AIP Journal of Mathematical Physics

Rigid motions: Action-angles, relative cohomology and polynomials with roots on the unit circle

J.-P. Françoise, P. L. Garrido, and G. Gallavotti

Citation: J. Math. Phys. **54**, 032901 (2013); doi: 10.1063/1.4794089

View online: http://dx.doi.org/10.1063/1.4794089

View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v54/i3

Published by the American Institute of Physics.

JOURNAL OF MATHEMATICAL PHYSICS **54**, 032901 (2013)

Rigid motions: Action-angles, relative cohomology and polynomials with roots on the unit circle

J.-P. Françoise,[1,a)] P. L. Garrido,[2,b)] and G. Gallavotti[3,c)]
[1]*Université P.-M. Curie, Laboratoire J.-L. Lions, UMR 7598 CNRS, 4 Pl. Jussieu, 75252 Paris, France*
[2]*Institute Carlos I for Computational and Theoretical Physics, Universidad de Granada, España*
[3]*Dipartimento di Fisica and INFN, Università di Roma "La Sapienza", Italia*

(Received 25 May 2012; accepted 12 February 2013; published online 13 March 2013)

Revisiting canonical integration of the classical solid near a hyperbolic or elliptic uniform rotation, normal canonical coordinates p, q are constructed so that the Hamiltonian becomes a function ("normal form") of $x_+ = pq$ or of $x_- = p^2 + q^2$: the two cases are treated simultaneously distinguishing them, respectively, by a label $a = \pm$, in terms of various power series with coefficients which are shown to be polynomials in a variable r_a^2 depending on the inertia moments. The normal forms are derived via the analysis of a relative cohomology problem and shown to be obtainable without reference to the construction of the normal coordinates via elliptic integrals (unlike the derivation of the normal coordinates p, q). Results and conjectures also emerge about the properties of the above polynomials and the location of their roots. In particular a class of polynomials with all roots on the unit circle arises. [http://dx.doi.org/10.1063/1.4794089]

I. OVERVIEW

Integration of the motion of a rigid body with a fixed point and no acting forces other than the ideal rigidity constraint is well known. We recall in Sec. II the kinematic description of the coordinates in which the system is obviously integrable by reduction to quadratures (Deprit's coordinates, described in Eq. (2.5)). We try to treat simultaneously the motions near all proper rotations, at the expense of having to append labels $a = \pm 1$ to most quantities thus making the notation a little heavier. This is done to keep always clear that the stable and unstable motions are in some sense described by the same analytic functions.

The classical integrations, (Ref. 1 and Sec. 69 of Ref. 2), and the modern versions based on the Arnold-Liouville theorem (See Sec. 50 of Ref. 3, also see Ref. 4), lead naturally to coordinates p', q' in which the motion is "linearized" (or "integrated") in the sense that the Hamiltonian becomes a function of $x'_+ = p'q'$ in the hyperbolic case or of $x'_- = p'^2 + q'^2$ in the elliptic case, and the "angles" $\alpha_+ = \text{arctanh}\frac{p'}{q'}$, respectively, $\alpha_- = \arctan\frac{p'}{q'}$ evolve linearly in time: recalling that a label $a = +$ refers to the hyperbolic coordinates and $a = -$ to the elliptic, it is $\alpha_a(t) = \alpha_a(0) + g_{0,a}(x'_a)t$ for a suitable speed $g_{0,a}(x')$. The integrations are rederived here (Secs. II and III with no claim of originality) as this is simpler than guiding the reader to the literature where they are scattered and to their conversion into a form usable for our purposes.

However the (p', q') are not canonical as their Jacobian $D_a(x')$ with respect to the original canonical coordinates is not 1. Such Jacobian is a complicated combination of the elliptic integrals which linearize the motion, see Eq. (3.2), and it is not even manifested that after computing the

---

[a)] jpf@math.jussieu.fr.
[b)] garrido@onsager.ugr.es.
[c)] giovanni.gallavotti@roma1.infn.it.

Jacobian a function depending only on the energy (and on the angular momentum) is obtained. We show that the Jacobian $D_a(x')$ can instead be directly derived via an application of the relative cohomology properties. *The method and the determination of the Jacobian $D_a(x')$ is our first (new) result*: in Sec. IV its expression is derived and normal coordinates are determined Eq. (4.11).

The normal coordinates are intimately related to the normal "action-angle" variables and to the "normal form": the normal "angles" are the above α_a while the normal "action" is defined as the function $\mathcal{A}(x'_a) = \int_0^{x'} D_a(y)dy$ whose x'_a-derivative is the Jacobian $D_a(x')$ (as it will be described in Eqs. (4.2) and (4.3) and computed in (5.4)): this is obtained as usual by requiring that the Hamiltonian expressed as a function of $\mathcal{A}$ has the speed $g_0(x')$ as $\mathcal{A}$-derivative when evaluated at $\mathcal{A}(x')$. The expression of the Hamiltonian as a function of x_a will be called "normal form" and x_a the "action" (respectively, $p^2 + q^2$ and pq in our notations).

In Sec. V the results of Sec. IV are used to express Jacobian, normal coordinates, and normal form for the Hamiltonian as power series in $x_+ = pq$ or in $x_- = p^2 + q^2$, respectively, with coefficients given by polynomials in $r_\pm^2$ (the action is analytic in p, q although the p, q are defined up to a sign in terms of the corresponding action angle coordinates).

Remarkably the polynomials seem to have their zeros on the unit circle: for some (infinite) classes of such polynomials we prove this property, and *this is our second (new) result*, and for others we conjecture it.

Finally in Sec. VI it is shown that the normal form of the Hamiltonian can be *alternatively* obtained without determining first normal coordinates: this means that studying elliptic integrals (as done in the analysis in Ref. 4) can be bypassed. It also does not require to use (a not yet existent generalization of) the Birkhoff's normal form perturbative algorithm which in any case would determine the normal form *simultaneously* with the normal coordinates (and only locally very close to the proper rotations). In this case the perturbation parameter would be the spread of the inertia moments, i.e., essentially what we call r_a (for $r_a = 0$ the system acquires cylindrical symmetry) which, furthermore, is not necessarily small.

The plan of the present work follows.

In Sec. III explicit expressions for the elliptic integrals are derived, Eq. (3.7): they express the motions in terms of variables p', q' evolving exponentially in time on the hyperbolae $p'q' = const$ or rotating uniformly on the circles $p'^2 + q'^2 = const$ and "reducing" the problem to determine a nontrivial scaling function C (depending on $p'q'$ or $p'^2 + q'^2$) which leads to the above mentioned canonical coordinates via $p = p'C$, $q = q'C$. In Appendix A details are given on the computation of appropriate elliptic integrals: this should also clarify the advantages of the cohomological analysis of the following Secs. IV and VI.

In Sec. IV the scaling function C is related to a Jacobian determinant between two differential forms, and it is computed: this is done, avoiding evaluation of other elliptic integrals, by computing the cohomology of the forms $dB \wedge d\beta$ (where B, β are two Deprit's coordinates, see Sec. II) and $dp' \wedge dq'$ relative to the functions $p'q'$ or $p'^2 + q'^2$. Normal canonical coordinates are therefore completely determined, Eqs. (4.10) and (4.11) (together with Eq. (3.7)).

The latter expressions are still quite implicit and in Sec. V we bring them to a form suitable for the evaluation of the integrating coordinates via computable power series; and as an example a few terms of the scaling functions C and of the normal form of the Hamiltonian are evaluated, Eqs. (5.4) and (5.6). Doing so various families of polynomials, in a variable r^2 depending on the inertia moments, arise. The polynomials appear to have zeros on very special locations suggesting possible conjectures.

To understand the properties of the polynomials which arise in the construction of the normal forms we show, Sec. VI, that the normal form can be generated by solving another problem of relative cohomology: this leads to an explicit determination of the normal forms in terms of a further family of polynomials. In this case we prove that some of the new polynomials have all roots on the unit circle (with the help of theorems on the location of zeros of symmetric polynomials): and we conjecture a relation with the Lee-Yang theorem for the zeros of "ferromagnetic polynomials".

In Sec. VII we determine the action angle coordinates for the other two Deprit variables (which either do not appear in the Hamiltonian or are constants of motion).

In Sec. VIII we apply the relative cohomology method of Secs. IV and VI to the pendulum, to illustrate the cohomological method. And to the geodesic flow on revolution ellipsoids, with the purpose of showing that in the latter case also appear natural families of polynomials in a parameter r^2 (depending on the ellipsoid equatorial and polar axes): but in this case the zeros are not located on the unit circle, *although almost so*: Eq. (8.6).

Appendices contain supplements (e.g., the proof of Chen's result on the zeros of symmetric polynomials).

II. SOLID WITH FIXED CENTER OF MASS

The theory of Jacobian elliptic functions, for reference see Ref. 5, yields a complete calculation for the motion of a solid with a fixed point. This is revisited here, to exhibit a few interesting properties of the relevant elliptic integrals.

The Hamiltonian of a solid with inertia moments I_1, I_2, I_3, in Deprit's canonical coordinates $(K_z, A, B, \gamma, \varphi, \beta)$, see problems in Sec. 4.11 of Ref. 6, is

$$\widetilde{H}(K_z, A, B, \gamma, \varphi, \beta) = \frac{1}{2}\frac{B^2}{I_3} + \frac{1}{2}\Big(\frac{\sin^2\beta}{I_1} + \frac{\cos^2\beta}{I_2}\Big)(A^2 - B^2) \tag{2.1}$$

and it will be convenient to order the proper axes of the solid body so that $I_3 < I_1 < I_2$. In this way the motions with $B = 0$ and $\beta = 0, \pi$ or $\pm\frac{\pi}{2}$ are stable or unstable rotations, respectively. For convenience the unstable rotations will be studied by shifting by $\frac{\pi}{2}$ the origin of the angle β, so that for the unstable rotations we shall use the Hamiltonian:

$$\widetilde{H}(K_z, A, B, \gamma, \varphi, \beta) = \frac{1}{2}\frac{B^2}{I_3} + \frac{1}{2}\Big(\frac{\cos^2\beta}{I_1} + \frac{\sin^2\beta}{I_2}\Big)(A^2 - B^2). \tag{2.2}$$

Motions can be described by referring them to the two node lines $\mathbf{m}$ and $\mathbf{n}$: $\mathbf{m}$ being the node between the angular momentum plane, orthogonal to the angular momentum $\mathbf{K} = A\,\mathbf{k}$, and the fixed reference plane $\bar{\mathbf{i}} - \bar{\mathbf{j}}$, and $\mathbf{n}$ being the node between the angular momentum plane and the inertial plane $1 - 2$. The angle γ locates the node $\mathbf{m}$ on the fixed plane and the angle φ locates $\mathbf{n}$ on the angular momentum plane with respect to $\mathbf{m}$ (Fig. 1).

