PRELIMINARY DRAFT:
Notes about the Macroscopic Fluctuating Theory

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Abstract

We take as starting point the assumption that a system at a mesoscopic scale is described by a field $\phi(x,t)$ that evolves by a Langevin equation that locally conserves or not the field. Its dynamic behavior may also depend on the action of external agents on the bulk or/and at the system's boundaries. We derive the corresponding Fokker-Planck equation and the probability of a path and we use them to study general properties of the system stationary state. In particular we focus on the study of the quasi-potential that defines the stationary distribution at the small noise limit. We argue that the system is at equilibrium when it is *macroscopic reversible*, that is when the most probable path to create a fluctuation from the stationary state is equal to the time reversed path that relaxes it. When this doesn't occurs the system is in a nonequilibrium stationary state whose quasi-potential presents some lack of differentiability. We also derive closed equations for the two-body correlations at the stationary state and we apply them to some typical cases. Finally we obtain generalized Green-Kubo class of formulas by using the Large Deviation Principle.

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I. INTRODUCTION

When we study systems at nonequilibrium states we inmediately realize the hard task we face in order to get some general result, a prediction to be checked by experiments (numerical or not) or just reproduce some observation. That is so even when we define very simple theoretical models at microscopic level that, once defined their interactions with some external agents or boundary conditions, they develop a colective nonlinear behavior that strongly depends on such external influences. A typical set of nonequilbrium models that have been extensively studied in the last decade are just driven by boundary agents. They have a microscopic bulk hamiltonian dynamics with boundary conditions that produce some type of current that go across the system. The typical example is a container filled with interacting gas particles submitted to thermal baths with different temperatures at some of its boundaries. These kind of systems are specially interesting because when the thermal baths have equal temperatures the system stationary state is the thermal equilibrium. We know that at equilibrium we have the Thermodynamics and the Boltzmann-Gibbs ensemble theories to understand and predict the system macroscopic and mesoscopic (fluctuating) behavior. The existence of just a set of parameters that drives a system from equilibrium to a nonequilibrium state make these system very interesting to be theoretically studied. In fact there has been much effort in the last decades in order to extend these wonderfull equilibrium theories to them. However the problem is so difficult even in those cases that, for instance, it is still unresolved how we can (rigorously) derive their hydrodynamic macroscopic description from their microscopic dynamics.

In the last years a very interesting effort has been done by Bertini at al. [1]. They have developed a mesoscopic theory for diffusive systems that they called *Macroscopic Fluctuating Theory (MFT)*. MFT is based on the existence of a well defined set of hydrodynamic equations togehter with the assumption that a Large Deviation Principle can be apply that describes the fluctuating behavior of the macroscopic fields and on the top of it they use a Fundamental Principle that connects the way the system relax to how it creates a fluctuation. All those items previously were rigurously proven in some one dimensional microscopic stochastic nonequilibrium models as for example the Symmetric Simple Exclusion Process (SSEP), the Weakly Asymetric Exclusion Process (WASEP) or the Kipnis Marchioro Presutti model (KMP) (see for instance the review by Bertini et al. [2]). From this starting point MFT intends to obtain general properties on such systems in a very serious attempt to globally understand the behavior of diffusive systems.

In this paper we want to extend (in a non-rigurous way) their seminal work to more general nonequilibrium systems. In order to do that we assume that of our system is defined at the mesoscopic level by a continuous Langevin equation with a local white noise field that is time and spatial uncorrelated. This set up let us to reproduce the results from MFT and to study more general conservative systems and/or nonconservative ones. Obviously in many cases we lack of any rigurous connection of such equations from a microscopic model. Nevertheless, the goal in this paperis to look for general properties and concepts that maybe useful to the overall undestanding of nonequilibrium system.

In our paper we have focused on one component Langevin equations with conserved or non-conserved dynamics. In section II we define the basic starting equations. We also define the corresponding Fokker-Planck equations and the Path Probability expressions. Section III is devoted to study the stationary probability distribution in the small noise limit that is defined by a functional of the fields that is called *quasi-potential* . We use the stationary Fokker-Planck equation to obtain a closed partial differential equation for the quasi-potential. We formally solve the corresponding Hamilton-Jacobi equation by using the method of characteristics. We show that the effective dynamic equations are describing the most probable path to create a given fluctuation (*the dual dynamics*). The dual dynamics differs, in general, from the time reversed deterministic dynamics. That is, the most probable path that the system uses to relax is different to the one it follows to create the fluctuation. We introduce in Section IV the macroscopic reversibility propery that a system has when the dual dynamics and the time reversed deterministic dynamics coincide. We show that the systems at equilibrium are macroscopic reversible and that their quasi-potentials has existing and continuous first and second functional derivatives with respect the fields. That was the Onsager's idea when study dynamic fluctuations of systems at equilibrium where microscopic reversibility was assumed to be extended to the mesoscopic level. Finally we also show that the original MFT Fundamental Principle holds in our context and we interpret it as a generalized detailed balance condition on paths. In Section V we study steady state correlations. We obtain the general set of closed equations to study them for the conserved and non-conserved case and we apply them to some well known situations. In section VI we use those equations to try to build the conditions in which conserved and non-conserved situations develope the same quasi-potential. That is an attempt to build nonequilibrium dynamical ensembles. Finally in section VII we show how to use the Large Deviation Principle to obtain generalized Green-Kubo relations.

II. LANGEVIN DESCRIPTION OF MESOSCOPIC SYSTEMS

Our system at a mesoscopic level of description is characterized by a scalar field $\phi(x,t) \in \mathbb{R}$ where $x \in \Lambda \subset \mathbb{R}^d$, d is the spatial dimension and t is the time. We assume that the system dynamics is given by a mesoscopic Langevin equation with a white noise. Along this work we are going to consider two separate models: the reaction (non-conserved local dynamics) and the diffusive (conserved local dynamics) cases. Their corresponding Langevin equations are:

- *Reaction dynamics (RD):*

$$\partial_t \phi(x,t) = F[\phi; x, t] + h[\phi; x, t]\xi(x,t) \tag{1}$$

- *Diffusion dynamics (DD):*

$$\partial_t \phi(x,t) + \nabla \cdot j = 0 \tag{2}$$

  with

$$j_\alpha[\phi; x, t] = G_\alpha[\phi; x, t] + \sum_{\beta=1}^{d} \sigma_{\alpha,\beta}[\phi; x, t]\psi_\beta(x,t) \quad \alpha = 1, \ldots, d \tag{3}$$

where F, G, h and σ are given local funcionals of $\phi(x,t)$, $\nabla\phi(x,t)$, ... We take $\xi(x,t)$ and $\psi_\alpha(x,t)$ to be uncorrelated gaussian random variables:

$$\begin{aligned}
\langle \xi(x,t) \rangle &= 0 \\
\langle \xi(x,t)\xi(x',t') \rangle &= \frac{1}{\Omega}\delta(x-x')\delta(t-t') \\
\langle \psi_\alpha(x,t) \rangle &= 0 \\
\langle \psi_\alpha(x,t)\psi_\beta(x',t') \rangle &= \frac{1}{\Omega}\delta_{\alpha,\beta}\delta(x-x')\delta(t-t')
\end{aligned} \tag{4}$$

where Ω is the parameter that controls the time and spatial separation between the mesoscopic and macroscopic descriptions. It is assumed that its value is large and therefore, the fluctuations are going to be just perturbations to the *deterministic* or macroscopic case ($\Omega \to \infty$). That is why we call this setup *Macroscopic Fluctuating Theory.*

Our set of equations are completely determined by also given the boundary conditions $(\phi(x,t), x \in \partial\Lambda)$, and the initial condition $(\phi(x,0), x \in \Lambda)$.

These Langevin equations in the limit $\Omega \to \infty$ become a set of partial differential equations:

$$\partial_t \phi_D(x,t) = F[\phi_D; x, t] \quad \text{(RD case) or} \quad \partial\phi_D(x,t) + \nabla \cdot G[\phi_D; x, t] = 0 \quad \text{(DD case)} \tag{5}$$

whose solutions, ϕ_D (for any given initial conditions) are called *Deterministic or Classical solutions*. A *stationary state*, ϕ^*, is defined by the stationary solution of the deterministic equation:

$$F[\phi^*; x] = 0 \quad \text{(RD case) or} \quad \nabla \cdot G[\phi^*; x] = 0 \quad \text{(DD case)} \tag{6}$$

From the Langevin equations we can construct some relevant descriptions very useful in order to analize general properties of the system behavior: the Fokker-Planck equation and the Path Probability.

A. The Fokker-Plank equation

The existence of the random variables ξ or ψ_α makes possible to characterize the system evolution by the probabilty to find the system at a configuration ϕ at time t, $P[\phi; t]$. This probability distribution can be derived from the Langevin equation:

$$P[\phi; t] = \langle \prod_{x \in \Lambda} \left[\delta \left(\bar{\phi}(x,t) - \phi(x,t) \right) \right] \rangle_{\xi,\psi} \tag{7}$$

where $\bar{\phi}(x,t)$ is the solution of the Langevin equation for a given noise realization, that is, it depends on the set of $\xi(x,t)$ or $\psi_\alpha(x,t)$ values. The average, $\langle \cdot \rangle_{\xi,\psi}$, is done with respect the gaussian disribution associated to the random variables. Obviously except for a few very simple cases we cannot solve the Langevin equation to get $P[\phi; t]$. Nevertheless we can construct a self contained differential equation from its definition. The idea is to discretize the time in the Langevin equation and to connect $P[\phi; t_{n+1}]$ with the previous one distribution, $P[\phi, t_n]$, by using the fact that the noise is time uncorrelated (truly a Marcov chain). There is not an unique way to discretize time and therefore the final form of the Fokker-Plank equation depends on the discretization scheme used. In any case we should stress that the averaged values of the observables obtained from the solution of any different schemes are always the same. That is, during their computation one should have in mind the kind of discretization used in order to solve some of the technical problems we can find (for instance what to do if we find a Heaviside function evaluated at zero). We have used in the Appendix I a generic discrete scheme and we have computed its corresponding Fokker-Planck equation to show explicitly the discretization effect. In this paper we are going to use the Ito's discretization scheme. The corresponding results for the RD and DD cases are in this case

- *Reaction dynamics (RD):*

$$\partial_t P[\phi;t] = \int_\Lambda dx \frac{\delta}{\delta\phi(x,t)} \left[-F[\phi;x,t]P[\phi;t] + \frac{1}{2\Omega}\frac{\delta}{\delta\phi(x,t)}\left(h[\phi;x,t]^2 P[\phi;t]\right)\right] \tag{8}$$

- *Diffusion dynamics (DD):*

$$\partial_t P[\phi;t] = \int_\Lambda dx \sum_{\alpha=1}^{d}\left(\partial_\alpha \frac{\delta}{\delta\phi(x,t)}\right)\Bigg[-G_\alpha[\phi;x,t]P[\phi;t] + \frac{1}{2\Omega}\sum_{\beta=1}^{d}\left(\partial_\beta\frac{\delta}{\delta\phi(x,t)}\right)\left(\chi_{\alpha,\beta}[\phi;x,t]P[\phi;t]\right)\Bigg] \tag{9}$$

where

$$\chi_{\alpha,\beta}[\phi;x,t] = \sum_{\gamma=1}^{d}\sigma_{\alpha,\gamma}[\phi;x,t]\sigma_{\beta,\gamma}[\phi;x,t] \tag{10}$$

and the operator is defined in the discrete version of the Langevin equation (see the Appendix I)

$$\left(\partial_\alpha\frac{\delta}{\delta\phi(x)}\right) = \lim_{a\to 0}\frac{1}{2a}\left(\frac{\partial}{\partial\phi(n+i_\alpha)} - \frac{\partial}{\partial\phi(n-i_\alpha)}\right) \tag{11}$$

where $x = na$. This operator has the nice property

$$\left(\partial_\alpha\frac{\delta}{\delta\phi(x,t)}\right)H[\phi;x,t] = \partial_\alpha\left(\frac{\delta}{\delta\phi(x,t)}H[\phi;x,t]\right) - \frac{\delta}{\delta\phi(x,t)}\left(\partial_\alpha H[\phi;x,t]\right) \tag{12}$$

B. The Path Probability

We can also ask ourselves about the probability to observe a particular time sequence of field values or *path*. Let us first to deduce it for the RD case. An arbitrary path is defined by the set: $\{\phi\}[t_0,t_1] = (\phi(x,t), x\in\Lambda, t\in[t_0,t_1])$. The probability to see such path is just the product of the probabilities to create the adequate set random values of $\xi(x,t)$ to create the path:

$$P[\{\phi\}[t_0,t_1]] = cte \int D\xi \exp\left[-\frac{\Omega}{2}\int_{-\infty}^{\infty} dt \int_\Lambda dx \xi(x,t)^2\right]$$
$$\prod_{t\in[t_0,t_1]}\prod_{x\in\Lambda}\delta\left(\xi(x,t) - \frac{\partial_t\phi(x,t) - F[\phi;x,t]}{h[\phi;x,t]}\right)$$
$$= cte \exp\left[-\frac{\Omega}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \left(\frac{\partial_t\phi(x,t) - F[\phi;x,t]}{h[\phi;x,t]}\right)^2\right] \qquad (13)$$

To get a similar equation for the DD case we should do a little more work. First, let us define the random variable $\nu(x,t)$ for a fix value of $\phi(x,t)$:

$$\nu(x,t) = \sum_{\alpha=1}^{d}\partial_\alpha\left(\sum_{\beta=1}^{d}\sigma_{\alpha,\beta}[\phi;x,t]\psi_\beta(x,t)\right) \qquad (14)$$

we observe that $\nu(x,t)$ is a sum of gaussian random variables and therefore it is a gaussian random variable. Its probability distribution is characterized just by its first two moments that we can compute explicitely:

$$\langle\nu(x,t)\rangle = 0$$
$$\langle\nu(x,t)\nu(x',t')\rangle = \frac{1}{\Omega}L[\phi;x,x',t]\delta(t-t') \qquad (15)$$

where

$$L[\phi;x,x',t] = \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\partial_\alpha\partial'_\beta\left[\chi_{\alpha\beta}[\phi;x,t]\delta(x-x')\right] \qquad (16)$$

Therefore the probability distribution for the $\nu(x,t)$ random variables is of the form:

$$P[\nu] = cte \exp\left[-\frac{\Omega}{2}\int_{-\infty}^{\infty} dt\int_\Lambda dx\int_\Lambda dx' M[\phi;x,x',t]\nu(x,t)\nu(x',t')\right] \qquad (17)$$

where $M[\phi;x,x',t]$ is the inverse of $L[\phi;x,x',t]$:

$$\int_\Lambda d\bar{x} L[\phi;x,\bar{x},t]M[\phi;\bar{x},x',t] = \delta(x-x') \qquad (18)$$

