{v} \exp(-Nf(\boldsymbol{v}|\boldsymbol{m}))}. \tag{3.18}$$

This is essential in order to determine the global stability of a set of locally stable stationary states $\{\boldsymbol{v}_i\}$.

Once the system has reached one of these stationary states, say $\boldsymbol{v}_i$, let us assume *that the most probable fluctuation process is the following: the system evolves to any* other state, say $\boldsymbol{m}$, with probability $P(\boldsymbol{v}_i \to \boldsymbol{m})$ and from there it evolves to the nearest attractor, say $\boldsymbol{v}_j$, by means of the deterministic equations (3.16) and (3.17). A more general formulation of the fluctuation process can be given by means of path integrals, but this is beyond the scope of this paper. In these terms, we can say that a given stationary state $\boldsymbol{v}_i$ is metastable with respect to another $\boldsymbol{v}_j$ whenever the following condition holds:

$$\int_{\boldsymbol{m}\in D(\boldsymbol{v}_i)} \mathrm{d}\boldsymbol{m}\, P(\boldsymbol{v}_j \to \boldsymbol{m}) < \int_{\boldsymbol{m}\in D(\boldsymbol{v}_j)} \mathrm{d}\boldsymbol{m}\, P(\boldsymbol{v}_i \to \boldsymbol{m}) \tag{3.19}$$

where $D(\boldsymbol{v})$ is the set of initial states evolving by means of the deterministic equations (3.16) and (3.17) to the stationary state $\boldsymbol{v}$. When $N \to \infty$, and because the functional $f(\boldsymbol{m}|\boldsymbol{v}_i)$ is in our case a monotonous function in $\boldsymbol{m}$ for any fixed $\boldsymbol{v}_i$, the equation (3.19) can be simplified by using the steepest descent method to

$$\exp[-N(f(\boldsymbol{x}_j|\boldsymbol{v}_j) - f(\boldsymbol{v}_j|\boldsymbol{v}_j))] < \exp[-N(f(\boldsymbol{x}_i|\boldsymbol{v}_i) - f(\boldsymbol{v}_i|\boldsymbol{v}_i))] \tag{3.20}$$

or, equivalently,

$$f(\boldsymbol{x}_j|\boldsymbol{v}_j) - f(\boldsymbol{x}_i|\boldsymbol{v}_i) - f(\boldsymbol{v}_j|\boldsymbol{v}_j) + f(\boldsymbol{v}_i|\boldsymbol{v}_i) > 0 \tag{3.21}$$

where $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are two points which minimize $f(\boldsymbol{m}|\boldsymbol{v}_i)$ and $f(\boldsymbol{m}|\boldsymbol{v}_j)$ respectively along the separatrix between both domains of attraction, $D(\boldsymbol{v}_i)$ and $D(\boldsymbol{v}_j)$.

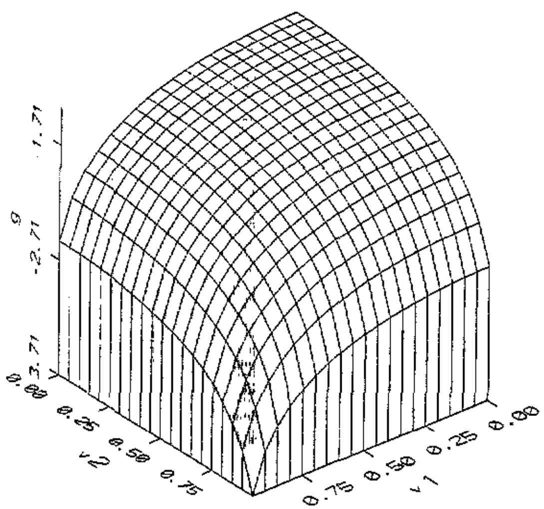


Figure 5. Behaviour of the free energy functional $g(v) \equiv g(v_F, v_A) = g((v_1+v_2)/2, (v_1-v_2)/2)$, defined in equation (3.17) for the same case as in figure 4.

It is easy to show that in our mean field model the contribution of the first two terms in equation (3.21) may be neglected. Therefore, the condition given by equation (3.21) is reduced to $g(\boldsymbol{v}_i) \equiv f(\boldsymbol{v}_i | \boldsymbol{v}_i) > g(\boldsymbol{v}_j)$. In figure 5 we show the behaviour of the function $g(\boldsymbol{v}_{\mathrm{st}})$. In particular, we find that $g(\boldsymbol{v}_{\mathrm{st}}^{\mathrm{I,II}}) = -2\log(2) + \log(1 - v_{\mathrm{F,A}}^2)$; from figure 2($a$) we see that $v_{\mathrm{F}} > v_{\mathrm{A}}$, implying that $g(\boldsymbol{v}_{\mathrm{st}}^{\mathrm{I}}) < g(\boldsymbol{v}_{\mathrm{st}}^{\mathrm{II}})$. Namely, we confirm our previous heuristic argument that $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{I}} = (v_{\mathrm{F}}, 0)$ is the 'true' stationary state, $\boldsymbol{v}_{\mathrm{st}}^{\mathrm{II}} = (0, v_{\mathrm{A}})$ being a metastable one.

4. Conclusions

We have analysed a relatively simple mean field model system for which it is possible to study in detail the dependence of the non-equilibrium stationary properties on the functional form of the microscopic dynamic rates. This dependence is studied at deterministic and fluctuating levels of description. It is checked that the system phase diagram changes not only quantitatively but also qualitatively from one rate to another. Finally, it is possible to build an effective dynamic mechanism which defines in a natural form a local free energy functional that allows us to distinguish between the stable and the metastable or transient states that appear in certain cases.

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