With reference to the Hamiltonian in the form Eq. (2.2), where $\beta = 0, \pi$ correspond to *unstable* rotations around the $\mathbf{i_1}$ proper axis, the pairs of canonical variables are (B, β), (A, φ) and (K_z, γ); $\sin\theta = \frac{B}{A}$ while the angle δ is constant. At the "fixed points" $B = 0$, $\beta = 0, \pi$ or $B = 0$, $\beta = \pm\frac{\pi}{2}$ the angles φ and γ rotate at constant speeds: $\dot\varphi = \frac{A}{I_1}$ and $\dot\gamma = 0$ or $\dot\varphi = \frac{A}{I_2}$ and $\dot\gamma = 0$ respectively; the angle between the angular momentum and the inertial axis $\mathbf{i}_3$ is $\frac{\pi}{2}$ and $\theta = 0$. For the stable

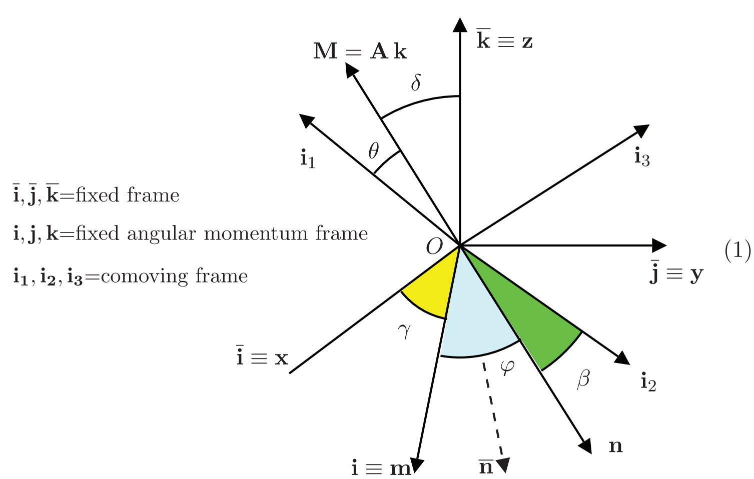

FIG. 1. The Deprit angles. Here $\bar{\mathbf{n}}$ is the node line $(\bar{\mathbf{i}}, \bar{\mathbf{j}}) \cap (\mathbf{i}_1, \mathbf{i}_2)$, $\mathbf{n}$ is the node line $(\mathbf{i}_2, \mathbf{i}_3) \cap (\mathbf{i}, \mathbf{j})$, and $\mathbf{m} \equiv \mathbf{i}$ is the node $(\mathbf{i}, \mathbf{j}) \cap (\bar{\mathbf{i}}, \bar{\mathbf{j}})$. The $\mathbf{j}$ axis is not drawn and $\mathbf{k} \| \mathbf{M}$. Colors are meant to suggest that the angles γ, φ, β lie on different planes.

rotations the role of the axes $\mathbf{i}_1$ and $\mathbf{i}_2$ are exchanged. Hence the coordinates are adapted to describe proper rotations around the axis 1 ($\beta = 0, \pi$, unstable) of intermediate inertia or 2 ($\beta = \pm\frac{\pi}{2}$, stable) of largest inertia.

The inertia moments $I_3 < I_1 < I_2$ will be used to define the quantities

$$
\begin{aligned}
&J_+ \stackrel{def}{=} I_1, \qquad J_- \stackrel{def}{=} I_2,\\
&J_{31}^{-1} \stackrel{def}{=} I_3^{-1} - I_1^{-1}, \qquad J_{32}^{-1} \stackrel{def}{=} I_3^{-1} - I_2^{-1}, \qquad J_{12}^{-1} \stackrel{def}{=} I_1^{-1} - I_2^{-1}.\\
&\widetilde{J}_+ \stackrel{def}{=} J_{31}, \qquad \widetilde{J}_- \stackrel{def}{=} J_{32}
\end{aligned}
\tag{2.3}
$$

The Lyapunov coefficients at the unstable fixed point are $\pm\lambda_+$ real with $\lambda_+ \stackrel{def}{=} \frac{A}{(\widetilde{J}_+ J_{12})^{\frac{1}{2}}}$ and at the stable are $\pm\lambda_-$ imaginary with $\lambda_- \stackrel{def}{=} \frac{i\,A}{(\widetilde{J}_- J_{12})^{\frac{1}{2}}}$.

We try to use a notation valid for *both* the stable and the unstable rotations: therefore a label $a = \pm 1$, often abridged into $a = \pm$, is appended to most quantities: with $a = +1$ referring to the unstable case and $a = -1$ referring to the stable one. Calling U the total energy, define for $a = \pm 1$:

$$
\begin{aligned}
&b_a^2 \stackrel{def}{=} \frac{2U - A^2/J_a}{\widetilde{J}_a^{-1}}, \qquad \overline{g}_a^2 \stackrel{def}{=} (2U - A^2/J_a)\widetilde{J}_a^{-1} \equiv b_a^2\,(\widetilde{J}_a^{-1})^2,\\
&s_a^2 \stackrel{def}{=} A^2 \frac{J_{12}^{-1}}{2U - A^2/J_a}, \qquad r_a^2 \stackrel{def}{=} \frac{J_{12}^{-1}}{\widetilde{J}_a^{-1}}, \qquad k_a^2 \stackrel{def}{=} \frac{a s_a^2 - a r_a^2}{1 + a s_a^2},\\
&g_a^2 = \overline{g}_a^2\,(1 + a s_a^2), \qquad b_a^2(1 + a s_a^2) = \frac{g_a^2}{(\widetilde{J}_a^{-1})^2}
\end{aligned}
\tag{2.4}
$$

then g_a is a characteristic inverse time scale and, referring to Eq. (2.2) for $a = +$ and to Eq. (2.1) for $a = -$,

$$
\begin{aligned}
&g_a^2 = A^2 J_{12}^{-1} \widetilde{J}_a^{-1}(a + \frac{1}{s_a^2}),\\
&\dot\beta = \pm\overline{g}_a\sqrt{(1 + a s_a^2 \sin^2\beta)(1 + a r_a^2 \sin^2\beta)},\\
&B = \pm b_a \sqrt{\frac{1 + a s_a^2 \sin^2\beta}{1 + a r_a^2 \sin^2\beta}} \equiv \pm\frac{\overline{g}_a}{\widetilde{J}_a^{-1}}\sqrt{\frac{1 + a s_a^2 \sin^2\beta}{1 + a r_a^2 \sin^2\beta}}
\end{aligned}
\tag{2.5}
$$

with A, $K_z \equiv A\cos\delta$ constants of motion. If $a = -$ it is $(2U - \frac{A^2}{J_-}) > 0$.

Suppose first $(2U - \frac{A^2}{J_a}) > 0$ and small: then

$$
\begin{aligned}
&\overline{g}_a^2, \qquad a\,g_a^2 > 0, \qquad b_a^2 > 0, \qquad \begin{cases} r_+^2 \in (0, +\infty) \\ r_-^2 \in (0, 1) \end{cases},\\
&s_a^2 \gg 1, \begin{cases} 0 < k_+^2 < 1 \\ k_-^2 > 1 \end{cases}, \qquad 1 + a r_a^2 = \frac{\widetilde{J}_{-a}^{-1}}{\widetilde{J}_a^{-1}} = \frac{r_a^2}{r_{-a}^2},\\
&a s_a^2 = \frac{k_a^2 + a r_a^2}{1 - k_a^2}, \qquad \frac{b_a^2}{1 - k_a^2} = \frac{g_a^2 \widetilde{J}_a^2}{1 + a r_a^2} = g_a^2 \widetilde{J}_+ \widetilde{J}_-,
\end{aligned}
\tag{2.6}
$$

and motions near $B = 0$, $\beta = 0$ and with energy U close to $A^2/2J_a$ and can be expressed in terms of Jacobian elliptic integrals (see Ref. 5 for notations).

Remark: Rotating the positions of the inertia moments in $\widetilde{H}$ the meaning of the variables changes but the following analysis remains essentially unchanged (up to renaming variables): it appears in this way that the proper rotations around axis $\mathbf{i}_1$ are (linearly) unstable while the rotations around the highest and lowest inertia axes ($\mathbf{i}_2$ and $\mathbf{i}_3$) are (linearly) stable.

III. HYPERBOLIC AND ELLIPTIC COORDINATES

"Linearization" of rigid body motions is, of course, well known to lead to elliptic integrals, see p. 144 of Ref. 2. Here we rederive the integration in a form suitable for our purposes of normal coordinates construction.

Let $\sqrt{-1} = -i$ so that $g_a = \sqrt{a}|g_a|$; for $a = +$ Eq. (2.3) describe the separatrix branches emerging from the proper rotation around $\mathbf{i}_1$ controlled by Eq. (2.2), while for $a = -$ they describe the motion near the stable rotation around $\mathbf{i}_2$ controlled by Eq. (2.1).

Let $U - A^2/2J_a > 0$ close to 0; from the definitions of the Jacobian elliptic integrals, see formulae 2.616 of Ref. 5, and setting

$$k'_a = \sqrt{1-k_a^2} \equiv \sqrt{\frac{1+ar_a^2}{1+as_a^2}}, \qquad u_a = \sqrt{1+as_a^2}\,\overline{g}_a t \equiv g_a t, \tag{3.1}$$

it follows (choosing the sign $+$ in Eq. (2.5), for instance) from (5, 8.153.3] for $a = +$ and 5, 8.153.1-8] for $a = -$ after correcting the obvious typo in 5, 8.153.8] and by $k'_- = -i\sqrt{k_-^2 - 1}$), see Appendix A for details,

$$\begin{aligned}
B(t) &= \frac{b_a}{\mathrm{dn}(u_a,k_a)} = b_a\frac{\mathrm{cn}(-iu_a,k'_a)}{\mathrm{dn}(-iu_a,k'_a)},\\
\sin^2\beta(t) &= \frac{\mathrm{sn}^2(u_a,k_a)}{1+as_a^2\mathrm{cn}^2(u_a,k_a)} = \frac{-\frac{\mathrm{sn}^2(-iu_a,k'_a)}{\mathrm{cn}^2(-iu_a,k'_a)}}{1+as_a^2\frac{1}{\mathrm{cn}^2(-iu_a,k'_a)}},\\
\dot\beta(t) &= \frac{(1+as_a^2)\overline{g}_a\mathrm{dn}(u_a,k_a)}{1+as_a^2\mathrm{cn}^2(u_a,k_a)} = \overline{g}_a\frac{\mathrm{dn}(-iu_a,k'_a)\mathrm{cn}(-iu_a,k'_a)}{1-\frac{1}{1+as_a^2}\mathrm{sn}^2(-iu_a,k'_a)},
\end{aligned} \tag{3.2}$$

where the last relation is deduced from the equations of motion and the expression for $\sin^2\beta$.