This last property can be easely proven in the discrete version. Let us assume that we have a gaussian distribution of the form:

$$P[\xi] = Z^{-1}\exp\left[-\frac{\Omega}{2}\sum_n\sum_{n'}A(n,n')\xi(n)\xi(n')\right] \quad Z = \frac{C(\Omega)}{(detA)^{1/2}} \qquad (19)$$

then

$$\langle\xi(n)\xi(n')\rangle = -\frac{2}{\Omega}\frac{\partial}{\partial A(n,n')}\log Z = \frac{1}{\Omega}(A^{-1})(n',n) \qquad (20)$$

Therefore, the Langevin equation corresponding to the DD case can be rewritten by:

$$\partial_t \phi(x,t) + \nabla \cdot G[\phi; x, t] + \nu(x,t) = 0 \tag{21}$$

where we have made use of the Ito's prescription, that is, the fields $\phi(x,t)$ in $\sigma_{\alpha\beta}[\phi; x, t]$ depend on the previous time ψ's. At this point we can copy the argument of the RD case to find:

$$P[\{\phi\}[t_0, t_1]] = cte \exp\Bigg[-\frac{\Omega}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \int_\Lambda dx' \left(\partial_t\phi(x,t) + \nabla G[\phi; x, t]\right) \\ M[\phi; x, x', t]\left(\partial_t \phi(x', t) + \nabla' G[\phi; x', t]\right)\Bigg] \tag{22}$$

Observe that in the DD case the current field $j(x,t)$ cannot be derived from the knowledge of $\phi(x,t)$ for dimensions higher than one (any current $j+q$ such that $\nabla q = 0$ implies the same Langevin Equation). Therefore we can naturally define the probability for a given ϕ and j path: $\{\phi, j\}[t_0, t_1] = ((\phi(x,t), j(x,t)), x \in \Lambda, t \in [t_0, t_1])$: $P[\{\phi, j\}[t_0, t_1]]$. That probability is related with $P[\{\phi\}[t_0, t_1]]$ by:

$$P[\{\phi\}[t_0, t_1]] = cte \int Dj P[\{\phi, j\}[t_0, t_1]] \tag{23}$$

where

$$P[\{\phi, j\}[t_0, t_1]] = cte \exp\Bigg[-\frac{\Omega}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \int_\Lambda dx' \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \left(j_\alpha(x,t) - G_\alpha[\phi; x, t]\right) \\ \left(\partial_\alpha \partial'_\beta M[\phi; x, x', t]\right)\left(j_\beta(x', t) - G_\beta[\phi; x', t]\right)\Bigg] \prod_{t\in[t_0,t_1]}\prod_{x\in\Lambda} \delta\left(\partial_t \phi + \nabla \cdot j\right) \tag{24}$$

This expression can be simplified. Let us substitute the definition of L in eq. (16) into eq. (18) :

$$-\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \partial_\alpha \left[\chi_{\alpha\beta}[\phi; x, t]\partial_\beta M[\phi; x, x', t]\right] = \delta(x - x') \tag{25}$$

We multiply both sides by a test function $f(x)$, integrate with respect x and derivate with respect ∂'_γ:

$$\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \int_\Lambda dx \partial_\alpha f(x) \chi_{\alpha\beta}[\phi; x, t]\partial'_\gamma \partial_\beta M[\phi; x, x', t] = \partial'_\gamma f(x') \quad \forall \gamma, f \tag{26}$$

therefore

$$\sum_{\beta=1}^{d} \chi_{\alpha\beta}[\phi; x, t]\partial'_\gamma \partial_\beta M[\phi; x, x', t] = \delta_{\alpha,\gamma}\delta(x - x') \tag{27}$$

and we find the relation

$$\partial_\alpha \partial'_\beta M[\phi; x, x', t] = (\chi[\phi; x, t]^{-1})_{\alpha\beta} \delta(x - x') \tag{28}$$

We can substitute this expression into eq.(24) and we find

$$P[\{\phi, j\}\,[t_0, t_1]] = cte \exp\left[-\frac{\Omega}{2} \int_{t_0}^{t_1} dt \int_\Lambda dx \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} (j_\alpha(x,t) - G_\alpha[\phi; x, t]) \right.$$
$$\left. (\chi[\phi; x, t]^{-1})_{\alpha\beta} \left(j_\beta(x,t) - G_\beta[\phi; x, t] \right) \right] \prod_{t \in [t_0, t_1]} \prod_{x \in \Lambda} \delta\left(\partial_t \phi + \nabla \cdot j \right) \tag{29}$$

III. THE STATIONARY STATE AND THE QUASI-POTENTIAL

Once we have defined the basic equations of our model system we can go forward and study first the stationary state of the system. The stationary probability distribution can be writen for large enough values of Ω as:

$$P_{st}[\phi] \simeq \exp\left[-\Omega V_0[\phi] \right] \tag{30}$$

where $V_0[\phi]$ is the so called *quasi-potential*. Observe that in the strict limit $\Omega \to \infty$ we get:

$$P_{st}[\phi] = \prod_{x \in \Lambda} \delta(\phi(x) - \phi^*(x)) \tag{31}$$

where $\phi^*(x)$ is solution of

$$\frac{\delta V_0[\phi^*]}{\delta \phi^*(x)} = 0 \quad \forall x \in \Lambda \tag{32}$$

Let us remark that ϕ^*, which is the minimum of the quasi-potential, coincides with the stationary deterministic solution given by eq. (6).

We can obtain V_0 from two different ways, by using the Fokker-Planck equation or from the path probability. Both give us some insight about the properties and structure of V_0.

A. V_0 for RD:

We substitute eq.(30) into eq.(8) and for $\Omega \to \infty$ we get :

$$\int_\Lambda dx \left[F[\phi; x] \frac{\delta V_0[\phi]}{\delta \phi(x)} + \frac{1}{2} h[\phi; x]^2 \left(\frac{\delta V_0[\phi]}{\delta \phi(x)} \right)^2 \right] = 0 \tag{33}$$

This is a *Hamilton-Jacobi* type of equation that it can be solved by the method of characteristics [5] (see Appendix II). The Hamiltonian associated to the Hamilton-Jacobi equation is:

$$H[\pi, \phi] = \int_\Lambda dx\, \pi(x) \left[F[\phi; x] + \frac{1}{2} h[\phi; x]^2 \pi(x) \right] \tag{34}$$

where

$$\pi(x) = \frac{\delta V_0[\phi]}{\delta \phi(x)} \tag{35}$$

The corresponding Hamilton evolution equations are then given by:

$$\begin{aligned} \partial_\tau \phi(x, \tau) &= \frac{\delta H}{\delta \pi(x, \tau)} = F[\phi; x, \tau] + h[\phi; x, \tau]^2 \pi[x, \tau] \\ \partial_\tau \pi(x, \tau) &= -\frac{\delta H}{\delta \phi(x, \tau)} = -\int_\Lambda dy\, \pi(y, \tau) \left[\frac{\delta F[\phi; y, \tau]}{\delta \phi(x, \tau)} + \frac{1}{2} \frac{\delta h[\phi; y, \tau]^2}{\delta \phi(x, \tau)} \pi(y, \tau) \right] \end{aligned} \tag{36}$$

The quasi-potential V_0 is obtained once we solve those evolution equations with initial conditions: $\phi(x, -\infty) = \phi^*(x)$ and $\pi(x, -\infty) = 0$ and then:

$$V_0[\phi] = V_0[\phi^*] + \int_{-\infty}^0 d\tau \int_\Lambda dx\, \pi(x, \tau) \partial_\tau \phi(x, \tau) \tag{37}$$

where $\phi(x, 0) = \phi(x)$. Let us mention that the evolution equations are nonlinear and it could be that for a given field ϕ there are several π-fields that pertain to the same time trajectory: $\{(\pi(\tau), \phi(\tau))\}_{\tau=-\infty}^0$. Obviously, all such values give rise to the same $V_0[\phi]$. In these cases, we should choose the π values that minimize the *action* that defines the probability of this path (as we will see). In other words, we will choose the path with higher probability. The main consequence of this phenomena is that at such ϕ the derivatives would be typically discontinuous (there are two different $\pi(x) = \delta V_0 / \delta \phi(x)$ depending on how we approach to ϕ with the time parameter τ).

Nevertheless there is a case in which we know the solution: the equilibrium that is attained by assuming $F[\phi; x] = \frac{1}{2} h[\phi; x]^2 \delta V[\phi] / \delta \phi(x)$. We can check directly in the evolution equations that for that value of F, $\pi(x, \tau) = \delta V[\phi] / \delta \phi(x)|_{\phi(x) = \phi(x, \tau)}$ and therefore $V_0[\phi] = V[\phi]$.

We can study the linearized dynamics near the initial condition $(0, \phi^*)$:

$$\begin{aligned} \partial_\tau \epsilon(x) &= \int_\Lambda dy A(x, y) \epsilon(y) + h[\phi^*; x]^2 \eta(x) \\ \partial_\tau \eta(x) &= -\int_\Lambda dy A(y, x) \eta(y) \end{aligned} \tag{38}$$

where $A(x,y) = \delta F[\phi;x]/\delta\phi(x)|_{\phi=\phi^*}$. The Lyapunov coefficients, λ, are solutions of the equations

$$det(A+\lambda\mathbb{I}) = 0 \quad , \quad det(-A+\lambda\mathbb{I}) = 0 \tag{39}$$

therefore the possible values of the Lyapunov exponents are 0 or $(-\lambda,\lambda)$ which is typical of a hamiltonian flow. That is, we can define a stable and unstable manifolds crossing the stationary point $(0,\phi^*)$. All the trajectories starting from the stationary point should pertain to the unstable manifold, M_u. This is important from a practical point of view if we want to solve the equations of motion: if we start from the stationary point we will be there forever and by other hand only the initial points (π_0,ϕ_0) that pertain to M_u will evolve to the stationary state when $\tau\to-\infty$. Therefore, the right strategy to obtain the quasi-potential is to reconstruct the unstable manifold around the stationary point and then to take any one pertaining to it as an initial point for solving the equations of motion (see for instance [9]).

We could also obtain the stationary state distribution by using the probability of a path. The main idea is to use the fact that the probability to go from any given starting and end points of a time time interval is just the sum of all the probabilities of each path that connects both states. Therefore we can write

$$P_{st}[\phi] = P_{st}[\phi^*]\int D\psi P[\{\psi\}[-\infty,0]]\prod_{x\in\Lambda}\delta(\psi(x,0)-\phi(x))\prod_{x\in\Lambda}\delta(\psi(x,-\infty)-\phi(x)^*) \tag{40}$$

where $P[\{\psi\}[t_0,t_1]]$ is given by eq.(13). In the limit $\Omega\to\infty$ the path integral is dominated by the most probable path $\{\bar\phi(x,t)\}[-\infty,0]$ which is solution of

$$\left.\frac{\delta L[\phi;-\infty,0]}{\delta\phi(x,t)}\right|_{\phi=\bar\phi} = 0 \quad \forall\, x\in\Lambda\,,\; t\in[-\infty,0] \tag{41}$$

where

$$L[\phi;t_0,t_1] = \frac{1}{2}\int_{t_0}^{t_1} d\tau\int_\Lambda dy\left(\frac{\partial_\tau\phi(y,\tau)-F[\phi;y,\tau]}{h[\phi;y,\tau]}\right)^2 \tag{42}$$

That is, the equation to be solved is:

$$\begin{aligned}&\partial_t\left(\frac{\partial_t\bar\phi(x,t)-F[\bar\phi;x,t]}{h[\bar\phi;x,t]^2}\right) = \\ &-\int_\Lambda dy\frac{\partial_t\bar\phi(x,t)-F[\bar\phi;x,t]}{h[\bar\phi;x,t]^2}\left[\frac{\delta F[\bar\phi;y,t]}{\delta\bar\phi(x,t)}+\frac{1}{2}\frac{\delta h[\bar\phi;y,t]^2}{\delta\bar\phi(x,t)}\frac{\partial_t\bar\phi(x,t)-F[\bar\phi;x,t]}{h[\bar\phi;x,t]^2}\right]\end{aligned} \tag{43}$$

with boundary conditions $\bar\phi(x,-\infty) = \phi(x)^*$ and $\bar\phi(x,0) = \phi(x)$. The quasi-potential is then:

$$V_0[\phi] = V_0[\phi^*] + L[\bar\phi;-\infty,0] \tag{44}$$

Let us remark two things:

- (1) $\bar{\phi}$ is the most probable path to *create* a given fluctuation ϕ and it has its own dynamics that it can be compared with the deterministic evolution equation (5) with boundary conditions $\phi_D(x,0) = \phi(x)$ and $\phi_D(x,\infty) = \phi(x)^*$ which describes the most probable path to *relax* from an arbitrary ϕ to the stationary state. The dynamics for ϕ is called the *dual dynamics* [2]. At equilibrium both dynammics are related by a time inversion operation: $\bar{\phi}(x,t) = \phi(x,-t)$ as we will see.

- (2) The equation (43) can be derived from from the Hamilton-Jacobi scheme (36) by just eliminating the π-field to build an unique effective evolution equation for ϕ's. That is, the effective *Hamiltonian* dynamics build from the Hamilton-Jacobi equation is equivalent to the *Lagrangian* dynamics build from the path integral scheme. In fact both objects are related by a Lagrange transformation.

$$\mathcal{L}[\phi] = \int_\Lambda dx\, \partial_t\phi(x,t)\pi(x,t) - H[\pi,\phi] \quad , \quad \frac{\delta H[\pi,\phi]}{\delta\pi(x,t)} = \partial_t\phi(x,t) \tag{45}$$

where

$$\mathcal{L}[\phi] = \frac{1}{2}\int_\Lambda dx \left(\frac{\partial_t\phi(x,t) - F[\phi;x,t]}{h[\phi;x,t]}\right)^2 \quad , \quad L[\phi;t_0,t_1] = \int_{t_0}^{t_1} dt\mathcal{L}[\phi] \tag{46}$$

The quasi-potential has a nice dynamic property: is a Lyapunov function for the deterministic and the dual dynamics. Let

$$S[\phi] = V_0[\phi] - V_0[\phi^*] \tag{47}$$

then

$$\frac{dS[\phi_D(t)]}{dt} \leq 0 \quad \text{and} \quad \frac{dS[\bar{\phi}(-t)]}{dt} \leq 0 \tag{48}$$

where $\phi_D(t) = \{\phi_D(x,t), x\in\Lambda\}$ and $\bar{\phi}(t) = \{\bar{\phi}(x,t), x\in\Lambda\}$ are the solutions of equations (5) and (43) respectively. Moreover,

$$\lim_{t\to\infty}\frac{dS[\phi_D(t)]}{dt} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{dS[\bar{\phi}(-t)]}{dt} = 0 \tag{49}$$

In other words, the deterministic and the time inverse dual dynamic evolutions tend to minimize the quasi-potential at all times. Let us show these relations. We can write in general:

$$\frac{dS[\phi(t)]}{dt} = \int_\Lambda dx \frac{\delta V_0[\phi(t)]}{\delta\phi(x,t)}\partial_t\phi(x,t) \tag{50}$$

we know that $\partial_t \phi_D(x,t) = F[\phi_D; x, t]$ and $\partial_t \bar{\phi}(x,-t) = -F[\bar{\phi}; x, -t] - h[\bar{\phi}; x, -t]^2 \pi(x,-t)$ for the deterministic and time reversed dual dynamics respectively. Therefore:

$$\begin{aligned} \frac{dS[\phi_D(t)]}{dt} &= \int_\Lambda dx \frac{\delta V_0[\phi_D(t)]}{\delta \phi_D(x,t)} F[\phi_D; x, t] \\ \frac{dS[\bar{\phi}(-t)]}{dt} &= \int_\Lambda dx \frac{\delta V_0[\bar{\phi}(-t)]}{\delta \bar{\phi}(x,-t)} \left(-F[\bar{\phi}; x, -t] - h[\bar{\phi}; x, -t]^2 \pi(x,-t)\right) \end{aligned} \tag{51}$$

we can use now the Hamilton-Jacobi equation (33) and we get the desired result:

$$\begin{aligned} \frac{dS[\phi_D(t)]}{dt} &= -\tfrac{1}{2} \int_\Lambda dx h[\phi_D; x, t]^2 \left(\frac{\delta V_0[\phi_D(t)]}{\delta \phi_D(x,t)}\right)^2 \leq 0 \\ \frac{dS[\bar{\phi}(-t)]}{dt} &= -\tfrac{1}{2} \int_\Lambda dx h[\bar{\phi}; x, -t]^2 \pi(x,-t)^2 \leq 0 \end{aligned} \tag{52}$$

B. V_0 for DD:

First we substitute eq.(30) into the Fokker-Planck equation for DD systems (eq.(9)) and we get the corresponding Hamilton-Jacobi type of equation:

$$\int_\Lambda dx \left[G[\phi] \cdot \nabla \frac{\delta V_0[\phi]}{\delta \phi(x)} + \frac{1}{2} \nabla \frac{\delta V_0[\phi]}{\delta \phi(x)} \cdot \chi[\phi] \nabla \frac{\delta V_0[\phi]}{\delta \phi(x)} \right] = 0 \tag{53}$$

where we remind that $G = (G_\alpha[\phi; x])_{\alpha=1}^d$, $\nabla = (\partial_\alpha)_{\alpha=1}^d$ and $\chi = (\chi_{\alpha,\beta}[\phi; x, t])_{\alpha,\beta=1}^d$. The asociated hamiltonian is now:

$$H(\pi, \phi) = \int_\Lambda dx \left[G[\phi] \cdot \nabla \pi + \frac{1}{2} \nabla \pi \cdot \chi[\phi] \nabla \pi \right] \tag{54}$$

The evolution equations for the dual dynamics are:

$$\begin{aligned} \partial_t \bar{\phi}(x,t) &= -\nabla \cdot G[\bar{\phi}; x, t] - \nabla \left(\chi[\bar{\phi}; x, t] \nabla \pi(x,t)\right) \\ \partial_t \pi(x,t) &= -\int_\Lambda dy \nabla \pi(y,t) \cdot \left[\frac{\delta G[\bar{\phi}; y, t]}{\delta \bar{\phi}(x,t)} + \frac{1}{2} \frac{\delta \chi[\bar{\phi}; y, t]}{\delta \bar{\phi}(x,t)} \nabla \pi(y,t) \right] \end{aligned} \tag{55}$$

and, as in the RD case, the initial conditions are: $\bar{\phi}(x,-\infty) = \phi^*(x)$ and $\pi(x,-\infty) = 0$. Now ϕ^* is solution of $\nabla \cdot G[\phi^*; x] = 0$. As in de RD case, we can show that these equations of movement are equal to the ones we obtain by looking for the *most probable path* from the definition of $P_{st}[\phi]$ by using the path probabilities (see eq.(40)). That is, now

$$\begin{aligned} L[\phi; t_0, t_1] = \frac{1}{2} \int_{t_0}^{t_1} dt \int_\Lambda dx \int_\Lambda dx' \left(\partial_t \phi(x,t) + \nabla \cdot G[\phi; x, t]\right) \\ M[\phi; x, x', t] \left(\partial_t \phi(x',t) + \nabla' \cdot G[\phi; x', t]\right) \end{aligned} \tag{56}$$

and the most probable path is given by eq.(41) that in this case reads:

$$\partial_t\left[\int_\Lambda dx' M[\phi;y,x',t]\left(\partial_t\phi(x',t)+\nabla'\cdot G[\phi;x',t]\right)\right]=\int_\Lambda dx\int_\Lambda dx'\left(\partial_t\phi(x',t)+\nabla'\cdot G[\phi;x',t]\right)$$
$$\left[\nabla\frac{\delta G[\phi,x,t]}{\delta\phi(y,t)}M[\phi;x,x',t]+\frac{1}{2}\left(\partial_t\phi(x,t)+\nabla\cdot G[\phi;x,t]\right)\frac{\delta M[\phi;x,x',t]}{\delta\phi(y,t)}\right] \tag{57}$$

All the comments and remarks we did for the RD model can be easely translated in this case. For instance one can show again that $V_0[\phi]$ is a Lyapunov function for the deterministic and the dual dynamics.

IV. EQUILIBRIUM VS NON-EQUILIBRIUM: THE MACROSCOPIC TIME SYMMETRY

We have exposed the way to compute the quasi-potential, V_0, from the Langevin equation that define the system mesoscopic behavior. At this point, it seems that the distinction between a system at equilibrium or in a nonequilibrium state is non-existent. In both cases we would need to build V_0 from our Hamilton-Jacobi type of equations. However we already commented above about the possibility that V_0 had some non-analyticities in its domain of definition. Is that the main difference between a system at equilibrium or in a nonequilibrium state? Onsager and Machlup theory about fluctuations and relaxation to equilibrium [10] used the time reversibility of the microscopic equations of motion to derive properties of the fluctuations near the equilibrium state. Has time reversibility any role on this mesoscopic description? We are going to argue about these issues and give some connections that could allow us to (maybe) distinguish *a priori* whether or not we are dealing with a system at equilibrium or not.

Let us first study separately the RD and DD cases.

A. RD case:

Let us make a definition first. A system is called to be *macroscopic reversible* when $\bar\phi(x,t)$ which is solution of the dual dynamics is also solution of the time reversed deterministic dynamics, that is

$$\partial_t\bar\phi(x,t)=-F[\bar\phi;x,t] \tag{58}$$

with $\bar{\phi}(x,-\infty) = \phi(x)^*$. In other words, the most probable path to create a fluctuation is just the time reversed one to relax the fluctuation using the deterministic dynamic equation.

There is a nice property that relates the pseudo-potential with the macroscopic reversibility. Let $F[\phi;x]$ be a given functional and let us assume that it has the property:

$$\frac{\delta}{\delta\phi(y)}\left[\frac{F[\phi;x]}{h[\phi;x]^2}\right] = \frac{\delta}{\delta\phi(x)}\left[\frac{F[\phi;y]}{h[\phi;y]^2}\right] \quad \forall\, x,y \in \Lambda \tag{59}$$

then (1) the system is macroscopic reversible and (2) the quasi-potential is regular, that is,

$$\frac{\delta^2 V_0[\phi]}{\delta\phi(x)\delta\phi(y)} = \frac{\delta^2 V_0[\phi]}{\delta\phi(y)\delta\phi(x)} \quad \forall\, x,y \in \Lambda \tag{60}$$

To show these assertions let us find the conditions under which a system is macroscopic reversible. In such case, the solution of eq. (58) should be also solution of the equations of motion we built from the Hamilton Jacobi equation (36). Then, from the first of such equations we get:

$$\partial_t\bar{\phi}(x,t) = F[\bar{\phi};x,t] + h[\phi;x,t]^2\pi(x,t) = -F[\bar{\phi};x,t] \Rightarrow \pi(x,t) = -\frac{2F[\bar{\phi};x,t]}{h[\bar{\phi};x,t]} \tag{61}$$

This $\pi(x,t)$ should be solution of the second of the equations of motion. After we substitute on it we get

$$\int_\Lambda dy F[\bar{\phi};y,t]\left(\frac{\delta}{\delta\bar{\phi}(y,t)}\left[\frac{F[\bar{\phi};x,t]}{h[\bar{\phi};x,t]^2}\right] - \frac{\delta}{\delta\bar{\phi}(x,t)}\left[\frac{F[\bar{\phi};y,t]}{h[\bar{\phi};y,t]^2}\right]\right) = 0 \tag{62}$$

That is, if the condition (59) applies then the most probable path is also solution of the time reversed deterministic dynamics. Moreover, we know that $\pi(x,t) = \delta V_0[\bar{\phi}]/\delta\bar{\phi}(x,t) = -2F[\bar{\phi};x,t]/h[\bar{\phi};x,t]^2$ and the regularity of V_0 is also proved.

In conclusion, there is a family of RD Langevin equations whose most probable fluctuations are macroscopic reversible and their quasi-potential is continuous and probably also are their first derivatives (NOTE: Clairaut's Theorem states: If $f(x,y)$, $\partial_x f(x,y)$, $\partial_y f(x,y)$, $\partial_x\partial_y f(x,y)$ and $\partial_y\partial_x f(x,y)$ are defined in an open region containing the point (a,b) and they are *continuous* at (a,b) then $\partial_x\partial_y f(x,y) = \partial_y\partial_x f(x,y)$. This indicates that equal mixed derivatives is a common property of functions with existent and continuous first derivatives. Obviously, different mixed derivatives would imply non continuity on the derivatives but we are unable to find an *if only in* result: equal mixed derivatives in an open set implies continuity of the derivatives on that set).

We observe how macroscopic reversibility is related with regularity of the potential. These properties are the ones expected for systems being at equilibrium states. At this point we can order, in some sense, the above ideas and introduce the following properties (without any rigurous general proof):

- *A system is at equilibrium if and only if the corresponding quasi-potential $V_0[\phi]$ is at least a C^2 functional with respect the fields $\phi(x)$.*

- *The systems at equilibrium are macroscopic reversible.*

Sometimes we could be interested in building $F[\phi; x, t]$ functionals having an *a priori* given equilibrium potential $V_0[\phi]$ and a noise intensity $h[\phi; x, t]$ by using

$$F[\phi; x, t] = -\frac{1}{2} h[\phi; x, t]^2 \frac{\delta V_0[\phi]}{\delta \phi(x, t)} \tag{63}$$

this is the so-called *detailed balance condition.* The systems so defined are macroscopic reversible with the appropiate set of boundary conditions. A typical example of equilibrium potentials are the ones of the form:

$$V_0[\phi] = \int_\Lambda dx\, v[\phi; x] \tag{64}$$

with $v[\phi; x]$ having the property

$$\frac{\delta^2 v[\phi; x]}{\delta\phi(v)\delta\phi(z)} = \frac{\delta^2 v[\phi; x]}{\delta\phi(z)\delta\phi(v)} \quad \forall \quad v, z \in \Lambda \tag{65}$$

then

$$F[\phi; x] = -\frac{1}{2} h[\phi; x]^2 \int_\Lambda dy \frac{\delta v[\phi; y]}{\delta\phi(x)} \tag{66}$$

For instance if we choose the Ginzburg-Landau form:

$$v[\phi; x] = \frac{1}{2}(\nabla\phi)^2 + w(\phi(x)) \tag{67}$$

with $w(\lambda)$ just any one dimensional function. Then we find

$$F[\phi; x] = \frac{1}{2} h[\phi; x]^2 \left(\Delta\phi(x) - \left.\frac{dv(\lambda)}{d\lambda}\right|_{\lambda=\phi(x)} \right) \tag{68}$$

The corresponding Langevin dynamics is the well known Hohenberg-Halpering model A [11].

B. DD case:

In this case, a system is *macroscopic reversible* when the dual dynamics is solution of the equation:

$$\partial\bar{\phi}(x,t) = \nabla \cdot G[\bar{\phi}; x, t] \tag{69}$$

with $\bar{\phi}(x,-\infty) = \phi(x)^*$. One can prove in this case a property anagolous to the one for the RD case. Let $G[\phi; x, t]$ and $\chi[\phi; x, t]_{\alpha,\beta}$ given functionals with the property:

$$\left(\partial_{x,\alpha}\frac{\delta}{\delta\phi(x)}\right)\left[\sum_{\gamma=1}^{d}\chi^{-1}_{\beta\gamma}[\phi; y]G_\gamma[\phi; y]\right] = \left(\partial_{y,\beta}\frac{\delta}{\delta\phi(y)}\right)\left[\sum_{\gamma=1}^{d}\chi^{-1}_{\alpha\gamma}[\phi; x]G_\gamma[\phi; x]\right] \tag{70}$$

and with boundary conditions such that $G[\phi^*; x] = 0$ then (1) the system is macroscopic reversible and (2) the quasi-potential is regular. To proof these asertions let us assume macroscopic reversibility. Then

$$G = -\frac{1}{2}\chi\nabla\pi \quad \Rightarrow \quad \nabla\pi = -2\chi^{-1}G \tag{71}$$

We do time derivatives to both sides of the last equation and we substitute the $\partial_t\pi$ term by the expression from the Hamilton-Jacobi equation (55) and we find that the above property (70) is a sufficient condition to obtain the equation compatibility. Finally we prove the regularity of $V_0[\phi]$ just by inserting $\pi = \delta V_0/\delta\phi$ into the property. Observe the the minimum of the potential ($\pi^* = 0$ correponds, in this case, to have all the currents equal to zero at the stationary state ($G[\phi^*; x] = 0$) that is a natural property for macroscopic systems being at equilibrium. Let us remark that there are nonequilibrium systems with zero net currents. In this context we can conclude that a system is at equilibrium if it is macroscopic reversible and their net currents are zero.