Notice that $\beta(0) = 0$ and if $(2U - A^2/J_a) \to 0^+$ it is $k_\pm^2 \to 1^\mp$, $a\,g_a^2 \to \widetilde{J}_a^{-1} J_{12}^{-1} A^2$, $b_a \to 0^+$ and motions with this energy are "like" the motions close to the separatrix or, respectively, close to equilibrium of a pendulum.

For $(2U - A^2/J_a) \le 0$, or for $k_a > 1$ hence k'_a imaginary, the above relations have to be interpreted via suitable analytic continuations. Some of the following formulae become singular as $U \to A^2/2J_a$: the singularity is only apparent and it will disappear from all relevant formulae derived or used in the following.

Introduce variables useful in the following (most of them appear in the theory of Jacobi's elliptic functions):

$$\begin{aligned}
&\mathbf{K}(k) \stackrel{def}{=} \int_0^{\frac{\pi}{2}} (1-k^2\sin^2\theta)^{-\frac12}d\theta,\\
&x'_a \stackrel{def}{=} ae^{-\pi\frac{\mathbf{K}(k_a)}{\mathbf{K}(k'_a)}}, \qquad I^2 \stackrel{def}{=} 4\widetilde{J}_+\widetilde{J}_-,\\
&g_a^2k_a'^2 = \overline{g}_a^2(1+ar_a^2) \quad u_a = g_a t, \qquad g_{0,a} = |g_a|\,\frac{\pi}{2\mathbf{K}(k'_a)}.
\end{aligned} \tag{3.3}$$

The g_a, $g_{0,a}$ depend on k_a and can, and will, be imagined as functions of x'_a.

Let $\alpha_a \stackrel{def}{=} \arcsin\frac{1}{\sqrt{1+ar_a^2}} - iv_a = \int_0^{\alpha_a}\frac{d\theta}{\sqrt{1-k_a'^2\sin^2\theta}}$ and the integral can be evaluated by series 2.511.2 and 8.113.1 of Ref. 5 , defining P_m and V

$$\begin{aligned}
-iv_a &= \frac{2\mathbf{K}(k'_a)}{\pi}\alpha_a - \sqrt{ar_a^2}\sum_{m=1}^\infty \binom{-\frac12}{m}(-k_a'^2)^m P_m(\frac{1}{1+ar_a^2})\\
&\stackrel{def}{=} \frac{2\mathbf{K}(k'_a)}{\pi}\Big(\alpha_a + \sqrt{ar_a^2}V(\frac{1}{1+ar_a^2},k_a'^2)\Big)
\end{aligned} \tag{3.4}$$

with $P_m(z) = \frac{1}{2m}(\sum_{k=0}^{m-1} \frac{(2m-1)!!\,(m-k-1)!}{2^k\,(2m-2k-1)!!\,\ell!} z^{\ell-k})$. This is well defined near the proper rotations , where $k_a \to 1$, as k'_a tends to 0. Therefore, as derived in detail in Appendix A,

$$\gamma_a \equiv \Gamma(ar_a^2, k_a'^2) \stackrel{def}{=} e^{v_a \frac{\pi}{2\mathbf{K}(k'_a)}} = \Big(\frac{i+\sqrt{ar_a^2}}{\sqrt{1+ar_a^2}}\Big) e^{i\sqrt{ar_a^2}V(\frac{1}{1+ar_a^2}, k_a'^2)} \stackrel{def}{=} \frac{\overline{\gamma}_a}{\sqrt{a}} \tag{3.5}$$

and $|\gamma_+| = 1, \gamma_- = ie^{\lambda}$ with λ real and $\overline{\gamma}_-$ real, with λ real.

Introduce the new variables p'_+, p'_- defined by

$$p'_\sigma \stackrel{def}{=} \sqrt{x'_a} e^{\sigma\sqrt{a} g_{0,a} t}, \quad \sigma = \pm, \qquad \text{hence} \quad x'_a \equiv p'_+ p'_- . \tag{3.6}$$

The above relations can be algebraically elaborated into the transformation of coordinates $(B, \beta) \longleftrightarrow (p'_+, p'_-)$, see again details in Appendix A:

$$\begin{aligned} B &= R(p'_+, p'_-) \stackrel{def}{=} a\, I g_{0,a}(x'_a) \sum_{\mu=\pm 1} \sum_{m=0}^{\infty} \frac{a^m\, x_a'^m\, p'_\mu}{1 + a(x_a'^m p'_\mu)^2}, \\ \beta &= S(p'_+, p'_-) \stackrel{def}{=} a \sum_{n=0}^{\infty} \sum_{\sigma,\eta=\pm 1} \frac{\eta^{\frac{1+a}{2}}\sigma}{i}\, a^n \operatorname{arctanh}(x_a'^n p'_\sigma \overline{\gamma}_a^\eta) \end{aligned} \tag{3.7}$$

and in the new coordinates the *flow* $t \to p'_\sigma e^{\sqrt{a}\sigma g_{0,a} t}$ *generates a solution* of the equations of motion if $|p_\pm|$ is small and if the real parts of $p'_\pm$ are >0. However the equations of motion are analytic and so are Eq. (3.7): hence Eq. (3.7) yield a solution under the only condition that $|p_\pm|$ are small, to be required for its convergence.

Equation (3.7) gives a complete parametrization of the motions *but the coordinates p_σ are not canonical*: the determination of the canonical coordinates will be discussed and done in Sec. IV.

Remarks: (1) via identities relating Jacobian elliptic functions it would be possible to avoid considering functions that are defined by analytic continuations, for instance when the modulus $k_a^2 > 1$ or $k_a'^2 < 0$, and reduce instead to considering only cases with the "simple" arguments (expressing functions by elliptic functions of other arguments, like $h_a^2 = 1 - k_a^{-2}$, $h_a'^2 = k_a^{-2}$, when necessary): we avoid this because it would hide the nice property that the stable and unstable cases can be seen as analytic continuations of each other.

(2) The formula for β reminds of one found by Jacobi which he commented by saying that "*inter formulas elegantissimas censeri debet*," p. 509 of Ref. 7 (i.e., "*it should be counted among the most elegant formulae,*" see also Ref. 8).

IV. CANONICAL COORDINATES CONSTRUCTION. JACOBIAN

The motions $p'_\sigma \to p'_\sigma e^{\sigma\sqrt{a} g_{0,a} t}$ solve the equations of motion if the real and imaginary parts of $p'_\pm$ are positive. But the equations of motion are analytic, hence the formulae in Secs. II and III, and the ones that will be discussed in the present Sec. IV give solutions of the equations independently of the sign of p'_σ, provided the series converge. Convergence requires $|p'_+|, |p'_-| \ll 1$: which represents many data, in particular those in the vicinity of the separatrix or of the stable proper rotation.

The coordinates can be called "hyperbolic" or "elliptic". We also see that time evolution preserves both volume elements $dBd\beta$ and $dp'dq'$; which means that the Jacobian determinant $\frac{\partial(B,\beta)}{\partial(p',q')}$ must be a function constant over the trajectories, hence a function $D_a(x')$ of $x' \stackrel{def}{=} p'_+ p'_-$ in the two cases. Note that $D_a(x')$ has dimension of an action.

Introduce the variables p', q', related to $p'_\pm$ in Eq. (3.6) by

$$\begin{cases} p' = p'_+, \ q' = p'_- \text{ if } a = + \\ p' + i\sigma q' = p'_\sigma \text{ if } a = - \end{cases}, \tag{4.1}$$

to deal with real coordinates when useful; it is then possible to change coordinates setting $p_\pm = C_a(x'_a)\, p'_\pm$, (or $p \stackrel{def}{=} C_a(x'_a)\, p', q \stackrel{def}{=} C_a(x'_a)\, q'$) and choose $C_a(x')$ so that the Jacobian determinant

$D_a(x')$ for $(B, \beta) \longleftrightarrow (p_+, p_-)$ is $\equiv 1$. A brief calculation shows that this is achieved by fixing

$$C_a^2(x') = \frac{1}{x'}\int_0^{x'} D_a(y)dy, \tag{4.2}$$

which is possible for x' small because, from the equations of motion it is $D_a(0)$ finite (e.g., $D_+(0) = 4Ig(0)\sin\frac{\pi\alpha_+}{2K(k'_+)} > 0$). Therefore map $(B, \beta) \longleftrightarrow (p, q)$ is area preserving, hence *canonical*. The Hamiltonian Eq. (2.1) becomes a function $\mathcal{U}_a(x_a)$ of $x_a = p_+p_-$ and the derivative of the energy with respect to x_a has to be $g_{0,a}(x'_a)$ (because the p_+, p_- are canonically conjugated to B, β). Note that x_a has the dimension of an action, while p, q are, dimensionally, square roots of action.

This allows us to find $D_a(x')$: by imposing that the equations of motion for the (p, q) canonical variables have to be the Hamiltonian's equations with Hamiltonian $\mathcal{U}_a(x_a) \stackrel{def}{=} U_a(x'_a) \equiv H(B, \beta)$ it follows that $\frac{d\mathcal{U}_a(x_a)}{dx_a} = g_{0,a}(x'_a)$, i.e., $\frac{dU_a(x'_a)}{dx'_a}\frac{dx'_a}{dx_a} = g_{0,a}$ or $\frac{dU_a(x'_a)}{dx'_a} = g_{0,a}\frac{d}{dx'_a}(x'_a C_a(x'_a)^2)2^{-\frac{1-a}{2}} = g_{0,a}D_a(x'_a)$ by the above expression for $C_a(x'_a)$. The just obtained relation gives

$$D_a(x'_a) = g_{0,a}(x'_a)^{-1}\frac{d}{dx'_a}U_a(x'_a), \tag{4.3}$$

which is an expression for the Jacobian $\frac{\partial(B,\beta)}{\partial(p',q')} \equiv \frac{\partial(R,S)}{\partial(p',q')} = \frac{\partial(p,q)}{\partial(p',q')}$ (note that the Jacobian between (B, β) and (p, q) is identically 1 by construction).