We can use this *detailed balance property* to build diffusive Langevin equations with an *a priori* equilibrium stationary state. In particular, if we choose $V_0[\phi]$ of the form (64) with $v[\phi; x]$ given by eq. (67), we get:

$$G_\alpha[\phi; x] = \frac{1}{2}\sum_{\beta=1}^{d}\chi_{\alpha\beta}[\phi; x]\partial_\beta\left(\Delta\phi(x) - \left.\frac{dv(\lambda)}{d\lambda}\right|_{\lambda=\phi(x)}\right) \tag{72}$$

This expression corresponds to the Hohenberg-Halpering model B [11].

C. The Fundamental Principle

Bertini and co-workers obtained the dual dynamics by generalizing the large deviation properties of some microscopic stochastic models at the mesoscopic scale [2]. They generalized the Einstein proposal about fluctuacions of systems at equilibrium in which it is connected the probability of having a fluctuation with the minimum reversible work necessary to create it. We generalize here their findings to any system described by the RD and DD Langevin dynamics.

Let us define the joint probability of a given path from t_0 to t_1 knowing that $\phi[t_0]$ is choosen from the stationary distribution:

$$P(\{\phi\}[t_0,t_1]|\phi[t_0]) = P_{st}[\phi[t_0]]P[\{\phi\}[t_0,t_1]] \tag{73}$$

For $\Omega \to \infty$ we can be write:

$$P(\{\phi\}[t_0,t_1]|\phi[t_0]) \propto \exp\Bigg[-\Omega V_0[\phi[t_0]] - \frac{\Omega}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \left(\frac{\partial_t\phi(x,t) - F[\phi;x,t]}{h[\phi;x,t]}\right)^2 + O(\Omega^0)\Bigg] \tag{74}$$

If we time reverse our system we do not expect that the corresponding dynamics being the same as de original one for any general non-equilibrium system. That is due to the irreversible and dissipative character of non-equililibrium phenomena. However, let us assume that such dynamics will follow a similar Langevin dynamics:

$$\partial_t\tilde{\phi}(x,t) = F^*[\tilde{\phi};x,t] + h^*[\tilde{\phi};x,t]\xi(x,t) \tag{75}$$

Therefore we can define the stationary probabilitiy, path probability,... in a similar way we did for the original dynamics. In particular we can define the joint probability as above:

$$\begin{aligned} P^*(\{\tilde{\phi}\}[t_0,t_1]|\tilde{\phi}[t_0]) &= P_{st}[\tilde{\phi}[t_0]]P^*[\{\tilde{\phi}\}[t_0,t_1]] \\ &\propto \exp\left[-\Omega V_0[\tilde{\phi}[t_0]] - \frac{\Omega}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \left(\frac{\partial_t\tilde{\phi}(x,t) - F^*[\tilde{\phi};x,t]}{h^*[\tilde{\phi};x,t]}\right)^2 + O(\Omega^0)\right] \end{aligned} \tag{76}$$

for any given path $\{\tilde{\phi}\}[t_0,t_1]$. Observe that we are using the fact that the time reversed dynamics relaxes to the same stationary probability as before: $P_{st}[\tilde{\phi}] \simeq \exp[-\Omega V_0[\tilde{\phi}]]$ because the boundary conditions or the action of external agents that characterizes the stationary state does not change.

Let us fix a path $\{\phi\}[t_0, t_1]$ and its time reversed image: $\{\tilde{\phi}\}[-t_1, -t_0]$ where $\tilde{\phi}(x,t) = \phi(x,-t)$. The *fundamental principle* states that

$$P(\{\phi\}[t_0, t_1]|\phi[t_0]) = P^*(\{\tilde{\phi}\}[-t_1, -t_0]|\tilde{\phi}[-t_1]) \tag{77}$$

We may consider this principle as a generalization of the *detailed balance condition* for the stationary marcovian Master equation: the probability to go from a stationary state to another arbitrary state is equal to the probability to go from the later being stationary to the first. However in that case the dynamics did not change for the time reversed path as it now occurs.

When we substitute the explicit form of P and P^* we get:

$$\begin{aligned} & V_0[\phi[t_0]] + \frac{1}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \left(\frac{\partial_t\phi(x,t) - F[\phi;x,t]}{h[\phi;x,t]}\right)^2 \\ = & V_0[\phi[t_1]] + \frac{1}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \left(\frac{\partial_t\phi(x,t) + F^*[\phi;x,t]}{h^*[\phi;x,t]}\right)^2 \end{aligned} \tag{78}$$

for any given path. This equation fixes the form of F^* and h^*, that is, the form of the time reversed dynamics. Let us assume that the path chosen does not contain any value where V_0 being non-differentiable. Then

$$V_0[\phi[t_1]] - V_0[\phi[t_0]] = \int_{t_0}^{t_1} dt\partial_t V_0[\phi[t]] = \int_{t_0}^{t_1} dt \int_\Lambda dx \frac{\delta V_0[\phi[t]]}{\delta\phi(x,t)}\partial_t\phi(x,t) \tag{79}$$

and then by using equation (78) we get:

$$\begin{aligned} & \int_{t_0}^{t_1} dt \int_\Lambda dx \Bigg\{ \frac{1}{2}\left[\left(\frac{1}{h^*[\phi;x,t]} - \frac{1}{h[\phi;x,t]}\right)\partial_t\phi(x,t) + \frac{F^*[\phi;x,t]}{h^*[\phi;x,t]} + \frac{F[\phi;x,t]}{h[\phi;x,t]}\right] \\ & \left[\left(\frac{1}{h^*[\phi;x,t]} + \frac{1}{h[\phi;x,t]}\right)\partial_t\phi(x,t) + \frac{F^*[\phi;x,t]}{h^*[\phi;x,t]} - \frac{F[\phi;x,t]}{h[\phi;x,t]}\right] \\ & + \frac{\delta V_0[\phi[t]]}{\delta\phi(x,t)}\partial_t\phi(x,t) \Bigg\} = 0 \end{aligned} \tag{80}$$

for any path and any time interval. Then, we can fix any t and we can take any arbitrary value for $\partial_t\phi(x,t)$. Therefore the coefficients of the time derivatives should be equal to zero and also the independent term:

$$\begin{aligned} & (\partial_t\phi(x,t))^2 : h^*[\phi;x,t] = h[\phi;x,t] \\ & (\partial_t\phi(x,t))^1 : F^*[\phi;x,t] = -F[\phi;x,t] - h[\phi;x,t]^2\frac{\delta V_0[\phi]}{\delta\phi(x,t)} \\ & (\partial_t\phi(x,t))^0 : \int_\Lambda dx \frac{F^*[\phi;x,t] - F[\phi;x,t]}{h[\phi;x,t]^2} = 0 \end{aligned} \tag{81}$$

The first two equations indicate that the time reversed dynamics have the same noise intensity as the direct dynamics but its deterministic part is different:

$$\partial_t \tilde{\phi}(x,t) = -F[\tilde{\phi};x,t] - h[\tilde{\phi};x,t]^2 \frac{\delta V_0[\tilde{\phi}]}{\delta \tilde{\phi}(x,t)} + h[\tilde{\phi};x,t]\xi(x,t) \tag{82}$$

This last equation is just the Hamilton-Jacobi equation (33). Observe that the deterministic part of eq.(82) (i.e. the most probable path) is just the time reversed most probable path that connects the deterministic stationary state with any other described by $P_{st}(\bar{\phi})$ (see eq.(36)). That is $\tilde{\phi}(x,t) = \bar{\phi}(x,-t)$ at the deterministic level. All this fits with our expectations that the effective dynamics that follows the most probable path to create a fluctuation is just the dynamics that defines the relaxation towards the stationary state for the time reversed system.

We can follow a similar argument for the DD Langevin type of equations. Let us assume that the time-reversed Langevin dynamics is:

$$\partial_t \tilde{\phi}(x,t) + \nabla \cdot \tilde{j}[\tilde{\phi};x,t] = 0 \tag{83}$$

where

$$\tilde{j}_\alpha[\tilde{\phi};x,t] = G^*_\alpha[\tilde{\phi};x,t] + \sum_{\beta=1}^{d} \sigma^*_{\alpha,\beta}[\tilde{\phi};x,t]\psi_\beta(x,t) \quad \alpha = 1,\ldots,d \tag{84}$$

The Fundamental Principle implies now:

$$\begin{aligned} & V_0[\tilde{\phi}[t_0]] + \frac{1}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \int_\Lambda dx' \left(\partial_t \tilde{\phi}(x,t) + \nabla G[\tilde{\phi};x,t]\right) \\ & \qquad M[\tilde{\phi};x,x',t] \left(\partial_t \tilde{\phi}(x',t) + \nabla' G[\tilde{\phi};x',t]\right) \\ & = V_0[\tilde{\phi}[t_1]] + \frac{1}{2}\int_{t_0}^{t_1} dt \int_\Lambda dx \int_\Lambda dx' \left(\partial_t \tilde{\phi}(x,t) + \nabla G^*[\tilde{\phi};x,t]\right) \\ & \qquad M^*[\tilde{\phi};x,x',t] \left(\partial_t \tilde{\phi}(x',t) + \nabla' G^*[\tilde{\phi};x',t]\right) \end{aligned} \tag{85}$$

and using eq.(79) and identifying powers of $\partial_t \tilde{\phi}$ we get:

$$\begin{aligned} M^*[\tilde{\phi};x,x',t] = M[\tilde{\phi};x,x',t] \Rightarrow \sigma^*[\tilde{\phi};x,t] = \sigma[\tilde{\phi};x,t] \\ G^*_\alpha[\tilde{\phi};x,t] = -G_\alpha[\tilde{\phi};x,t] - \sum_{\beta=1}^{d} \chi_{\alpha\beta}[\tilde{\phi};x,t]\partial_\beta \frac{\delta V_0[\tilde{\phi}[t]]}{\delta \tilde{\phi}(x,t)} \end{aligned} \tag{86}$$

and the last (order zero) is again the Hamilton-Jacobi equation for the V_0 potential. The same comments done in the RD case apply here.

Let us remark that the Fundamental Principle implies that all macroscopic reversible systems follows the detailed balance condition (63). We have shown that the Fundamental Principle holds for all the RD and DD Langevin type of equations (at least for the dynamics of the most probable path). It remains an open issue to see if it is a more general principle that goes beyond this Langevin mesoscopic description.

V. CORRELATIONS

We have seen some global properties of systems described by continuous Langevin equations. At some point it is necessary to connect the theory with measurements and observations done in real systems. We are going to focus into stationary two body correlations that are directly related to the V_0 potential. Let us define first the *Generating Functional*:

$$Z[b] = Z[0]\int D\phi P_{st}[\phi]\exp\left[\Omega\int_\Lambda dx\, b(x)\phi(x)\right] \tag{87}$$

where $b(x)$ is a kind of external field. We know from this expression that the n-body correlations at the stationary state (without external field) are given by

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{st} = \frac{1}{\Omega^n Z[0]}\frac{\delta^n Z[b]}{\delta b(x_1)\ldots\delta b(x_n)}\bigg|_{b=0} \tag{88}$$

The truncated n-body correlations are defined by:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{st}^c = \frac{1}{\Omega^n}\frac{\delta^n W[b]}{\delta b(x_1)\ldots\delta b(x_n)}\bigg|_{b=0} \tag{89}$$

where $W[b] = \ln Z[b]$,

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{st}^c = \langle(\phi(x_1)-\langle\phi(x_1)\rangle_{st})\ldots(\phi(x_n)-\langle\phi(x_n)\rangle_{st})\rangle_{st} \quad n>1 \tag{90}$$

and $\langle\phi(x)\rangle_{st}^c = \langle\phi(x)\rangle_{st}$.

Therefore, in order to obtain the correlations we just need to compute the generating functional that depends on V_0. There are two main strategies: (1) assume that V_0 is a convex analytic function around the deterministic stationary solution ϕ^* and then to use the Legendre transformation to solve the corresponding Hamilton-Jacobi equation or (2) obtain explicitly the quasi-potential V_0 around ϕ^* by solving the linearized Hamilton equations that define the most probable path. The second strategy is more general and it includes the computation of correlations when V_0 is non analytic near the deterministic stationary solution. Let us show here the first path of reasoning for the RD case and we leave the other one to the Appendix III.

A. RD case:

Let us assume that $P_{st}[\phi] \propto \exp[-\Omega V_0[\phi]]$. Then the Generating functional can be written:

$$Z[b] \propto \int D\phi \exp[-\Omega \mathcal{F}[\phi]] \quad , \quad \mathcal{F}[\phi] = V_0[\phi] - \int_\Lambda dx\, b(x)\phi(x) \tag{91}$$

Let us define $\phi^*[b]$ as the field that minimizes $\mathcal{F}$ and let us assume that $\mathcal{F}[\phi]$ is differentiable around $\phi^*[b]$, then

$$\mathcal{F}[\phi] = \mathcal{F}[\phi^*[b]] + \frac{1}{2}\int_\Lambda dxdy\, \frac{\delta^2 \mathcal{F}[\phi]}{\delta\phi(x)\phi(y)}\Big|_{\phi=\phi^*[b]} (\phi(x) - \phi^*[x;b])(\phi(y) - \phi^*[y;b]) + \dots \tag{92}$$

where $\phi^*[b]$ is solution of

$$\frac{\delta \mathcal{F}[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi^*[b]} = 0 \quad \Rightarrow \quad \frac{\delta V_0[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi^*[b]} = b(x) \tag{93}$$

we observe that $\phi^*[0] = \phi^*$, the minimum of $V_0[\phi]$.

We can substitute this expansion on the Generating Functional and we obtain:

$$Z[b] \propto e^{-\Omega\mathcal{F}[\phi^*[b]]} \int D\omega \exp\left[-\frac{1}{2}\int_\Lambda dxdy \frac{\delta^2 V_0[\phi]}{\delta\phi(x)\phi(y)}\Big|_{\phi=\phi^*[b]} \omega(x)\omega(y) + O(\Omega^{-1/2})\right] \tag{94}$$

where we have done the change of variables $w(x) = \sqrt{\Omega}(\phi(x) - \phi[x;b])$. We see that this expression has some meaning whenever V_0 is differentiable and convex around $\phi^*[b]$. Convexity also guarantees that there is a one to one relation between b and $\phi^*[b]$. We can also show that

$$\frac{\delta \mathcal{F}[\phi^*[b]]}{\delta b(x)} = -\phi^*[x;b] \tag{95}$$

That is, $\mathcal{F}[b] \equiv \mathcal{F}[\phi^*[b]]$ is the Legendre transform of $V_0[\phi]$. We can now relate $\mathcal{F}$ with the correlations:

$$W[b] = -\Omega\mathcal{F}[\phi^*[b]] + O(\Omega^0) \tag{96}$$

and

$$\lim_{\Omega\to\infty} \Omega^{n-1}\langle\phi(x_1)\dots\phi(x_n)\rangle_{st}^c = -\frac{\delta^n \mathcal{F}[\phi^*[b]]}{\delta b(x_1)\dots\delta b(x_n)}\Big|_{b=0} \equiv C_n(x_1,\dots,x_n) \tag{97}$$

where $\langle\phi(x)\rangle_{st}^c = \langle\phi(x)\rangle_{st} = \phi^*(x) = \phi^*[x;0]$.