To proceed it is necessary to determine the function $C_a(x'_a)^2$. The idea in Ref. 8 is to make use of the general theory of relative cohomology classes to determine $D_a(x'_a)$ and, therefore, to find a complete expression of normal variables following a procedure employed in the theory of limit cycles of planar vector fields.[9,10]

A derivation can be based directly on the equalities on symplectic forms

$$\begin{aligned} dB \wedge d\beta &= D_a(x'_a)dp' \wedge dq' = \frac{d}{dx'_a}[x'_aC_a(x'_a)^2]\, dp' \wedge dq' \\ &= \Big(\frac{i}{2}\Big)^{\frac{1-a}{2}}\frac{d}{dx'_a}[x'_aC_a(x'_a)^2]\, dp'_+ \wedge dp'_-. \end{aligned} \tag{4.4}$$

To simplify the notation we drop for a while the label a from x''s. The idea is to compute only the *cohomology class* of the volume forms in the relative cohomology of the function $x' = p'_+p'_-$.[9,10]

For any 2-form $\varphi(p', q')dp' \wedge dq'$ with $\varphi(p', q') = \sum_{m,n} f_{m,n}p'^m q'^n$ the "*cohomology class*" relative to $x' = p'_+p'_-$ is the function defined by $\psi(x') \stackrel{def}{=} \sum_m f_{m,m}(p'q')^m$. Therefore an expression for the Jacobian $D_a(x')$ is obtained by finding an expression for the cohomology class of the form $dB \wedge d\beta$ relative to $x' = p'_+p'_-$.

Remark that Eq. (3.7) imply

$$\begin{aligned} R'(p'_+, p'_-)\, dS'(p'_+, p'_-) = Ig_0(x') \sum_{m,n=0}^{+\infty}\sum_{\mu,\sigma,\eta=\pm1}\frac{\eta^{\frac{1+a}{i}}\sigma}{i} \\ \cdot\frac{a^{m+n}x'^m\, x'^n\overline{\gamma}_a^\eta}{(1+a(x'^m p'_\mu)^2)(1-(x'^n p'_\sigma\gamma_a^\eta)^2)}\, p_\mu dp'_\sigma + \mathbf{0}, \end{aligned} \tag{4.5}$$

where $\mathbf{0}$ is a form with 0 cohomology (i.e., the first term is RdS "modulo $d(p'_+p'_-)$" because the cohomology of $d\Big(G(p'_+p'_-)d(p'_+p'_-)\Big)$ is, for all G's, 0).

The terms with p_+dp_+ and p_-dp_- do not contribute to the cohomology and, up to terms (denoted $\mathbf{0}$) not contributing to the cohomology, the form in Eq. (4.4) becomes (as only terms with $\mu = -\sigma$ contribute)

$$\begin{aligned} R'(p'_+, p'_-)\, dS'(p'_+.p'_-) = Ig_0(x')\tfrac{1}{i}\sum_{m,n=0}^{+\infty}\sum_{\eta,\sigma=\pm1} x'\frac{\sigma dp'_\sigma}{p'_\sigma} \\ \cdot\frac{a^{m+n}x'^{m+n}\,\eta^{\frac{1+a}{2}}\,\overline{\gamma}_a^\eta}{(1+a(x'^m p'_{-\sigma})^2)(1-(x'^n\overline{\gamma}^\eta p'_\sigma)^2)} + \mathbf{0}. \end{aligned} \tag{4.6}$$

If $RdS = x'F(p'_+, p'_-)\frac{1}{2}(\frac{dp'_-}{p'_-} - \frac{dp'_+}{p'_+}) + \mathbf{0}$ and $F(p'_+, p'_-) = \sum_{h,k=0}^{\infty} F_{h,k} p'^h_+, p'^k_-$, only the terms with $k = h$ contribute and we can take

$$F = I g_0(x')\frac{1}{i} \sum_{m,n=0}^{\infty} \sum_{\eta,\sigma=\pm 1} \frac{a^{m+n} x'^{m+n+1}\, \eta^{\frac{1+a}{2}} \overline{\gamma}_a^\eta}{(1 + a(x'^m p'_\sigma)^2)\,(1 - (x'^n \overline{\gamma}_a^\eta p'_{-\sigma})^2)}\,, \tag{4.7}$$

hence the cohomology with respect to $p_+ p_-$ is that of $(\frac{dp_+}{p_+} - \frac{dp_-}{p_-})$ times

$$\begin{aligned} &\frac{2I g_0(x')}{i} \sum_{\eta=\pm 1} \sum_{m,n,h} x' \eta^{\frac{1+a}{2}} a^{m+n} x'^{m+n} \overline{\gamma}_a^\eta (x'^{m+n+1} \overline{\gamma}_a^\eta)^{2h} \frac{(-a)^h}{2} \\ &= I g_0(x')\, x' \sum_{\eta=\pm} \frac{\eta^{\frac{1+a}{2}}}{i} \sum_{\ell=0}^{\infty} \frac{(\ell+1)\, a^\ell\, x'^\ell\, \overline{\gamma}_a^\eta}{1 + a(x'^{(\ell+1)}\, \overline{\gamma}_a^\eta)^2}, \end{aligned} \tag{4.8}$$

which contributes

$$-\frac{d}{dx'}\Big(2I\, x'\, g_0(x') \sum_{\eta=\pm 1} \frac{\eta^{\frac{1+a}{2}}}{i} \sum_{\ell=0}^{\infty} \frac{(\ell+1) x'^\ell\, a^\ell\, \overline{\gamma}_a^\eta}{1 + a(x'^{(\ell+1)} \overline{\gamma}_a^\eta)^2}\Big)\, dp'_+ \wedge dp'_-, \tag{4.9}$$

because the cohomology of $d(x' f(x')(\frac{dp'_+}{p'_+} - \frac{dp'_-}{p'_-}))$ is $-2\frac{d}{dx'}(x' f(x'))dp'_+ \wedge dp'_-$. Therefore, by Eq. (4.2) and by $dp'_+ \wedge dp'_- = (\frac{2}{i})^{\frac{1-a}{2}} dp' \wedge dq'$, see Eq. (3.5):

$$\begin{aligned} C_a(x')^2 &= 2\, I\, g_{0,a}(x') \sum_{\eta=\pm 1} \Big(\frac{\eta}{i}\Big)^{\frac{1+a}{2}} \sum_{\ell=0}^{\infty} \frac{(\ell+1) a^\ell x'^\ell \overline{\gamma}_\alpha^\eta}{1 + a(x'^{(\ell+1)} \overline{\gamma}_\alpha^\eta)^2} \\ &\equiv 2\, I\, g_{0,a}(x') \sum_{\eta=\pm} \frac{\eta}{i} \sum_{\ell=0}^{\infty} \frac{(\ell+1) a^\ell x'^\ell \Gamma(ar_a^2, k'^2_a)^\eta}{1 + (x'^{(\ell+1)} \Gamma(ar_a^2, k'^2_a)^\eta)^2} \\ &\stackrel{def}{=} 2Ar_a C(ax'_a, ar_a^2)^2. \end{aligned} \tag{4.10}$$

Summarizing we set, with $p'_\pm$ defined in Eqs. (4.1) and (3.6),

$$\begin{aligned} &p = \sqrt{2Ar_a}\, p'\, C(ax'_a, ar_a^2), \quad q = \sqrt{2Ar_a}\, q'\, C(ax'_a, ar_a^2), \\ &x' = f(p'_+ p'_-), \ \frac{p_+ p_-}{2Ar_a} = x_a = x'_a C^2(ax'_a, ar_a^2), \end{aligned} \tag{4.11}$$

where $p'_+ p'_- = p'q'$ if $a = +$ and $p'_+ p'_- = (p'^2 + q'^2)$ if $a = -$.

V. EVALUATION OF JACOBIANS AND NORMAL FORMS:

It it possible to evaluate C_a^2 and the normal form for the energy $U = H$ in the canonical coordinates (p_+, p_-) up to large orders in x_a. This is done simply by deriving power series expansions, in x'_a first and then x'_a in terms of x_a, for the functions in Eq. (5.3).

Consistently with Eq. (4.11) C_a^2 can be expanded into a power series of x or x' whose coefficients are functions of the parameter r_a^2. There is a simple relation between the cases $a = \pm$ and the coefficients of the expansions in x can be computed exactly with a finite computational algorithm up to any prefixed order: this is described below.

The analysis of the Jacobian $C_a(x, r)^2$ can be based on the computable expressions of $\overline{\gamma}_a$ in Eq. (3.5).

Normal forms express $\widetilde{H}_a \stackrel{def}{=} \frac{2H - A^2 J_a^{-1}}{a J_{12}^{-1}}$. From Eqs. (2.1) and (2.2) we derive $A^2 s_a^{-2}$ (see Eq. (2.6)) in terms of k_a^2 and, by Eq. (3.3), x'_a in terms of k_a^2:

$$x'_a = a e^{-\pi \frac{\mathbf{K}(k_a)}{\mathbf{K}(k'_a)}}, \qquad a = \pm \tag{5.1}$$

which can be inverted in terms of the functions $F(z)=\frac{2\sum_{n=1}^{\infty}z^{(2n-1)^2}}{1+2\sum\limits_{n=0}^{\infty}z^{(2n)^2}}$ as

$$k_a(x')^2=\Big(\frac{1-F(ax')}{1+F(ax')}\Big)^4\stackrel{def}{=}W(ax'),\qquad a=\pm1 \tag{5.2}$$

as shown in Sec. 8.198 of Ref. 5. Therefore the above relations give explicitly the functions $\widetilde H_a$ in terms of x', hence they determine the functions $U_a(x')$.