We can write down the Hamilton-Jacobi equation (33) applied to $\phi(x) = \phi^*(x;b)$:

$$\int_\Lambda dx \left[F[\phi^*[b];x]\frac{\delta V_0[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi^*[b]} + \frac{1}{2}h[\phi^*[b];x]^2\left(\frac{\delta V_0[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi^*[b]}\right)^2\right] = 0 \tag{98}$$

that by using the Lengendre transform relations it is converted to:

$$\int_\Lambda dx \left[F[-\frac{\delta \mathcal{F}[\phi^*[b]]}{\delta b(x)}; x] b(x) + \frac{1}{2} h[\frac{\delta \mathcal{F}[\phi^*[b]]}{\delta b(x)}; x]^2 b(x)^2 \right] = 0 \tag{99}$$

that is valid for any value of b. We can obtain closed equations for the correlations by expanding these functionals around $b = 0$. We observe that:

$$\begin{aligned} \frac{\delta \mathcal{F}[b]}{\delta b(x)} &= -\phi^*[x; b] = -\phi^*(x) - \int_\Lambda dy\, C_2(x,y) b(y) + O(b^2) \\ F[\phi^*[b]; x] &= \int_\Lambda dy dz \frac{\delta F[\phi; x]}{\delta \phi(z)}\bigg|_{\phi=\phi^*} C_2(y,z) b(y) + O(b^2) \end{aligned} \tag{100}$$

and the expansion on b of eq.(99) becames:

$$\begin{aligned} \int_\Lambda dx dy\, b(x) b(y) \Bigg[\quad \int_\Lambda dz \left[\frac{\delta F[\phi; x]}{\delta \phi(z)}\bigg|_{\phi=\phi^*} C_2(z,y) + \frac{\delta F[\phi; y]}{\delta \phi(z)}\bigg|_{\phi=\phi^*} C_2(z,x) \right] \\ + h[\phi^*; x]^2 \delta(x-y) \Bigg] + O(b^3) = 0 \quad \forall b \end{aligned} \tag{101}$$

and therefore, the two body correlations $C_2(x,y)$ is solution of the equation:

$$\int_\Lambda dz \left[\frac{\delta F[\phi; x]}{\delta \phi(z)}\bigg|_{\phi=\phi^*} C_2(z,y) + \frac{\delta F[\phi; y]}{\delta \phi(z)}\bigg|_{\phi=\phi^*} C_2(z,x) \right] = -h[\phi^*; x]^2 \delta(x-y) \tag{102}$$

or in a more compact form:

$$C_2(x,y) = h[\phi^*; x] h[\phi^*; y] \bar{C}(x,y) \tag{103}$$

with $\bar{C}(x,y)$ solution of

$$\int_\Lambda dz \left[B(x,z) \bar{C}(z,y) + B(y,z) \bar{C}(z,x) \right] = -\delta(x-y) \tag{104}$$

with

$$B(x,y) = \frac{h[\phi^*; y]}{h[\phi^*; x]} \frac{\delta F[\phi; x]}{\delta \phi(y)}\bigg|_{\phi=\phi^*} \tag{105}$$

Observe that B maybe non-symmetric on its arguments while $\bar{C}$ is symmetric by construction. We can think this equation as a representation of the linear operator equation:

$$B\bar{C} + \bar{C}B = -I \tag{106}$$

with I the identity operator. The formal solution can be found by using the fact that $\partial/\partial\alpha e^{\alpha B} = B e^{\alpha B}$. Then

$$\frac{\partial}{\partial \alpha} \left[e^{\alpha B} \bar{C} e^{\alpha B^T} \right] = -e^{\alpha B} e^{\alpha B^T} \Rightarrow \bar{C} = \int_0^\infty d\alpha\, e^{\alpha B} e^{\alpha B^T} \tag{107}$$

where we have assumed that B is negative defined. We can write explicitely this equation by doing the assumption that B is *diagonalizable* or, in other words, that we can apply to B some spectral theorem. Let $v(x;\lambda_n)$ and $w(x;\lambda_n)$ be the set of *right* and *left* eigenvectors of B with eigenvalues λ_n and λ_n^* (complex conjugate of λ_n) respectively:

$$
\begin{aligned}
\int_\Lambda dy\, B(x,y)v(y;\lambda_n) &= \lambda_n v(x;\lambda_n) \\
\int_\Lambda dy\, B(y,x)w(y;\lambda_n) &= \lambda_n^* w(x;\lambda_n)
\end{aligned}
\tag{108}
$$

The eigenvalues may have real or complex values but because B is real valued then they appear in pairs when they are complex: $(\lambda, v(x;\lambda)), (\lambda^*, v(x;\lambda^*) = v(x;\lambda)^*)$. We assume that each set form a complete basis on the functional space and that they follow the ortogonality conditions:

$$
\begin{aligned}
\int_\Lambda dx\, w(x;\lambda_n)^* v(x;\lambda_m) &= \delta_{n,m} \\
\sum_n w(x;\lambda_n)^* v(y;\lambda_n) &= \delta(x-y)
\end{aligned}
\tag{109}
$$

The solution then reads:

$$
\bar{C}(x,y) = -\sum_{n,m} \frac{v(x;\lambda_n)v(y;\lambda_m)}{\lambda_n+\lambda_m} \int_\Lambda dz\, \bar{w}(z;\lambda_n)\bar{w}(z;\lambda_m) \tag{110}
$$

Finally, we see in this case that the solution is symmetric, $\bar{C}(x,y) = \bar{C}(y,x)$, and real, $\bar{C}(x,y)^* = \bar{C}(x,y)$, due to the pairing property of the eigenvalues.

The above solution simplifies when B is symmetric: $B(x,y) = B(y,x)$. In this case the right and left eigenvalues and eigenvectors coincide, all of them are real and the eigenvectors form an ortonormal base on the functional space. Therefore

$$
\bar{C}(x,y) = -\frac{1}{2}\sum_n \frac{1}{\lambda_n} v(x;\lambda_n)v(y;\lambda_n) = -\frac{1}{2}B^{-1}(x,y) \tag{111}
$$

where

$$
\int_\Lambda dz B(x,z)B^{-1}(z,y) = \delta(x-y) \tag{112}
$$

Let us study some particular cases.

- *Equilibrium:* Let us choose

$$
F[\phi;x] = -\frac{1}{2}h[\phi;x]^2 \frac{\delta V_0[\phi]}{\delta\phi(x)} \tag{113}
$$

We know in this case that for a given C^2 potential $V_0[\phi]$ the stationary state of the system is the equilibrium state. Let us compute the two-body correlations by using the equation (104). First we see that

$$B(x,y) = -\frac{1}{2}h[\phi^*;x]h[\phi^*;y]\frac{\delta^2 V_0[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi^*} \equiv -\frac{1}{2}h[\phi^*;x]h[\phi^*;y]V_2(x,y) \quad (114)$$

and by inspection we get that:

$$\bar{C}(x,y) = \frac{V_2^{-1}(x,y)}{h[\phi^*;x]h[\phi^*;y]} \quad \Rightarrow \quad C_2(x,y) = V_2^{-1}(x,y) \quad (115)$$

Let us remark that for all differentiable quasi potentials around the deterministic stationary state the last relation always holds. We know that in general

$$C_2(x,y) = -2\frac{\delta \log Z_0[V_2]}{\delta V_2(x,y)} \quad , \quad Z_0[V_2] = \int D\omega \exp\left[-\frac{1}{2}\int_\Lambda dxdy\, V_2(x,y)\omega(x)\omega(y)\right] \quad (116)$$

and we can compute Z_0 explicitly because is a gaussian-like integral:

$$Z_0[V_2] = cte(\det(V_2))^{-1/2} \quad (117)$$

and after we do the derivative we get the result: $C_2(x,y) = V_2^{-1}(x,y)$ that connects the two body correlations with the second derivatives of the quasi-potential around the deterministic stationary state.

- *Small deviations from the equilibrium:* Let us assume that our system sligthly deviates from the equilibrium due to a unbalanced noise term:

$$F[\phi;x] = -\frac{1}{2}\tilde{h}[\phi;x]^2\frac{\delta\tilde{V}[\phi]}{\delta\phi(x)} \quad , \quad h[\phi;x]^2 = \tilde{h}[\phi;x]^2\tilde{g}[\phi;x] \quad (118)$$

with

$$\tilde{g}[\phi;x] = 1 + \epsilon g[\phi;x] \quad (119)$$

$\tilde{V}[\phi]$, $g[\phi;x]$ and $\tilde{h}[\phi;x]$ are given functionals and ϵ can be used as a perturbative parameter. When $\epsilon = 0$ $V_0[\phi] = \tilde{V}[\phi]$ and $C_2 = V_2^{-1}$. When $\epsilon \neq 0$ we see that the deterministic solution ϕ^* is solution of the equation $\delta/\delta\phi(x)\tilde{V}[\phi]|_{\phi=\phi^*} = 0$ and the extremal points of $\tilde{V}$ and the quasi-potential V_0 coincide. The matrix B is in this case:

$$B(x,y) = \tilde{g}[\phi;x]\tilde{B}(x,y) \quad , \quad \tilde{B}(x,y) = -\frac{1}{2}h[\phi^*;x]h[\phi^*;y]\frac{\delta^2\tilde{V}[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi^*} \quad (120)$$

and the equation for the correlations is now:

$$\tilde{G}\tilde{B}\bar{C} + \bar{C}\tilde{G}\tilde{B} = -I \tag{121}$$

where $\tilde{G}(x,y) = \tilde{g}[\phi;x]\delta(x-y)$. We look for perturbative solutions of this equation:

$$\bar{C} = \sum_{n=0}^{\infty} \epsilon^n \bar{C}_n \tag{122}$$

and we can decompose the equation for $\bar{C}$:

$$\begin{aligned} &\tilde{B}\bar{C}_0 + \bar{C}_0\tilde{B} = -I \\ &\tilde{B}\bar{C}_n + \bar{C}_n\tilde{B} = -G\tilde{B}\bar{C}_{n-1} - \bar{C}_{n-1}\tilde{B}G \quad n > 0 \end{aligned} \tag{123}$$

where $\tilde{G} = I + \epsilon G$ and $G(x,y) = g[\phi;x]\delta(x-y)$. The solutions are:

$$\begin{aligned} &\bar{C}_0 = -\frac{1}{2}\tilde{B}^{-1} \\ &\bar{C}_n = \int_0^\infty d\alpha\, e^{\alpha\tilde{B}} \left(G\tilde{B}\bar{C}_{n-1} + \bar{C}_{n-1}\tilde{B}G\right) e^{\alpha\tilde{B}} \quad n > 0 \end{aligned} \tag{124}$$

and, in particular,

$$\bar{C}_1 = -\int_0^\infty d\alpha\, e^{\alpha\tilde{B}} G e^{\alpha\tilde{B}} = QAQ^T \tag{125}$$

where Q is the matrix that diagonalizes $\tilde{B}$: $\tilde{B} = QDQ^T$, that is $Q_{ij} = v_i(\lambda_j)$ (in discrete notation) and

$$A_{i,j} = \frac{(Q^T G Q)_{ij}}{\lambda_i + \lambda_j} \tag{126}$$

B. DD case:

We can follow the same scheme here as we did in the RD case. We apply the Hamilton-Jacobi equation (53) to the field $\phi(x) = \phi^*(x;b)$ and we expand the equation up to second order in b-fields. The equation for the two-body correlations is now:

$$\int_\Lambda dz\ [K(x,z)C_2(z,y) + K(y,z)C_2(z,x)] = -\sum_{i,j} \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\left[\chi_{ij}[\phi^*;x]\delta(x-y)\right] \tag{127}$$

where

$$K(x,y) = \frac{\delta}{\delta\phi(y)}\left(-\nabla\cdot G[\phi;x]\right)\Big|_{\phi=\phi^*} \tag{128}$$

with ϕ^* solution of $\nabla\cdot G[\phi^*;x] = 0$.

In the DD case, nonequilibrium states may happen due to the bulk dynamic mechanism and/or the action of boundary conditions. Let us discuss both cases separately.

- **Nonequilibrium via boundary conditions:** Let us assume first that the stationary state of our system is the equilibrium one with a given $V_0[\phi]$ for an appropiate set of boundary conditions. Let us assume that the bulk dynamics is:

$$G_i[\phi; x] = -\frac{1}{2}\sum_j \chi_{ij}[\phi; x]\partial_j \frac{\delta V_0[\phi]}{\delta\phi(x)} \tag{129}$$

We know that the corresponding two-body correlations are given by:

$$C_2^{eq}(x, y) = V_2^{-1}(x, y; \phi^*_{eq}) \quad , \quad V_2(x, y; \phi) = \frac{\delta^2 V_0[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi^*_{eq}} \tag{130}$$

with ϕ^*_{eq} solution of $G[\phi^*_{eq}; x] = 0$ (the current is equal to zero). Let us assume that changing the boundary conditions the system develops non zero currents and therefore we have a nonequilibrium stationary state. The deterministic solution ϕ^* is given now by the solution of the equation:

$$-\frac{1}{2}\sum_j \chi_{ij}[\phi; x]\partial_j \frac{\delta V_0[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi^*} = J_i \quad , \quad +\text{boundary conditions} \tag{131}$$

where J_i are constants that depend on the boundary conditions.