To express $\widetilde H_a$ in terms of x we have to determine the Jacobian function and invert the relation $x=x'\widetilde C_a^2(x')$ in the two cases. This is done by remarking that $g_{0,a}=\sqrt{a}\,g_a\frac{\pi}{2\mathbf{K}(k'_a)}$ and $U_a(x')$ can be written as

$$\begin{aligned}
&\frac{g_{0,a}(x')}{\sqrt{a}\,g_a(x')}=\frac{\pi}{2\mathbf{K}(k'_a)}\qquad a g_a(x')^2=\frac{A^2r_{-a}^2}{k_a^2+ar_a^2}\frac{1}{J_{12}^2},\\
&\widetilde H(x')\stackrel{def}{=}\frac{2U_a(x')-A^2/J_a}{aJ_{12}^{-1}}=A^2\frac{1}{as_a^2}=A^2\frac{(1-k_a^2)}{k_a^2+ar_a^2}\\
&C^2(x')=(W(ax')+ar_a^2)^{-\frac12}\frac{\pi}{2\mathbf{K}(k'_a)}\\
&\qquad\cdot 2\sum_{h=0}^{\infty}\sum_{\eta=\pm}\frac{\eta}{i}\frac{(h+1)(ax')^h\Gamma(ar_a^2,k_a'^2)^\eta}{1+((ax')^{h+1}\Gamma(ar_a^2,k_a'^2)^\eta)^2}
\end{aligned} \tag{5.3}$$

where the relations $r_{-a}^2=\frac{r_a^2}{1+ar_a^2}$, $\frac{Ir_{-a}}{J_{12}}=2r_a$, following from Eq. (2.6) or (2.3), have been used and the function Γ is defined in Eq. (3.5).

This is remarkable because the summation over $\eta=\pm$ is analytic in ar_a^2 in spite of the fact that each addend has a branch point in $\sqrt{ar_a^2}$ at $r_a=0$. Therefore Eq. (5.3) shows that the two cases $a=\pm$ are described by the same functions (*of course evaluated at completely different points*) and the power series for C and U can be computed setting $a=+$ and then the case $a=-$ is obtained by replacing x by $-x$ and r^2 by $-r^2, Ar_+$ by Ar_-. The simple relation between the unstable and stable cases implies that that we only need to study one of the two cases: we shall only make computations in the hyperbolic case $a=+$.

At this point the Jacobian and the normal forms can be computed as formal series in x after solving the implicit function problem $x=2Arx'C^2(x',r^2)$ in the form $x'=x\overline{C}^2(x,r^2)$ (in a power series in x).

An algebraic numerical evaluation for $a=+$ (unstable case with, for simplicity, $r=r_+, x'_+=x'$) yields results that for $n\le 7$, if $\xi\equiv\frac{x'}{(1+r^2)}$, are

$$\begin{aligned}
x'\,C(x',r^2)^2&=4\xi-24(-1+r^2)\xi^2+32(3-14r^2+3r^4)\xi^3\\
&-16(-1+r^2)(19-242r^2+19r^4)\xi^4\\
&+24(35-1140r^2+3026r^4-1140r^6+35r^8)\xi^5\\
&-192(-1+r^2)(11-740r^2+3426r^4-740r^6+11r^8)\xi^6\\
&+64(77-10206r^2+103635r^4-211460r^6+103635r^8\\
&\quad-10206r^{10}+77r^{12})\xi^7+\dots
\end{aligned} \tag{5.4}$$

Finally the polynomials in r^2 in Eq. (5.4), coefficients of order $n>1$ for the power expansion in ξ^n, *can be conjectured to have all roots real*: this has been numerically checked for $n\le 15$. Furthermore if we define the inverse function $x'=x'(x)$ via $x=x'C(x')$ it seems that the corresponding polynomials in r have, instead, all roots on the unit circle as we have checked.

The normal forms of the Hamiltonians (i.e., their expression in the variables p, q) are derived by expressing s^2 in Eq. (2.1) in terms of r^2 and k^2 and then expanding k^2 in powers of x' using $x'=e^{\pi\frac{\mathbf{K}(k)}{\mathbf{K}(k')}}$ in Eq. (5.1), via Sec. 8.198 of Ref. 5 (notice that in our hyperbolic case x' is the q of the quoted tables with k and k' exchanged and finally computing $x' = x\,\overline{C}^2(x, r^2)$ by inverting the Jacobian relation $x = x'\,C^2(x', r^2)$. This can be implemented algebraically on a computer.

With the definitions in Eq. (2.3) $\widetilde{H}_a \stackrel{def}{=} \frac{2H - A^2 J_a^{-1}}{a\,J_{12}^{-1}}$, in the unstable case ($a = +1$) or in the stable ($a = -1$) case, can be expressed as function of $x = \frac{pq}{2Ar_+}$ or, respectively, of $x = \frac{p^2+q^2}{2Ar_-}$ obtained by solving $x = x'C^2(x', r^2)$ as $x' = \Xi(x, r^2)$ and replacing x' by $\Xi(x, r^2)$ in $U(x')$ in Eq. (5.3) (notice that $J_{12}^{-1} = r_a^2 \widetilde{J}_a^{-}$). The resulting functions are normal forms for the Hamiltonian since p, q are conjugated variables. Hence

$$H(x_a) = \frac{A^2}{2J_a} + \frac{A^2\,ar_a^2}{2\widetilde{J}_a}\,\mathcal{H}(ax_a, ar_a^2) \tag{5.5}$$

for $a = \pm 1$ where $\mathcal{H}(x, r^2) \stackrel{def}{=} \frac{1}{as_a^2} = \frac{1 - W(\Xi(x_a, ar_a^2))}{W(\Xi(x_a, ar_a^2)) + ar_a^2}$, where $W(x)$ is in Eq. (5.2). One obtains

$$\begin{aligned}
\mathcal{H}(x, r^2) &= 4x - 2(-1 + r^2)x^2 - (1 + r^2)^2 x^3 \\
&- \frac{5}{4}(-1 + r^2)(1 + r^2)^2 x^4 - \frac{3}{16}(1 + r^2)^2(11 - 10r^2 + 11r^4)x^5 \\
&- \frac{7}{16}(-1 + r^2)(1 + r^2)^2(3 - 4r + 3r^2)(3 + 4r + 3r^2)x^6 \\
&- \frac{1}{64}(1 + r^2)^2(527 - 332r^2 + 330r^4 - 332r^6 + 527r^8)x^7 \\
&- \frac{9}{512}(-1 + r^2)(1 + r^2)^2(1043 + 548r^2 + 1058r^4 \\
&\quad + 548r^6 + 1043r^8)x^8 + \ldots
\end{aligned} \tag{5.6}$$

It is remarkable that *the polynomials in r appearing as coefficients of order $2 \le n \le 15$ in the power series expansion in x of the normal form have, for all n's, just roots of unity* as it appears by evaluating them numerically.

VI. NORMAL FORM WITHOUT DETERMINATION OF NORMAL COORDINATES

The function $\mathcal{H}(x)$ being the normal form of the Hamiltonian in is uniquely determined, while the normal coordinates themselves are not unique. Therefore it makes sense to see if the normal form itself could be derived without actually determining the normal coordinates. Here we show that this can be achieved quite directly without having to deal with elliptic integrals: while in the elliptic case, for instance, the classical prescription, Sec. 50 of Ref. 3, would be to study the Hamiltonian as a function of the integral $\oint B \frac{d\beta}{2\pi}$ with B defined in Eq. (2.5)).

As it is well known the full determination of the normal coordinates (or the action-angle coordinates) is much more difficult than the determination of the normal form of the Hamiltonian, see for instance the "conjecture" in Ref. 11 and its proof in Ref. 12. Therefore a direct determination of the normal form is worth investigating: and as it will appear here, the construction yields unexpected insights into our problem.

Consider first the normal form near the stable rotation. In contrast with our first method, we do not start with the Jacobi's coordinates in which the motion is linear.

The Hamiltonian value U will be written as (see Sec. II, $a = -$):

$$\frac{2U - A^2/I_2}{J_{32}^{-1}} = B^2(1 - r_a^2 \sin^2\beta) + r_a^2 A^2 \sin^2\beta. \tag{6.1}$$

For simplicity of notation the label $a = -$ will be omitted. Introducing $X = B\sqrt{1 - r^2(\sin\beta)^2}$, $Y = rA\sin\beta$ the Hamiltonian becomes $U = \frac{1}{2J_{32}}(X^2 + Y^2) + \frac{A^2}{2I_2}$ and the form $\omega = dB \wedge d\beta$ is, if

$X_\pm \stackrel{def}{=} \frac{X\pm iY}{\sqrt{2}}$,

$$\begin{aligned} dB\wedge d\beta &= \frac{1}{\sqrt{(1-r^2(\frac{Y}{rA})^2)(1-(\frac{Y}{rA})^2)}}\frac{dX\wedge dY}{Ar}\\ &= \sum_{n=0}^\infty\sum_{h+k=n}\binom{-\frac12}{h}\binom{-\frac12}{k}(-1)^{k+h}r^{2h}\Big(\frac{X_+-X_-}{i\sqrt{2}Ar}\Big)^{2(k+h)}\frac{dX\wedge dY}{Ar}\\ &= \sum_{n=0}^\infty P_n(r^2)\Big(\frac{X_+X_-}{2A^2r^2}\Big)^n\frac{dX\wedge dY}{Ar}+\mathbf{0}, \qquad \text{where } P_n(r^2)\\ &= (-1)^n\binom{2n}{n}\sum_{h+k=n}\binom{-\frac12}{h}\binom{-\frac12}{k}r^{2h}=\binom{2n}{n}\sum_{h+k=n}\binom{2h}{h}\binom{2k}{k}r^{2h} \end{aligned} \tag{6.2}$$

and $\mathbf{0}$ is a form with 0–cohomology relative to X_+X_-.

Therefore, see Proposition 1 of Ref. 13, Ref. 14, or Theorems 1.1 and 1.2 in Ref. 15, there is a change of coordinates $X, Y \to X', Y'$ which, if $H' = \frac{(X'^2+Y'^2)}{(2Ar)^2}$ and $H = \frac{(X^2+Y^2)}{(2Ar)^2}$, keeps H equal to H' and makes ω proportional to $dX'\wedge dY'$:

$$\omega = \sum_{n=0}^\infty P_n(r^2)H'^n\frac{dX'\wedge dY'}{Ar}\stackrel{def}{=} D(H')\frac{dX'\wedge dY'}{Ar}\,. \tag{6.3}$$

Let $\widehat{p} = \frac{X'}{\sqrt{2Ar}}\widehat{C}(H')$, $\widehat{q} = \frac{Y'}{\sqrt{2Ar}}\widehat{C}(H')$ with $\widehat{C}$ chosen, see also Sec. IV, so that $dB\wedge d\beta = d\widehat{p}\wedge d\widehat{q}$, i.e., as $\widehat{C}(z)^2 + z\frac{d\,\widehat{C}(z)^2}{dz} = D(z)$ or

$$\widehat{C}(H)^2 = \sum_{n=0}^\infty\frac{P_n(r^2)}{(n+1)}H^n. \tag{6.4}$$

For instance, up to $O(z^7)$:

$$\begin{aligned} &z\,\widehat{C}(z)^2 = z-\tfrac12(-1-r^2)z^2+\tfrac14(3+2r^2+3r^4)z^3\\ &-\frac{5}{16}(-1-r^2)(5-2r^2+5r^4)z^4\\ &+\frac{7}{64}(35+20r^2+18r^4+20r^6+35r^8)z^5\\ &-\frac{21}{128}(-1-r^2)(63-28r^2+58r^4-28r^6+63r^8)z^6\\ &+\frac{33}{256}(231+126r^2+105r^4+100r^6+105r^8+126r^{10}+231r^{12})z^7 \end{aligned} \tag{6.5}$$

and a normal form of the Hamiltonian is

$$U(\widehat{p},\widehat{q}) = \frac{A^2}{2I_2}+\frac{A^2r^2}{2J_{32}}\mathcal{H}(x,r^2), \tag{6.6}$$

where $\mathcal{H}(x, r^2)$ is 4 times the inverse function to $x = z\,\widehat{C}(z)^2$ and $x = \frac{\widehat{p}^2+\widehat{q}^2}{2Ar}$ (with $r = r_-$): this gives a normal form, because the variables $\widehat{p}, \widehat{q}$ are canonically conjugated to B, β. For instance (and in agreement with Eq. (5.6)):

$$\begin{aligned}\mathcal{H}(x) &= 4x - 2(1+r^2)x^2 - (-1+r^2)^2x^3 - \tfrac{5(1+r^2)(-1+r^2)^2}{4}x^4\\ &-\tfrac{3(-1+r^2)^2}{16}(11+10r^2+11r^4)x^5 - \tfrac{7(-1+r^2)^2(1+r^2)}{16}(9-2r^2+9r^4)x^6\\ &-\tfrac{(-1+r^2)^2}{64}(527+332r^2+330r^4+332r^6+527r^8)x^7\\ &-\tfrac{9(1+r^2)(-1+r^2)^2}{512}(1043-548r^2+1058r^4-548r^6+1043r^8)x^8+\dots\end{aligned} \tag{6.7}$$

It should be noted that the above analysis does not determine the canonical map into action-angle variables of the initial coordinates and, in particular, a further change of coordinates which transforms the plane $\widehat{p}, \widehat{q}$ into itself by rigidly rotating all circles centered at the origin by a radius dependent (and even r dependent) angle will lead to a normal form. For instance calling $\zeta = p + iq$ the change of variables $\zeta' = \zeta e^{i|\zeta|Q(r^2)}$ is a canonical map, for any choice of $Q(r^2)$, and it keeps the system in normal form.

Since the Hamiltonian is uniquely cast in normal form, the polynomials $P_n(r^2)$, coefficients of the expansion of $\mathcal{H}$ in powers of x, have an intrinsic significance. It is therefore interesting that they seem to enjoy the property of having all zeros on the unit circle: we have checked this for $1 < n < 15$: are really the *zeros on the unit circle for all n?*

The location on the unit circle of the roots of our polynomials is certainly related to the property that the polynomials

$$\overline{P}_n(r^2) \stackrel{def}{=} \sum_{h+k=n} \binom{-\frac{1}{2}}{h}\binom{-\frac{1}{2}}{k} r^{2h} \equiv \frac{1}{2^n}\sum_{h+k=n}\binom{2h}{h}\binom{2k}{k} r^{2h} \tag{6.8}$$

appearing in the above expression for $\widehat{C}^2$ have all the roots on the unit circle.

The latter is a special case of the general result on polynomials, that we call "polynomials with *symmetric, positive and monotonic coefficients,*" of the form $Q(z) = \sum_{h=0}^n F_h z^h$, with $F_h = F_{n-h}$ and $F_{h-1} > F_h > 0$ for $0 \le h \le [\frac{n+2}{2}]$:

Theorem (Ref. 16): *Polynomials with symmetric, positive and monotonic coefficients have all roots on the unit circle.*

The theorem can be found in Ref. 16, and it is described, for completeness in Appendix B.

This is analogous to the conjecture formulated after Eq. (5.4), and it is an algebraic property which could, we think, also be seen as follows. Consider the polynomials

$$Q(z^2) = z^n \sum_{\sigma_1,\dots,\sigma_n=\pm1} e^{\sum_{i,j} J_{ij}(\frac{\sigma_i+\sigma_j}{2})^2} z^{\sum_i \sigma_i} \tag{6.9}$$

with $J_{ij} > 0$. Then the coefficients Q_h of z^{2h} in Q have the form of a sum of $\binom{n}{h}$ exponentials of sums of some of the J_{ij}: the sums define quantities $a_h = a_{n-h} \stackrel{def}{=} e^{p(h)}$. By adjusting the constants J_{ij} the parameters $p(h-1) - p(h)$ for $0 < h \le [\frac{n+2}{2}]$ can independently take any value >0 so that $e^{p(h)-p(n)} = \frac{F_h}{F_n}$ can be achieved by suitable choices of the $J_{ij} > 0$. Therefore Q would have all zeros in the unit circle by Lee-Yang's theorem,[17] achieving a alternative proof of the above theorem.

The latter is a conjecture that can be checked rigorously for $n \le 5$ and we found some preliminary numerical evidence that it should also work at least for $n \le 8$. The proof for $n = 4$ emerged from a discussion with A. Giuliani; he also has a proof solving the case $n = 5$ and gave us the reference to Chen's theorem.

The polynomials in r^2 appearing as coefficients of x^k in the normal form Eq. (6.7) are sums of products of polynomials P_{n_j} of degrees which add up to $2(k-1) = \sum_j n_j$: an expression for the inverse function to $z\widehat{C}(z)^2$ can be found in terms of a "tree expansion" (quickly sketched in Appendix C).

Sums of products of symmetric polynomials with symmetrically decreasing positive coefficients is a polynomial with the same symmetry but it will not be, in general, decreasing. Hence the above theorem on the zeros of polynomials does not allow us any conclusion on the location of the zeros of the coefficients of the normal form: *however we see empirically that the zeros are apparently still on the unit circle* (up to $n = 15$ at least).

This is a special property of our polynomials P_n because, in general, the inverse function to the map $x = z + \sum_{n=1}^{\infty} P_n(w)z^{n+1}$ with $P_n(w)$ a degree n symmetric decreasing polynomial with positive coefficients (hence with all zeros on the unit circle) can be written $z = x + \sum_{n=1}^{\infty} Q_n(w)x^{n+1}$ but the Q_n *do not have*, in general, all zeros on the unit circle.

The analysis of the hyperbolic case is essentially identical: introducing the variable $X_\pm = B\sqrt{1 + r^2 \sin^2\beta} \pm Ar\sin\beta$ the form $dB \wedge d\beta$ is

$$\frac{dX_+ \wedge dX_-}{2Ar\sqrt{(1-\sin^2\beta)(1+r^2\sin^2\beta)}} = \frac{1}{2Ar}\sum_{n=0}^{\infty}\Big(\frac{-X_+X_-}{(2Ar)^2}\Big)^n P_n(-r^2) \tag{6.10}$$

and the normal form is related to the elliptic case and given by $U(p,q) = \frac{A^2}{2I_2} - \frac{xA^2r^2}{2J_{31}}\mathcal{H}(-x, -r^2)$. where $x = \frac{pq}{2Ar}$ and $r = r_+$.

Remarks: (1) The normal form in this section coincides with the one in Sec. V because of the normal form *uniqueness*. The agreement continues up to order 15: however the above analysis shows that this *must hold* to all orders: i.e., $\frac{H}{4A^2r_a^2}$ coincides with x_a and our construction in Secs. II–V determines explicitly one among the changes (*which are not unique*) of variables $(X, Y) \longleftrightarrow (X', Y')$ realizing the cohomological equivalence of the symplectic forms $dX' \wedge dY'$ and $dX \wedge dY$.

(2) It would be interesting to find a mechanical interpretation of the properties of the roots of the polynomials P_n.

VII. ACTION-ANGLES FOR THE "MOMENTUM" DEGREES OF FREEDOM (A, φ)

Consider the elliptic motions, i.e., rotations close to the proper rotation about the axis $\mathbf{i}_2$. Here we compute the second pair of canonical variables corresponding to the second degree of freedom and described canonically by (A, φ). Setting $\mathrm{sn}' \stackrel{def}{=} \mathrm{sn}(-iuh'^{-1}, h)$, $\mathrm{cn}' \stackrel{def}{=} \mathrm{cn}(-iuh'^{-1}, h)$, $\mathrm{dn}' \stackrel{def}{=} \mathrm{dn}(-iuh'^{-1}, h)$,

Equation (3.2) implies

$$\begin{aligned}\sin^2\beta &= h'^2\frac{\mathrm{sn}'^2/\mathrm{dn}'^2}{1 - s^2\,\mathrm{dn}'^2/\mathrm{cn}'^2} = \frac{1}{k^2}\frac{\mathrm{sn}'^2}{(1-h^2\,\mathrm{sn}'^2)(1-s^2\,\mathrm{dn}'^2/\mathrm{cn}'^2)}) \\ &= \frac{1}{k^2}\frac{\mathrm{sn}'^2(1-\mathrm{sn}'^2)}{(1-h^2\,\mathrm{sn}'^2)(1-\mathrm{sn}'^2 - s^2(1-s^2\,\mathrm{sn}'^2))} \\ &= \frac{1}{k^2}\frac{\mathrm{sn}'^2(1-\mathrm{sn}'^2)}{(1-h^2\,\mathrm{sn}'^2)(1-s^2)(1-\mathrm{sn}'^2)} = \frac{1}{k^2(1-s^2)}\frac{\mathrm{sn}'^2}{\mathrm{dn}'^2} = \frac{1}{r^2-s^2}\frac{\mathrm{sn}'^2}{\mathrm{dn}'^2}\end{aligned} \tag{7.1}$$

and this yields

$$\dot\varphi = \frac{A}{I_2} + \frac{A}{I_2}(\frac{I_2}{I_1} - 1)\sin^2\beta \tag{7.2}$$

therefore, Sec. 8.146.11 of Ref. 5, the expression of $\varphi(t)$ and of the uniformly rotating angle $\psi(t)$ with angular velocity $\omega_0(A, x)$ (with $x = x(x')^2 x'$) are

$$\begin{aligned}\psi &= \varphi - (\frac{g_{0,a}}{g})^2 \frac{1}{1-r^2} \sum_{\sigma,\sigma'=\pm 1} \sum_{n\sigma+m\sigma'\neq 0} (-1)^{n+m+1}\\ &\cdot\sigma\sigma' \frac{x'^{n+m-1}}{(1+x'^{2n-1})(1+x'^{2m-1})} e^{i(\sigma(2n-1)+\sigma'(2m-1))g_{0,a}t}\\ \omega_0(A,x') &= \frac{A}{I_2}\Big(1 + \frac{2(I_2 I_1^{-1}-1)}{1-r^2}\frac{g_{0,a}^2}{g^2}\sum_{n=1}^{\infty}\frac{x'^{2n-1}}{(1+x'^{2n-1})^2}\Big)\end{aligned} \tag{7.3}$$

with $n, m \geq 1$.