Let us break the two body correlation into two terms:

$$C_2(x, y) = C_2^{leq}(x, y) + C_D(x, y) \quad , \quad C_2^{leq}(x, y) = V_2^{-1}(x, y; \phi^*) \tag{132}$$

The first term is the *local equilibrium correlation*. It corresponds to consider that the correlation are the equilibrium one evaluated with the local values of the field ϕ^*. $C_D(x, y)$ evaluates the deviation from the local equilibrium. Obviously, $C_D = 0$ when $J_i = 0$. When we substitute eq. (132) into (127) we get the closed equation for C_D:

$$\begin{aligned}
&\sum_i \frac{\partial}{\partial x_i}\left[\alpha_i[\phi^*; x]C_D(x, y) + \frac{1}{2}\sum_j \chi_{ij}[\phi^*; x]\frac{\partial}{\partial x_j}\int_\Lambda dz\, C_2^{leq,-1}(x, z)C_D(z, y)\right] \\
&\sum_i \frac{\partial}{\partial y_i}\left[\alpha_i[\phi^*; y]C_D(x, y) + \frac{1}{2}\sum_j \chi_{ij}[\phi^*; y]\frac{\partial}{\partial x_j}\int_\Lambda dz\, C_2^{leq,-1}(y, z)C_D(z, x)\right] \\
&= -\sum_i \frac{\partial}{\partial x_i}\left[\alpha_i[\phi^*; x]C_2^{leq}(x, y)\right] - \sum_i \frac{\partial}{\partial y_i}\left[\alpha_i[\phi^*; y]C_2^{leq}(x, y)\right]
\end{aligned} \tag{133}$$

where α is a d-dimensional vector

$$\alpha[\phi; x] = \chi'[\phi; x]\chi^{-1}[\phi; x]J \tag{134}$$

and we have considered $\chi_{ij}[\phi;x]$ being a function dependent only on $\phi(x)$, that is $\chi_{ij}[\phi;x] = \chi_{ij}(\phi(x))$. Therefore $\chi'_{ij}[\phi;x] = \partial\chi_{ij}(u)/\partial u|_{u=\phi(x)}$.

The solution of these equations is very complex and it depends on the particular system and boundary conditions used. Let us work out explicitly a well known particular case: the pure *diffusive system* by taking:

$$V_0[\phi] = \int_\Lambda dx\ [V(\phi(x)) - 2E\cdot x\phi(x)] \tag{135}$$

where E is an external constant vector. With this choice we get:

$$G_i[\phi;x] = -\sum_j [D_{ij}[\phi;x]\partial_j\phi(x) - \chi_{ij}[\phi;x]E_j] \tag{136}$$

where

$$D[\phi;x] = \frac{1}{2}V''(\phi(x))\chi[\phi;x] \tag{137}$$

that is the so called *Einstein Relation*. We observe that in equilibrium (with the appropiate boundary conditions) we find that $\phi^*_{eq}(x)$ is solution of the *barometric equation*:

$$\nabla\phi^*_{eq}(x) = -\frac{2}{V''(\phi^*_{eq}(x))}E \tag{138}$$

Moreover,

$$C_2^{eq}(x,y) = \frac{1}{V''(\phi^*_{eq}(x))}\delta(x-y) \tag{139}$$

In a non equilibrium setup we obtain that the stationary state is solution of the equation:

$$-\sum_j [D_{ij}[\phi^*;x]\partial_j\phi^*(x) - \chi_{ij}[\phi^*;x]E_j] = J_i \tag{140}$$

and the equation for C_D is, in this case:

$$\begin{aligned}&\sum_{ij}\left[\frac{\partial}{\partial x_i}\left[\frac{\partial(D_{ij}[\phi^*;x]C_D(x,y))}{\partial x_j} - \chi'_{ij}[\phi^*;x]C_D(x,y)\right]\right.\\&\left.+\frac{\partial}{\partial y_i}\left[\frac{\partial(D_{ij}[\phi^*;y]C_D(x,y))}{\partial y_j} - \chi'_{ij}[\phi^*;y]C_D(x,y)\right]\right]\\&= \frac{1}{2}\left(\nabla\cdot\bar{\alpha}[\phi^*;x]\right)\delta(x-y)\end{aligned} \tag{141}$$

where

$$\bar{\alpha}[\phi;x] = \chi'[\phi;x]D^{-1}[\phi;x]J \tag{142}$$

In particular for 1-d, $D = \text{cte}$, $E = 0$ and $\chi[\phi; x]$ a positive second order polynomial of the form $\chi[\phi; x] = c_0 + c_1\phi(x) + c_2\phi(x)^2$ we find that $J = -Dd\phi^*(x)/dx$. That implies $\phi^*(x) = \phi^*(0) - Jx/D$, $J = D(\phi^*(L) - \phi^*(0))/L$, where we have fixed the values of ϕ at the boundaries of the segment $[0, L]$. Moreover

$$C_2(x, y) = \frac{\chi[\phi^*; x]}{2D}\delta(x - y) + C_D(x, y) \tag{143}$$

where the equation for C_D is:

$$\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right] C_D(x, y) = -2\frac{J^2}{D^3}c_2\delta(x - y) \tag{144}$$

and the solution is

$$C_D(x, y) = -2\frac{J^2}{2D^3}c_2\Delta^{-1}(x, y) \tag{145}$$

with

$$\frac{d^2\Delta^{-1}(x, y)}{dx^2} = \delta(x - y) \tag{146}$$

whose solution is (see for instance Ref.[3]):

$$\Delta^{-1}(x, y) = -\frac{1}{L}\left[(L - x)y\theta(x - y) + x(L - y)\theta(y - x)\right] \tag{147}$$

where $\theta(x)$ is the Heaviside function. Observe that the sign of the correction to local equilibrium depends on the sign of c_2.

We can also study the fluctuations of the averaged field value:

$$\rho[\phi] = \frac{1}{L}\int_0^L dx\phi(x) \tag{148}$$

$$\Sigma \equiv \Omega\langle(\rho[\phi] - \rho^*)^2\rangle_{st} = \frac{1}{L^2}\int_0^L dx \int_0^L dy\, C_2(x, y) \tag{149}$$

where $\rho^* = \rho[\phi^*]$. In this particular case we obtain:

$$\Sigma = \Sigma_{leq} + \Sigma_D \tag{150}$$

where

$$\begin{aligned}
\Sigma_{leq} &= \frac{1}{2DL}\left[c_0 + c_1\rho^* + \frac{c_2}{3}(\phi^*(0)^2 + \phi^*(0)\phi^*(1) + \phi^*(1)^2)\right] \\
\Sigma_D &= \frac{c_2}{12DL}\left(\phi^*(0) - \phi^*(L)\right)^2
\end{aligned} \tag{151}$$

Observe that the deviation from the local equilibrium is proportional to the square of the external gradient.

- **Bulk nonequilibrium:** Let us focus in a very simple nonequilibrium model at the bulk level that develops highly nontrivial correlations. Let

$$G[\phi; x] = -D\nabla\phi \tag{152}$$

where we assume that D and χ are constant arbitrary d-dimensional matrices. One can easely check that this system is reversible if D is proportional to χ. $\phi^*(x) = cte$ is solution for any homogeneous boundary condition. Therefore the currents are zero: $J = 0$. The equation for C_2 is:

$$\sum_{ij} D_{ij} \frac{\partial^2 \bar{C}_2(x-y)}{\partial x_i \partial x_j} = \frac{1}{2} \sum_{ij} \chi_{ij} \frac{\partial^2 \delta(x-y)}{\partial x_i \partial x_j} \tag{153}$$

where we have assumed that the correlations are translational invariant: $C_2(x, y) = \bar{C}_2(x - y)$. Then, the solution is given by:

$$\bar{C}_2(u) = \int dk e^{iku} \hat{C}_2(k) \quad , \quad \hat{C}_2(k) = \frac{k \cdot \chi k}{k \cdot Dk} \tag{154}$$

When D is not proportional to χ then $\hat{C}_2$ is non-analyitic at $k = 0$ and therefore $\bar{C}_2(u)$ has a typical power law decay behavior [7].

VI. AN INITIAL APPROACH TO DEFINE NONEQUILIBRIUM DYNAMICAL ENSEMBLES

We know from the ensemble theory for systems at equilibrium that there are several probability densities defined in the microstate space that give rise to the same macroscopic equilibrium description in the thermodynamic limit. For instance we know the microcanonical, canonical and grand canonical ensembles. Moreover, we can also build different stochastic dynamics (conserved or non-conserved) by using, for instance, the detailed balance condition in such a way that all of them drive the system to the same equilibrium state. Here we question ourselves about the possibility to build a couple of RD and DD dynamics having the same nonequilibrium stationary state. This problem so defined is highly nontrivial due to the way we formaly obtain V_0: by solving a Hamilton-Jacobi equation in each case which is equivalent (as we already saw above) to solve a set of Hamilton nonlinear kinetic equations.

We know that near the a deterministic solution ϕ^* the nonequilibrium quasi-potential V_0 is characterized by the correlations C_2 (assuming differentiability of it around ϕ^*). Then the first approach to the general problem is to look for the conditions on the RD and DD dynamics to get equal correlation functions. This problem is hard and we are going to simplify it a step further. Let us to assume that the RD dynamics has the following property:

$$B(x,y) \equiv \frac{h[\phi_1^*;y]}{h[\phi_1^*;x]} \frac{\delta F[\phi;x]}{\delta\phi(y)}\bigg|_{\phi=\phi_1^*} = B(y,x) \tag{155}$$

where ϕ_1^* is solution of $F[\phi_1^*;x]=0$. This class of dynamics include the equilibrium ones. We know that in this case the two body correlations are given by:

$$C_2(x,y) = -h[\phi_1^*;x]h[\phi_1^*;y]B^{-1}(x,y) \tag{156}$$

If we substitute this C_2 on the equation for the two body correlations in the DD case we get the relation between both dynamics in order to have the same C_2 correlation:

$$h[\phi_1^*;y]\int_\Lambda dz\, K(x,z)h[\phi_1^*;z]B^{-1}(z,y) = \sum_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\left[\chi_{ij}[\phi_2^*;x]\delta(x-y)\right] \tag{157}$$

where

$$K(x,y) = \frac{\delta}{\delta\phi(y)}\left(-\nabla\cdot G[\phi;x]\right)\bigg|_{\phi=\phi_2^*} \tag{158}$$

with ϕ_2^* solution of $\nabla\cdot G[\phi_2^*;x]=0$. We are assuming that the boundary conditions are equal in both cases. After some trivial algebra we get the *sufficient condition* that relates RD and DD dynamics to have the same C_2 correlation function:

$$\frac{\delta G_i[\phi;x]}{\delta\phi(y)}\bigg|_{\phi=\phi_2^*} = \sum_j \chi_{ij}[\phi_2^*;x]\frac{\partial}{\partial x_j}\left[\frac{1}{h[\phi_1^*;x]^2}\frac{\delta F[\phi;x]}{\delta\phi(y)}\bigg|_{\phi=\phi_1^*}\right] \tag{159}$$

If we ask $\phi_1^* = \phi_2^* = \phi^*$ we can find a more restrained relation:

$$G_i[\phi;x] = \sum_j \chi_{ij}[\phi;x]\frac{\partial}{\partial x_j}\left[\frac{F[\phi;x]}{h[\phi;x]^2}\right] \tag{160}$$

In the equilibrium case we give a F of the form:

$$F[\phi;x] = -\frac{1}{2}h[\phi;x]^2\frac{\delta V[\phi]}{\delta\phi(x)} \tag{161}$$

for any arbitrary h and V functionals. The corresponding conservative dynamics with equal two body correlations is the expected:

$$G_i[\phi;x] = -\frac{1}{2}\sum_j \chi_{ij}[\phi;x]\frac{\partial}{\partial x_j}\left[\frac{\delta V[\phi]}{\delta\phi(x)}\right] \tag{162}$$

obviously in this case the quasi.potential is $V_0[\phi] = V[\phi]$. If we choose a nonequilibrium setup, for instance,

$$F[\phi; x] = -\frac{1}{2} g[\phi; x]^2 \frac{\delta V[\phi]}{\delta \phi(x)} \quad \text{with} \quad h[\phi; x] \tag{163}$$

then

$$G_i[\phi; x] = -\frac{1}{2} \sum_j \chi_{ij}[\phi; x] \frac{\partial}{\partial x_j} \left[\frac{g[\phi; x]^2}{h[\phi; x]^2} \frac{\delta V[\phi]}{\delta \phi(x)} \right] \tag{164}$$

for any given χ_{ij}.

These simple examples show the possibility to have dynamics of different character (conservative or non-conservative) that have the same two-body correlations and, by extension, they have the same quasi-potential at the fluctuating quadratic level.

VII. LARGE DEVIATIONS AND GREEN-KUBO RELATIONS

A natural set of magnitudes whose averages are useful to be studied are the space and time averages of fields or functions of them when the system is at the stationary state. They are naturally computed in the path's probability framework by using the Large Deviation Principle (assuming that it can be applied). These relations are a kind of generalized Green-Kubo relations for systems at nonequilibrium stationary states.

Let us define the bulk average of an observable $a[\phi; x, t]$ for a given time t :

$$a[\phi; t] = \frac{1}{\Lambda} \int_\Lambda dx\, a[\phi; x, t] \tag{165}$$

Its time average over the time interval $[0, T]$ is then

$$a_T[\phi] = \frac{1}{T} \int_0^T dt\, a[\phi; t] \tag{166}$$

If the stochastic model is well behaved we can apply the Law of Large Numbers in the sense that

$$a^* \equiv \langle a[\phi; 0] \rangle_{ss} = \lim_{T \to \infty} a_T[\phi] \tag{167}$$

Let us assume that at time $t = 0$ the system is at the stationary state and it fluctuates around it for later times. Under this condition, the probability to observe a certain value of $a_T[\phi] = a$ is given by:

$$P(a; T) = \int D\phi[0, T] P_{ss}[\phi(0)] P[\{\phi\}[0, T]] \delta(a - a_T[\phi]) \tag{168}$$

The Large Deviation Principle states that for large values of T this distribution should be very peaked around a^*. In fact in such limit it should be of the form:

$$P(a;T) \simeq e^{-TR(a)} \qquad T \to \infty \tag{169}$$

with

$$R(a^*) = 0 \qquad R'(a^*) = 0 \tag{170}$$

Therefore

$$\lim_{T\to\infty} T\left\langle (a-a^*)^2 \right\rangle_P = R''(a^*) \tag{171}$$

where $\langle\cdot\rangle_P$ means the the average is done with the $P(a;T)$ distribution. We can now substitute $P(a,T)$ by its path definition and we get the Green-Kubo relation:

$$\frac{1}{2R''(a^*)} = \int_0^\infty d\tau \, \langle (a[\phi;0]-a^*)(a[\phi;\tau]-a^*)\rangle \tag{172}$$

where now $\langle\cdot\rangle$ is the path average defined above.