Since the generating function of the canonical map integrating the motions has the form $S(A', x, \beta) = A'\varphi + S_0(A', x, \beta)$, see Problem 4.11.5 of Ref. 6, it is $A' = A$ and the angle conjugated to A must differ by a constant from ψ and therefore it can be taken equal to ψ, i.e.:

$$\begin{aligned}\psi &= \varphi - (\frac{g_{0,a}}{g})^2 \frac{1}{1-r^2} \sum_{\sigma,\sigma'=\pm 1} \sum_{n\sigma+m\sigma'\neq 0} (-1)^{n+m+1}\\ &\cdot\sigma\sigma' \frac{z_\sigma^{2n-1} z_{\sigma'}^{2m-1}}{(1+x'^{2n-1})(1+x'^{2m-1})}\end{aligned} \tag{7.4}$$

with $z_\sigma = (p' + i\sigma q')$, $x' = p'^2 + q'^2$ and $p' = pa(x')^{-1}$, $q' = qa(x')^{-1}$.

This completes the construction of the elliptic canonical coordinates conjugated to (B, β, A, φ). The hyperbolic case, i.e., rotations close to the proper rotation around $\mathbf{i}_1$, is treated in the same way.

The case of the stable rotations close to the stable proper rotation around $\mathbf{i}_3$ is also interesting and could be treated in a similar way leading to a complete parametrization of the motion along the unstable manifolds emerging from the rotations around the axis $\mathbf{i}_3$ which correspond $A = B$. However the coordinates (B, β) have a singularity on such rotations (notice that the β coordinates disappear from H): hence new (B, β) coordinates have to be introduced. The simplest is to use the symmetry on the permutations of the inertia moments and make use of the equivalent Hamiltonian $H(B,\beta) = \frac{1}{2}\frac{B^2}{I_2} + (\frac{\sin^2\beta}{2I_3} + \frac{\cos^2\beta}{2I_1})(A^2 - B^2)$ in which the coordinates B, β have a suitably different meaning but can be used to describe the motions close to proper rotations around the axes $\mathbf{i}_3$ and $\mathbf{i}_1$ (respectively, stable and unstable). The analysis is then a repetition of the case $a = +$ with the Hamiltonian Eq. (2.2).

VIII. PENDULUM AND ELLIPSOID OF REVOLUTION

The method of Sec. VI can be applied to the case of the pendulum ($H = \frac{B^2}{2I} + Ig^2(1 - \cos\beta)$, Ref. 8) or to the geodesic motion on a revolution ellipsoid ($H = \frac{B^2}{2(b^2\sin^2\theta + a^2\cos^2\theta)} + \frac{A^2}{2a^2\sin^2\theta}$, Sec. 4.12.6 of Ref. 6).

The first case is considered in an attempt at proving the sign property of the normal form coefficients suggested in Ref. 8. In the second case it is of interest because the normal form has the form of a series with coefficients polynomials in the ratio $r^2 = \frac{a^2}{b^2}$ of the equatorial axis a to the polar axis b and in this case the zeros of the polynomials do not seem to have special properties (in particular they are not located on the unit circle).

The normal form at the stable equilibrium of the pendulum is easily evaluated (details are skipped) to be $8Ig^2\mathcal{H}(\frac{\tilde{p}^2+\tilde{q}^2}{16Ig})$ if $\mathcal{H}(x)$ is the inverse function to $x = z\widetilde{C}(z)^2$ with

$$\widetilde{C}^2(z) \stackrel{def}{=} \sum_{n=0}^{\infty} \binom{2n}{n}\binom{-\frac{1}{2}}{n}\frac{(-z)^n}{n+1} \equiv \sum_{n=0}^{\infty}\binom{2n}{n}^2 \frac{z^n}{4^n(n+1)}, \tag{8.1}$$

(which is $F(\frac{1}{2},\frac{1}{2};2;4z)$, see Eq. 9.100 of Ref. 5). And to order $O(x^8)$ the $\mathcal{H}$ is

$$\mathcal{H}(x)=x-\frac{1}{2}x^2-\frac{1}{4}x^3-\frac{5}{16}x^4-\frac{33}{64}x^5-\frac{63}{64}x^6-\frac{527}{256}x^7-\frac{9387}{2048}x^8-\dots \tag{8.2}$$

This is consistent with the first few terms computed in Sec C.8 of Ref. 8: due to different meaning of the symbols the relation between the function $\mathcal{H}(x)$ above and $W(x)$ in the quoted reference is $W(x)=\frac{1}{4}\mathcal{H}(4x)$. However constructing $\mathcal{H}(x)$ via the algorithm in Appendix C the sign property suggested in Ref. 8 that $\mathcal{H}(x)$ might have all coefficients, but the first, negative does not follow; see the final comments in Appendix C.

In the case of the geodesic motion near the equatorial circle of the ellipsoid let a, b be the equatorial and polar radii of an ellipsoid of revolution. The Hamiltonian for the geodesic motion is

$$H=\frac{1}{2}\frac{B^2}{b^2\cos^2\beta+a^2\sin^2\beta}+\frac{1}{2}\frac{A^2}{a^2\cos^2\beta}, \tag{8.3}$$

where A is the polar component of the angular momentum and $\beta=\theta-\frac{\pi}{2}$ the polar angle, Problem 4.11.5 of Ref. 6. Let $r^2\stackrel{def}{=}\frac{a^2}{b^2}$, $\varepsilon\stackrel{def}{=}r^2-1$, $X=\frac{B}{b\sqrt{1+\varepsilon\sin^2\beta}}$, $Y=-\frac{A}{a}\tan\beta$, $X_\pm=X\pm iY$; then $H=\frac{1}{2}(X^2+Y^2)+\frac{A^2}{2a^2}$ and:

$$\begin{aligned}dB\wedge d\beta&=\frac{ab}{A}\frac{\sqrt{1+\varepsilon\sin^2\beta}}{1+\tan^2\beta}dX\wedge dY\\&=\frac{ab}{A}\frac{(1+r^2\tan^2\beta)^{\frac{1}{2}}}{(1+\tan^2\beta)^{\frac{3}{2}}}dX\wedge dY=\frac{ab}{A}\frac{(1+r^2(\frac{Ya}{A})^2)^{\frac{1}{2}}}{(1+(\frac{Ya}{A})^2)^{\frac{3}{2}}}dX\wedge dY\\&=\frac{ab}{A}\sum_{h,k}\binom{\frac{1}{2}}{h}\binom{-\frac{3}{2}}{k}r^{2h}(\frac{(X_+-X_-)\,a}{2iA})^{2(h+k)}dX\wedge dY\\&=\frac{ab}{A}\sum_{n=0}^{\infty}\Big(\frac{X_+X_-a^2}{4A^2}\Big)^n\binom{2n}{n}\sum_{h+k=n}\binom{\frac{1}{2}}{h}\binom{-\frac{3}{2}}{k}r^{2h}dX\wedge dY+\mathbf{0}.\end{aligned} \tag{8.4}$$

Therefore the normal form is $H=\frac{A^2}{2a^2}+\frac{2A^2}{a^2}\mathcal{H}(x)$ with $x=\frac{a}{4Ab}(p^2+q^2)$ and $\mathcal{H}(x)$ the inverse function to $x=zC^2(z)$ with $C^2(z)\stackrel{def}{=}\sum_{n=0}^{\infty}P_n(r^2)z^n$ and

$$P_n(r^2)\stackrel{def}{=}\frac{1}{n+1}\binom{2n}{n}\sum_{h+k=n}\binom{\frac{1}{2}}{h}\binom{-\frac{3}{2}}{k}r^{2h}, \tag{8.5}$$

which is interesting as it shows that the location of the zeros on the unit circle is a peculiarity of the particular case of the rigid body.

Nevertheless a numerical investigation for $n\le 1000$ with an algebraic manipulator indicates, as a reasonable conjecture, that the zeros become asymptotically located on the unit circle: for n large they are within an annulus of radii $r_n'^2<r_n''^2$ with

$$\begin{cases}r'^2\ge_{n\to\infty}1+\frac{a'}{n^{c'}}\\ r''^2\le_{n\to\infty}1+\frac{a''}{n^{c''}}\end{cases},\qquad c'\simeq 1,\,c''\simeq .88 \tag{8.6}$$

with $\log a''\simeq 1.992$, $c''\simeq 0.880$, $\log a'\simeq 0.462$, $c'\simeq 1.000$.

ACKNOWLEDGMENTS

We are indebted to G. Gentile, A. Giuliani for discussions and to A. Giuliani for the suggestions quoted in Sec. VI.

APPENDIX A: HYPERBOLIC AND ELLIPTIC COORDINATES: DETAILS

In this appendix we give details for the derivation of the formulae in Sec. III.