We can apply this scheme to our RD and DD models and obtain (for a given $a[\phi;x,t]$) the function $R(a)$. Let us first start with a generic the RD system where we want to study the averaged global density:

$$\begin{aligned} a[\phi;x,t] &\to \phi(x,t) \\ a[\phi;t] &\to \rho[\phi;t] = \frac{1}{\Lambda}\int_\Lambda dx\, \phi(x,t) \\ a_T[\phi] &\to \rho_T[\phi] = \frac{1}{T}\int_0^T dt\, \rho[\phi;t] \end{aligned} \tag{173}$$

and $P[\{\phi\}[0,T]]$ is given by eq.(13). The probability to observe a given $\rho_T[\phi]=\rho$ is:

$$P[\rho;T] = \int D\phi[0,T]P_{ss}[\phi(0)]\int_{c-i\infty}^{c+i\infty}\frac{d\lambda}{2\pi i}\exp\left[-\Omega T R[\{\phi\}[0,T]]\right] \tag{174}$$

where

$$R[\{\phi\}[0,T],\lambda] = \frac{1}{2T}\int_0^T dt\int_\Lambda dx\left(\frac{\partial_t\phi(x,t)-F[\phi;x,t]}{h[\phi;x,t]}\right)^2 + \lambda\left(\rho-\rho_T[\phi]\right) \tag{175}$$

and we have used the representation of the Dirac delta by the integral on λ. When $T\to\infty$ the integrals are dominated by its minimum value over the fields and λ. That is

$$P[\rho,T] \simeq \exp[-\Omega T R[\{\tilde\phi\}[0,T],\tilde\lambda]] \tag{176}$$

where $\tilde{\phi}$ and $\tilde{\lambda}$ are solutions of

$$\frac{\delta R}{\delta\phi(y,\tau)}\bigg|_{\phi=\tilde{\phi},\lambda=\tilde{\lambda}} = 0 \quad , \quad \frac{\partial R}{\partial\lambda}\bigg|_{\phi=\tilde{\phi},\lambda=\tilde{\lambda}} = 0 \tag{177}$$

In general, these set of equations have many different solutions (see for instance [12]), static and dynamics ones that are local extremals of R. It is a daunting practical task to get some solutions and to check which is the one that is the absolute minimum for R. Let us assume the simplest case in which the deterministic solution of the Langevin equation is constant in space: $\phi^*(x,t) = \rho^* = cte$. Obviously, when $\rho = \rho^*$ we expect that $\tilde{\phi}(x,t) = \rho^*$. For values of ρ near the stationary state solution ρ^* we can assume by continuity that $\tilde{\phi}(x,t)$ is still constant and then

$$\tilde{\phi}(x,t) = \rho \tag{178}$$

is a solution of the equations. This is equivalent to the so-called *additivity principle*[4]. In this case

$$R[\{\tilde{\phi}\}[0,T],\tilde{\lambda}] = \Lambda\frac{F[\rho]^2}{2h[\rho]^2} \equiv R[\rho] \tag{179}$$

Therefore

$$R''[\rho^*] = \Lambda\frac{F'[\rho^*]^2}{h[\rho^*]} \tag{180}$$

and the Green-Kubo relation reads

$$\frac{h[\rho^*]^2}{F'[\rho^*]^2} = 2\Omega\int_{R^d} dx \int_0^\infty d\tau\,\langle(\phi(0,0)-\phi^*)(\phi(x,\tau)-\phi^*)\rangle \tag{181}$$

in the limit $\Lambda\to\infty$ and assuming spatial traslational invariance.

We can study in the same way different observables. In the DD case it has been studied extensively the time averaged mean current in some 1-d systems [13]:

$$J_T[\phi] = \frac{1}{T\Lambda}\int_0^T dt \int_\Lambda dx\, j(x,t) \tag{182}$$

In one dimension it is shown that the additivity principle is correct when we look for fluctuations of the current near the stationary value but, in general, it fail for large current fluctuations where the solutions that minimize the functional R are much more complex that the uniform solution. For instance, that occurs when we use periodic boundary conditions where such solutions are soliton-like moving around the system at constant speed. Moreover, it has been shown that in two dimensions the KMP model [14] with a thermal gradient in one direction and periodic boundary conditions in the other, presents a solution (*weak additivity principle*) that is not spatially uniform but is better minimizer than the uniform solution even near the stationary value [15]. That is, one should be very careful when using this technique for dimensions larger than one.

VIII. CONCLUSIONS

We have made an attempt to describe general properties of nonequilibrium systems at stationary states. We studied the stationary measure at the small noise limit through the quasi-potential. We show how the effective dynamics to create a fluctuation from the stationary state is, in general and for nonequilbrium dissipative system, different from the one to relax it. Moreover we have explicitley built the equations to derive the two body correlations at the stationary state which are related to the quasi-potential second derivatives around the stationary deterministic state. After the overall work we see that there is a systematic way to approach the study of nonequilibrium systems by taking as starting point the continuous Langevin equations. Nevertheless we observe that each concrete nonequilibrium system contains a very large amount of details, phenomena and complex structures that are hidden in the set of nonlinear differential equations one derives form the theory and that they are very difficult to solve because they are typically strongly dependent on the system's boundary conditions. Therefore one of the main questions to be solved is to know at to some extend the underlying microscopic details influence the mesoscopic description. We know that in some cases of boundary driven nonequilibrium systems (as for instance Fluctuating Hydrodynamics) the mesoscopic theory contains most of the necessary elements to describe correctly many observed phenomena. Nevertheless we would like to have an a priori predictive way to connect safely the microscopic and mesoscopic descriptions.

IX. ACKNOWLEDGEMENTS

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APPENDIX I: FROM LANGEVIN TO FOKKER PLANCK EQUATIONS THROUGH A FAMILY OF DISCRETIZATION SCHEMES

A. RD case:

Let us assume that the Langevin equation (1) is the continuous limit of its time discrete version:

$$\phi(x,s+1)=\phi(x,s)+\epsilon\left[F[\phi;x,s,v]+h[\phi;x,s,v]\xi(x,s)\right] \tag{183}$$

where we also assume:

$$\begin{aligned}F[\phi;x,s,v]&=F[v\phi(x,s)+(1-v)\phi(x,s+1);x]\\ h[\phi;x,s,v]&=h(v\phi(x,s)+(1-v)\phi(x,s+1))\end{aligned} \tag{184}$$

with $v\in[0,1]$, $x\in\Lambda\subset \mathrm{I\!R}^d$, $s\in Z$ and $F[\phi;x]$ is a functional on $\phi(y)$'s with y pertaining to a finite open region around x while. $h(\lambda)$ is a function. The random field ξ is a gaussian white noise characterized by:

$$\langle\xi(x,s)\rangle=0 \quad , \quad \langle\xi(x,s)\xi(x',s')\rangle=\frac{1}{\epsilon\Omega}\delta(x,x')\delta_{s,s'} \tag{185}$$

Observe that for any value $v\in[0,1]$ the limit $\epsilon\to 0$ of this discrete equation gives rise to the continuous Langevin equation (1).

We can expand the Langevin equation (183) in powers of ϵ:

$$\begin{aligned}\phi(x,s+1)=\phi(x,s)&+\epsilon h(\phi(x,s))\xi(x,s)+\epsilon F[\phi;x,s]\\ &+(1-v)\epsilon^2 h(\phi(x,s))h'(\phi(x,s))\xi(x,s)^2+O(\epsilon^{3/2})\end{aligned} \tag{186}$$

where we have assumed that ξ is of order $\epsilon^{-1/2}$.

The probability to find a given configuration ϕ at time s is defined by

$$P[\phi;s+1]=\langle\prod_{x\in\Lambda}\delta(\phi(x)-\phi(x,s+1))\rangle_\xi \tag{187}$$

where $\phi(x,s)$ is the solution of the Langevin equation for a given random noise realization and $\langle\cdot\rangle_\psi$ is the average over all noise realizations with their corresponding gaussian weight. We can substitute the ϵ expanded Langevin equation into eq.(187) and after some algebraic manipulation we get

$$\begin{aligned}P[\phi;s+1]=\int\prod_{x\in\Lambda}\left[d\bar\phi(x)\right]P[\bar\phi;s]\langle\prod_{x\in\Lambda}\delta\Big(&\phi(x)-\bar\phi(x)-\epsilon h(\bar\phi(x))\xi(x,s)-\epsilon F[\bar\phi;x]\\ &-(1-v)\epsilon^2 h(\bar\phi(x))h'(\bar\phi(x))\xi(x,s)^2+O(\epsilon^{3/2})\Big)\rangle_\xi\end{aligned} \tag{188}$$

where we have used the fact that $\phi(x,s)$ only depend on ξ's of previous times $s' < s$ and we can break the averages over ξ's. We now expand the last expression for $\epsilon \le 1$ by using the formulae

$$\prod_n \delta(a(n)+b(n)\eta+c(n)\eta^2) = \left(\prod_n \delta(a(n))\right) + \eta \sum_m \left(\prod_{n\neq m} \delta(a(n))\right) \delta'(a(m))b(m)$$
$$+\frac{1}{2}\eta^2 \sum_m \left[\left(\prod_{n\neq m} \delta(a(n))\right) \left[\delta''(a(m))b(m)^2 + 2\delta'(a(m))c(m)\right]\right.$$
$$\left. + \sum_{m'\neq m} \left(\prod_{n\neq m,m'} \delta(a(n))\right) \delta'(a(m))\delta'(a(m'))b(m)b(m')\right] + O(\eta^3) \tag{189}$$

that we get by doing the first two derivatives with respect η and then using the Taylor expansion up to second order in η. In our case we indentify $\eta = \epsilon^{1/2}$.

Finally, we can make the averages over ξ's and we get (in the limit $\epsilon \to 0$) the Fokker-Planck equation:

$$\partial_t P[\phi;t] = \int_\Lambda dx \frac{\delta}{\delta\phi(x)} \left[-(F[\phi;x] + \frac{(1-v)}{\Omega} h(\phi(x))h'(\phi(x)))P[\phi;t] \right.$$
$$\left. + \frac{1}{2\Omega}\frac{\delta}{\delta\phi(x)} \left(h(\phi(x))^2 P[\phi;t]\right)\right] \tag{190}$$

For $v = 1$ (Ito's discretization) we obtain the Fokker-Planck equation (8).

We also can compute the Lagrangian defining the path integral for a general v:

$$P[\{\phi\}[t_0,t_1]] = cte \exp\left[-\Omega L[\phi;t_0,t_1;v]\right] \tag{191}$$

where

$$L[\phi;t_0,t_1;v] = \int_{t_0}^{t_1} dt \int_\Lambda dx \left[\frac{(\partial_t\phi(x,t) - F[\phi;x,t])^2}{2h(\phi(x,t))^2} + (1-v)\frac{\delta F[\phi;x,t]}{\delta\phi(x,t)} \right.$$
$$\left. + \frac{(1-v)^2}{2} h'(\phi(x,t))^2 + \frac{1-v}{h(\phi(x,t))} h'(\phi(x,t))\left(\partial_t\phi(x,t) - F[\phi;x,t]\right)\right] \tag{192}$$

One can show that the observables (averages) computed with this lagrangian do not depend on the v used [8].

B. DD case:

In this case it is necessary to define an space and time discretizations. The field at lattice site $n \in Z^d$ at discrete time $s \in Z$, $\phi(n,s)$, is solution of the discrete Langevin equation:

$$\phi(n,s+1) = \phi(n,s) - \frac{\epsilon}{2a} \sum_{\alpha=1}^{d} \left[j_\alpha(\phi; n+i_\alpha, s) - j_\alpha(\phi; n-i_\alpha, s)\right] \tag{193}$$

where i_α is the unit vector in the direction α and

$$j_\alpha(\phi;n,s) = G_\alpha[\phi;n,s] + \sum_{\beta=1}^{d} \sigma_{\alpha\beta}[\phi;n,s]\psi_\beta(n,s) \tag{194}$$

$$\langle \psi_\alpha(n,s)\psi_\beta(n',s')\rangle = \frac{1}{\tilde{\Omega}\epsilon a^d}\delta_{\alpha,\beta}\delta_{n,n'}\delta_{s,s'} \tag{195}$$

where a and ϵ are the lattice node separation in space and time respectively. For simplicity we are considering just the Ito scheme.

The probability to find a given configuration ϕ at time s is defined by

$$P[\phi;s+1] = \langle \prod_{n\in\Lambda} \delta(\phi(n) - \phi(n,s+1))\rangle_\psi \tag{196}$$

where $\phi(n,s)$ is the solution of the Langevin equation for a given random noise realization, that is, it depends on ψ and $\langle\cdot\rangle_\psi$ is the average over all noise realizations with their corresponding gaussian weight. We can insert the right hand side of the Langevin equation and we introduce an auxiliary field $\bar{\phi}$:

$$P[\phi;s+1] = \langle \int \prod_{n\in\Lambda} \left[d\bar{\phi}(n)\delta(\bar{\phi}(n) - \phi(n,s))\right] \prod_{n\in\Lambda} \delta\left(\phi(n) - \bar{\phi}(n) + \frac{\epsilon}{2a}\sum_{\alpha=1}^{d}\left[j_\alpha(\bar{\phi};n+i_\alpha) - j_\alpha(\bar{\phi};n-i_\alpha)\right]\right)\rangle_\psi \tag{197}$$

now we use the fact that the noise ψ is time uncorrelated and the Ito's prescription. Moreover, $\phi(n,s)$ only depend on ψ's with times strictly smaller than s. Therefore we can break the average over ψ and we get:

$$P[\phi;s+1] = \int \prod_{n\in\Lambda} \left[d\bar{\phi}(n)\right] P[\bar{\phi};s] \langle\prod_{n\in\Lambda} \delta\left(\phi(n) - \bar{\phi}(n) + \frac{\epsilon}{2a}\sum_{\alpha=1}^{d}\left[j_\alpha(\bar{\phi};n+i_\alpha) - j_\alpha(\bar{\phi};n-i_\alpha)\right]\right)\rangle_\psi \tag{198}$$

We now expand the last expression for $\epsilon \leq 1$ taking into account that ψ is of order $\epsilon^{-1/2}$. We can use the formulae (189) with

$$\begin{aligned} a(n) &= \phi(n) - \bar{\phi}(n) \\ b(n) &= \frac{\epsilon^{1/2}}{2a}\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\left[\sigma_{\alpha\beta}[\bar{\phi};n+i_\alpha]\psi_\beta(n+i_\alpha,s) - \sigma_{\alpha\beta}[\bar{\phi};n-i_\alpha]\psi_\beta(n-i_\alpha,s)\right] \\ c(n) &= \frac{1}{2a}\sum_{\alpha=1}^{d}\left[G_\alpha[\bar{\phi};n+i_\alpha] - G_\alpha[\bar{\phi};n-i_\alpha]\right] \end{aligned} \tag{199}$$

After substituting this expansion into eq.(198) we can do explicitly the averages over ψ and after some algebra we get

$$
\begin{aligned}
P[\phi; s+1] = P[\phi; s] + \epsilon \sum_{m\in\Lambda} \frac{\partial}{\partial\phi(m)} \Bigg[P[\phi; s] \frac{1}{2a} \sum_{\alpha=1}^{d} \left(G_\alpha[\phi; n+i_\alpha] - G_\alpha[\phi; n-i_\alpha]\right) \\
+ \frac{1}{8\tilde{\Omega} a^{d+2}} \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \Bigg(\frac{\partial}{\partial\phi(m+i_\alpha-i_\beta)} \left(P[\phi; s]\chi_{\alpha\beta}[\phi; m+i_\alpha]\right) \\
- \frac{\partial}{\partial\phi(m+i_\alpha+i_\beta)} \left(P[\phi; s]\chi_{\alpha\beta}[\phi; m+i_\alpha]\right) \\
- \frac{\partial}{\partial\phi(m-i_\alpha-i_\beta)} \left(P[\phi; s]\chi_{\alpha\beta}[\phi; m-i_\alpha]\right) \\
+ \frac{\partial}{\partial\phi(m-i_\alpha+i_\beta)} \left(P[\phi; s]\chi_{\alpha\beta}[\phi; m-i_\alpha]\right) \Bigg) \Bigg] + O(\epsilon^2)
\end{aligned}
\tag{200}
$$

where

$$
\chi_{\alpha\beta}[\phi; n] = \sum_{\gamma=1}^{d} \sigma_{\alpha\gamma}[\phi; n]\sigma_{\beta\gamma}[\phi; n]
\tag{201}
$$

This expression can be written in a more compact form by using the definition:

$$
\left(\partial_\alpha \frac{\partial}{\partial\phi(n)}\right) \equiv \frac{1}{2a} \left(\frac{\partial}{\partial\phi(n+i_\alpha)} - \frac{\partial}{\partial\phi(n-i_\alpha)} \right)
\tag{202}
$$

where implicitly it is shown the action of the discrete partial derivative operator.

$$
\begin{aligned}
\frac{1}{\epsilon}\left[P[\phi; s+1] - P[\phi; s]\right] = \sum_{\alpha=1}^{d} \sum_{m\in\Lambda} \left(\partial_\alpha \frac{\partial}{\partial\phi(m)}\right) \Bigg[-G_\alpha[\phi; m] P[\phi; s] \\
+ \frac{1}{2\tilde{\Omega} a^d} \sum_{\beta=1}^{d} \left(\partial_\beta \frac{\partial}{\partial\phi(m)}\right) \left(\chi_{\alpha\beta}[\phi; m] P[\phi; s]\right) \Bigg] + O(\epsilon)
\end{aligned}
\tag{203}
$$

where we have used the property:

$$
\sum_{m\in\Lambda} \frac{\partial}{\partial\phi(m)} \left(\partial_\alpha F[\phi; m]\right) = -\sum_{m\in\Lambda} \left(\partial_\alpha \frac{\partial}{\partial\phi(m)}\right) F[\phi; m]
\tag{204}
$$

Also observe the useful relation:

$$
\left(\partial_\alpha \frac{\partial}{\partial\phi(m)}\right) \left(Q[\phi] F(\phi(m))\right) = F(\phi(m)) \partial_\alpha \left(\frac{\partial Q[\phi]}{\partial\phi(m)}\right)
\tag{205}
$$

with $F(\lambda)$ being a function.

In the limit $\epsilon \to 0$ and $a \to 0$ and defining $\tilde{\Omega} = a^d \Omega$ we recover the Fokker-Planck equation for diffusive systems.

APPENDIX II: THE METHOD OF CHARACTERISTICS TO SOLVE HAMILTON-JACOBI EQUATIONS

We just reproduce the page 233 in Gallavotti's book *Elements of Mechanics* [5]. Let $S(q,t)$ to be solution of the Hamilton-Jacobi equation

$$H(\frac{\partial S(q,t)}{\partial q}, q, t) + \frac{\partial S(q,t)}{\partial t} = 0 \tag{206}$$

where $H = H(p,q,t)$ is a given function on its arguments. Let us assume the following differential equation:

$$\frac{dq}{dt} = \left.\frac{\partial H}{\partial p}\right|_{p=\frac{\partial S}{\partial q}} \tag{207}$$

with the initial condition $q(t_0) = q_0$. Then, we can show that if we take

$$p(t) = \left.\frac{\partial S}{\partial q}\right|_{q=q(t)} \tag{208}$$

with $q(t)$ solution of eq.(207), then the functions $(q(t), p(t))$ are *solutions* of the Hamilton equations with Hamiltonian $H(p,q,t)$ and initial values: $q(t_0) = q_0$ and $p(t_0) = \partial S/\partial q|_{q=q_0}$. That is, each solution of the Hamilton-Jacobi equation (206) corresponds to a hamiltonian dynamics.

In order to show this assertion we just check that $p(t)$ so defined is solution of the corresponding Hamilton equation: $dp/dt = -\partial H/\partial q$:

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\left.\frac{\partial S}{\partial q_i}\right|_{q=q(t)}\right) = \sum_j \frac{\partial^2 S}{\partial q_i \partial q_j}\frac{dq_j}{dt} + \frac{\partial^2 S}{\partial t \partial q_i} \tag{209}$$

but deriving the Hamilton-Jacobi equation by $\partial/\partial q_i$ we find the relation:

$$\sum_j \frac{\partial^2 S}{\partial q_i \partial q_j}\frac{dq_j}{dt} + \left.\frac{\partial H(p,q,t)}{\partial q_i}\right|_{p=\frac{\partial S}{\partial q}} + \frac{\partial^2 S}{\partial t \partial q_i} = 0 \tag{210}$$

that we can use in eq.(209) to get the desired result:

$$\frac{dp_i}{dt} = -\frac{\partial H(p,q,t)}{\partial q_i} \quad , \quad \frac{dq_i}{dt} = \frac{\partial H(p,q,t)}{\partial p_i} \tag{211}$$

with the above metioned initial conditions.

We can find $S(q,t)$ just by studying the time behavior of $S(q(t),t)$ with $q(t)$ solution of the Hamilton equations:

$$\frac{dS(q(t),t)}{dt} = \sum_i \left.\frac{\partial S(q,t)}{\partial q_i}\right|_{q=q(t)}\frac{dq_i}{dt} + \left.\frac{\partial S(q,t)}{\partial t}\right|_{q=q(t)} \tag{212}$$

We do a time integration to it and we get:

$$S(q(t),t) - S(q(t_0),t_0) = \int_{t_0}^{t} d\tau \sum_i p_i(\tau)\frac{dq_i(\tau)}{d\tau} + \int_{t_0}^{t} d\tau \frac{\partial S(q,\tau)}{\partial \tau}\Big|_{q=q(\tau)} \tag{213}$$

where $q((\tau), p(\tau))$ are the solutions of the Hamilton equations with initial conditions $q(t_0) = q_0$ and $p(t_0) = \partial S/\partial q|_{q=q_0}$. It is convenient to choose: $p(t_0) = 0$, that is, the value of $q_0 = q^*$ in which $S(q, t_0)$ has an extreme: $\partial S/\partial q|_{q=q^*} = 0$.

APPENDIX III: PATH INTEGRAL METHOD TO OBTAIN THE CORRELATIONS

Let us begin for the RD case. In order to obtain $V_0[\phi]$ from the path integral formalism we have to solve the evolution equations for $(\bar{\phi}(x,t), \pi(x,t))$ given by eqs. (36) with boundary conditions $(\bar{\phi}(x,-\infty), \pi(x,-\infty)) = (\phi^*(x), 0)$ and $\bar{\phi}(x,0) = \phi(x)$. The quasi potential is obtained by using eqs. (42) and (44). We know that the two body correlation $C_2(x,y)$ is related with the second derivative of the quasi-potential at the deterministic stationary state (whenever V_0 is differentiable at such point). Therefore we want to solve the dynamic equations when $\bar{\phi}(x,0) = \phi^*(x) + \Omega^{-1/2}\omega(x)$. Obviously we can linearize the evolution equations around ϕ^* and we can study with detail the quasi-potential. However a priori we have no guarantee that the dynamic trajectories that connect the initial condition ϕ^* at time $-\infty$ to a small deviation from it do not have multiple solutions near ϕ^* along the path. In fact, we expect that the strong non-analyticities on $V_0[\phi]$ near ϕ^* will imply that the path that minimizes the Lagrangian functional is the one whose trajectory makes tours far from the initial point. An analytic solution for such types of situations are far from our actual knowledge. Let us focus then on the assumption that the linearized dynamics that connect the initial state with the final perturbed one is the correct. As we will see, this assumption is, in practice, equivalent to the local differentiability of the quasi-potential.

Let us linearize the evolution eqs. (36) assuming:

$$\bar{\phi}(x,t) = \phi^*(x) + \frac{1}{\sqrt{\Omega}}h[\phi^*;x]\bar{\omega}(x,t) \quad , \quad \pi(x,t) = \frac{1}{\sqrt{\Omega}}h[\phi^*;x]\bar{\eta}(x,t) \tag{214}$$

then

$$\begin{aligned} \partial_t\bar{\omega}(x,t) &= \int_\Lambda dy B(x,y)\bar{\omega}(y,t) + \frac{1}{2}\bar{\eta}(x,t) \\ \partial_t\bar{\eta}(x,t) &= -\int_\Lambda dy B(y,x)\bar{\eta}(y,t) \end{aligned} \tag{215}$$

where $B(x,y)$ is defined in eq.(105). And the initial conditions are: $(\bar{\omega}(x,-\infty),\bar{\eta}(x,-\infty)) = (0,0)$ and $\bar{\omega}(x,0)=\omega(x,0)$. The quasi-potential is, in this approximation given by:

$$V_0[\phi] = V_0[\phi^*] + \frac{1}{8}\int_{-\infty}^{0} dt\,\bar{\eta}(x,t)^2 \tag{216}$$

let us remark that the trajectory $\bar{\eta}(x,t)$ contains the boundary conditions and therefore the $\omega(x)=\sqrt{\Omega}(\phi(x)-\phi^*(x))$ field.

In order to solve the time evolution equations it is convenient to formally discretize them to simplify its handling:

$$\begin{aligned}\partial_t\bar{\omega} &= B\bar{\omega} + \frac{1}{2}\bar{\eta}\\ \partial_t\bar{\eta} &= -B^T\eta\end{aligned} \tag{217}$$

where $\bar{\epsilon}$ and $\bar{\eta}$ are vectors and B a matrix and B^T its transposed. The general solution is then

$$\begin{aligned}\bar{\eta}(t) &= e^{-tB^T}\bar{\eta}_0\\ \bar{\omega}(t) &= e^{tB}a_0 + \frac{1}{2}\int^t d\tau\, e^{(t-\tau)B}e^{-\tau B^T}\bar{\eta}_0\end{aligned} \tag{218}$$

where $\bar{\eta}_0$ and a_0 are constant vectors to be determined. First we assume that $\bar{\omega}(0)=\omega$, then

$$\omega = a_0 + C(0)\bar{\eta}_0 \quad , \quad C(t) = \frac{1}{2}\int^t d\tau\, e^{-\tau B}e^{-\tau B^T} \tag{219}$$

then

$$\bar{\eta}_0 = C(0)^{-1}(\omega - a_0) \tag{220}$$

Now we assume that B can be diagonalized (even without being symmetrical in general), that is, there exists a Q matrix such that $B=QDQ^{-1}$ with $D_{ij}=\lambda_i\delta_{i,j}$, $Q_{ij}=v_i(\lambda_j)$ where $(\lambda, v(\lambda))$ are the right eigenvalues and eigenvectors of B: $Bv(\lambda)=\lambda v(\lambda)$, $Q^{-1}_{ij}=w^*_j(\lambda_i)$ where $(\lambda^*, w(\lambda))$ are the left eigenvalues and eigenvectors of B:$B^Tw(\lambda)=\lambda^*w(\lambda)$ (a^* stands for the complex conjugate of a). Notice that the set of eigenvalues of B and B^T are the same. Two useful orthogonal properties can be derived from $QQ^{-1}=Q^{-1}Q=1$:

$$w^*(\lambda_i)\cdot v(\lambda_j) = \delta_{i,j} \quad , \quad \sum_k w^*_i(\lambda_k)v_j(\lambda_k) = \delta_{i,j} \tag{221}$$

Observe that if B is non-symmetric the set of eigenvectors may not be an orthonormal vector base.

With all these information we may introduce the boundary conditions to our general solutions. First we see that

$$\bar{\eta}(t) = (Q^{-1})^T e^{-tD} Q^T \bar{\eta}_0 \tag{222}$$

we know that $\bar{\eta}(-\infty) = 0$ implying that the real part of all the eigenvalues of B should be negative:

$$Re(\lambda_i) < 0 \quad \forall i \tag{223}$$

this is a "stability condition" over the dynamics and it is equivalent to ask that arbitrary and small perturbation to the deterministic stationary state will relax to it. The second condition is $\bar{\omega}(-\infty) = 0$. Let us write $\bar{\omega}(t)$ solution in function of its eigenvalues:

$$\bar{\omega}(t) = Qe^{tD}Q^{-1}a_0 + \frac{1}{2}\int^t d\tau\, Qe^{(t-\tau)D}Q^{-1}(Q^{-1})^T e^{-\tau D}Q^T\bar{\eta}_0 \tag{224}$$

First we can show that the integral term tends to zero when $t \to -\infty$ because:

$$(\int^t d\tau e^{(t-\tau)D}Q^{-1}(Q^{-1})^T e^{-\tau D})_{ij} = -\frac{(Q^{-1}(Q^{-1})^T)_{ij}}{\lambda_i + \lambda_j} e^{-t\lambda_j} \tag{225}$$

and we are assuming $Re(\lambda_i) < 0\ \forall i$. In the other hand the first term always diverge when applied to nonzero a_0 when $t \to \infty$. Therefore $a_0 = 0$ and the solution compatible with the boundary conditions is:

$$\bar{\eta}(t) = (Q^{-1})^T e^{-tD} Q^T C(0)^{-1}\omega \quad , \quad \bar{\omega}(t) = Qe^{Dt}Q^{-1}C(t)C(0)^{-1}\omega \tag{226}$$

where

$$C(t)_{ij} = -\frac{1}{2}\sum_{ks} Q_{ik}(Q^{-1}(Q^{-1})^T)_{ks}(Q^T)_{sj}\frac{e^{-(\lambda_k+\lambda_s)t}}{\lambda_k + \lambda_s} \tag{227}$$

Finally, the quasi potential is:

$$\Omega V_0[\phi] = \frac{1}{4}\omega^T (C(0)^{-1})^T\omega \tag{228}$$

and the two body correlation is

$$\bar{C} = 2C(0)^T \tag{229}$$

that in the continuum limit is equation (110).

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