Let $\sqrt{-1} = -i$ so that $g_a = \sqrt{a}|g_a|$; for $a = +$ Eq. (2.3) describe the separatrix branches emerging from the proper rotation around the axis $\mathbf{i}_1$ and for $a = -$ they describe the motion near the stable rotation around the axis $\mathbf{i}_2$. For $U - A^2/2J_a > 0$ close to 0 and from the definitions of the Jacobi elliptic integrals, see formulae 2.616 of Ref. 5, it follows (choosing the sign $+$ in the Eq. (2.5), for instance):

$$\frac{\sqrt{1+as_a^2}\ \sin\beta(t)}{\sqrt{1+as_a^2\ \sin^2\beta(t)}} = \mathrm{sn}(g_a t, k_a) \longleftrightarrow\ \sin\beta(t) = \frac{\mathrm{sn}(g_a t, k_a)}{\sqrt{1+as_a^2\mathrm{cn}^2(g_a t, k_a)}} \tag{A1}$$

and define h_a, h'_a in terms of k_a, k'_a so that $k_+, k'_+, h_-, h'_- \in (0, 1)$ as

$$h'^2_a = \frac{1}{k_a^2}, \qquad h_a^2 = 1 - \frac{1}{k_a^2} \tag{A2}$$

Then (Sec. 8.153.3 of Ref. 5 for $a = +$ and Sec. 8.153.1-8 of Ref. 5 for $a = -$ after correcting the obvious typo in Sec. 8.153.8 of Ref. 5 and by $k'_- = -i\sqrt{k_-^2 - 1}$):

$$\begin{aligned}
B(t) &= \frac{b_a}{\mathrm{dn}(u_a, k_a)} = b_a \frac{\mathrm{cn}(-iu_a, k'_a)}{\mathrm{dn}(-iu_a, k'_a)} = b_a \mathrm{cn}(-iu_a h'^{-1}_a, h_a),\\
\sin^2\beta(t) &= \frac{\mathrm{sn}^2(u_a, k_a)}{1+as_a^2\mathrm{cn}^2(u_a, k_a)} = \frac{-\frac{\mathrm{sn}^2(-iu_a, k'_a)}{\mathrm{cn}^2(-iu_a, k'_a)}}{1+as_a^2\frac{1}{\mathrm{cn}^2(-iu_a, k'_a)}},\\
\dot\beta(t) &= \frac{(1+as_a^2)\overline{g}_a\mathrm{dn}(u_a, k_a)}{1+as_a^2\mathrm{cn}^2(u_a, k_a)} = \overline{g}_a\frac{\mathrm{dn}(-iu_a, k'_a)\mathrm{cn}(-iu_a, k'_a)}{1-\frac{1}{1+as_a^2}\mathrm{sn}^2(-iu_a, k'_a)},
\end{aligned} \tag{A3}$$

where the last relation is derived from the equations of motion and the previous expression for $\sin^2\beta$. Also the second and third relations in Eq. (A3) could be given expressions involving the moduli h_a, h'_a so that one could choose to use the formulae with k_a for $a = +$ and with h_a for $a = -$.

Remark also that, Secs. 8.128.1 and 8.128.3 of Ref. 5, $\mathbf{K}(k') = h'\mathbf{K}(h)$, and also that $\frac{h'_-|k'_-|}{h_-} = 1$.

Notice that $\beta(0) = 0$ and if $(2U - A^2/J_a) \to 0^+$ it is $k_\pm^2 \to 1^\mp$, $a\, g_a^2 \to \widetilde{J}_a^{-1} J_{12}^{-1} A^2$, $b_a \to 0^+$ and motions with this energy are "like" the motions close to the separatrix or, respectively, close to equilibrium of a pendulum.

For $(2U - A^2/J_a) \le 0$, or for $k_a > 1$ hence k'_a imaginary, all the above relations have to be interpreted via suitable analytic continuations. Some of the following formulae become singular as $U \to A^2/2J_a$, but the singularity is only apparent and it will disappear from all relevant formulae derived or used in the following.

Denoting $\mathrm{cn} = \mathrm{cn}(u_a, k_a)$, $\mathrm{cn}' = \mathrm{cn}(-iu_a, k'_a)$, $\mathrm{dn}' = \mathrm{dn}(-iu_a, k'_a)$, $\mathrm{sn}' \stackrel{def}{=} \mathrm{sn}(-iu_a, k'_a)$, from the equation of motion (and $\mathrm{dn}^2\ \cdot \equiv 1 - k_a^2\mathrm{sn}^2\cdot$) it follows

$$\dot\beta = \frac{\overline{g}_a}{\mathrm{sn}(-iv, k'_a)}\frac{\mathrm{sn}(-iv, k'_a)\mathrm{dn}(-iu_a, k'_a)\mathrm{cn}(-iu, k'_a)}{1-\frac{1}{1+as_a^2}\mathrm{sn}^2(-iu_a, k'_a)} \tag{A4}$$

identically for all v: convenient choice of v is $v = v_a$ with $k'^2_a\mathrm{sn}^2(-iv, k'_a) = (1 + as_a^2)^{-1}$, i.e., $\mathrm{sn}(-iv_a, k'_a) = \frac{1}{\sqrt{1+ar^2}}$ and $-iv_a \stackrel{def}{=} \mathrm{am}(\arcsin\frac{1}{\sqrt{1+ar_a^2}}, k'_a)$, then (notice that $g_a^2 k'^2_a = \overline{g}_a(1 + ar_a^2)$):

$$\dot\beta = \tfrac{1}{2}\overline{g}_a(1+ar^2)^{\frac{1}{2}}\sum_{\eta=\pm1}\eta\ \mathrm{sn}(-i(u_a+\eta v_a), k'_a) \tag{A5}$$

which follows by the addition formulae, Sec. 8.156.1 of Ref. 5, from the last of Eq. (2.5).

If $\alpha_a \stackrel{def}{=} \arcsin\frac{1}{\sqrt{1+ar_a^2}}$, i.e.,$e^{i\alpha_a} = \frac{i+\sqrt{ar_a^2}}{\sqrt{1+ar_a^2}}$, it is $-i v_a = \int_0^{\alpha_a} \frac{d\theta}{\sqrt{1-k_a'^2 \sin^2\theta}}$ and the integral can be evaluated by series, 2.511.2 and 8.113.1 of Ref. 5, leading to Eqs. (3.4) and (3.5). As remarked in Sec. III $|\gamma_+| = 1, \gamma_- = ie^{\lambda}$ with λ real and $\overline{\gamma}_-$ real, with λ real.

It is useful to check that if $I = \sqrt{4\widetilde{J}_+\widetilde{J}_-}$:

$$g_{0,a} = \frac{\pi|g_a|}{2\mathbf{K}(k_a')}, \qquad \frac{2\pi b_a}{k_a'\mathbf{K}(k_a')} = 2I\frac{g_{0,a}}{\sqrt{a}}, \qquad \frac{\pi\overline{g}_a(1+r_a^2)^{\frac{1}{2}}}{2|k_a'|\mathbf{K}(k_a')} = g_{0,a}, \tag{A6}$$

From Eq. (2.6), via Secs. 8.146.1 and 8.146.2 of Ref. 5 and $g_{0,\pm}$ in Eq. (2.4), we get

$$\begin{aligned}\dot\beta_a &= \frac{g_{0,a}}{\sqrt{a}}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\sum_{\sigma,\eta=\pm1}\Big(\frac{\eta\sigma}{i}\Big)\Big(\sqrt{x_a'}(ax_a')^k\frac{\sqrt{a}}{\sqrt{a}^{\eta\sigma}}e^{\sigma\sqrt{a}\,g_{0,a}t}\overline{\gamma}_a^{\eta\sigma}\Big)^{2n-1}\\ &= \frac{d}{dt}\frac{1}{a}\sum_{k=1}^{\infty}\sum_{\sigma,\eta=\pm1}\frac{\eta^{\frac{1+a}{2}}\sigma}{i}\operatorname{arctanh}(p_\sigma(ax_a')^k\overline{\gamma}_a^{\eta}) \qquad \text{if}\\ p_\sigma' &\stackrel{def}{=} \sqrt{x_a'}e^{\sigma\sqrt{a}g_{0,a}t}, \qquad x_a' \equiv p_+'p_-'\end{aligned} \tag{A7}$$

Therefore this implies:

$$\beta(t)_a = a\sum_{k=0}^{\infty}\sum_{\sigma,\eta=\pm1}\frac{\eta^{\frac{1+a}{2}}\sigma}{i}\,a^k\ \operatorname{arctanh}(x_a'^k p_\sigma'\overline{\gamma}_a^{\eta}) \tag{A8}$$

i.e., the second of Eq. (3.7). Likewise from the first of Eq. (2.3) we derive:

$$\begin{aligned}B_a &= I\frac{g_{0,a}}{\sqrt{a}}\sum_{\mu=\pm1}\sum_{n=1,k=0}^{\infty}(-1)^{k-1}\Big((\sqrt{ax_a'})\,(ax_a')^k e^{\sigma g_{0,a}\sqrt{a}t}\Big)^{2n-1}\\ &= -aIg_{0,a}\sum_{\sigma=\pm}\sum_{m=0}^{\infty}\frac{p_\sigma'(ax_a')^m}{1+a(p_\sigma'x_a'^m)^2},\end{aligned} \tag{A9}$$

via Sec. 8.146.11 of Ref. 5, hence we obtain the first of Eq. (3.7).

Remark: via identities relating Jacobian elliptic functions it would be possible to avoid considering functions that are defined by analytic continuations, for instance when the modulus $k_a^2 > 1$ or $k_a'^2 < 0$, and reduce instead to considering only cases with the "simple" arguments (expressing functions by elliptic functions of the arguments h, h', see for instance the first of Eq. (A3), when necessary): we avoid this as it would hide the nice property that the stable and unstable cases are in a sense analytic continuation of each other.

Summarizing the transformation of coordinates $(B,\beta)\longleftrightarrow(p_+', p_-')$ in Eq. (3.7) is such that $t \to p_\sigma' e^{\sigma\sqrt{a}\,g_{0,a}t}$ generates a solution of the equations of motion (if $p_\pm' > 0$).

APPENDIX B: POLYNOMIALS ROOTS

The proof of the theorem is in the literature: first remark (a classical result, see pp. 107 and 301 of Ref. 18) that if $a_0 \ge a_1 \ge \ldots \ge a_m \ge 0$, and $Q_m(z) = \sum_{k=0}^m a_k z^k \not\equiv 0$, then $Q_m(z)$ can only vanish if $|z| \ge 1$. Because

$$\begin{aligned}|(1-z)\ Q_m(z)| &= |a_0 + (a_1-a_0)z + \ldots + (a_n - a_{n-1})z^n + a_n z^{n+1}|\\ &\ge a_0 - (a_0-a_1)|z| - \ldots - (a_{n-1}-a_n)z^n - a_n|z|^{n+1}\\ &> a_0 - (a_0-a_1) - \ldots - (a_{n-1}-a_n) - a_n = 0.\end{aligned} \tag{B1}$$

Following word by word the proof in Ref. 16, write the polynomials $P_n(z), n \geq 1$

$$a_0 + a_1 z + \ldots a_n z^n + a_n z^{n+1} + \ldots + a_0 z^{2n+1}, \qquad \text{or} \tag{B2}$$
$$a_0 + a_1 z + \ldots a_{n-1} z^{n-1} + \frac{1}{2} a_n z^n + \frac{1}{2} a_n z^n + a_{n-1} z^{n+1} + \ldots + a_0 z^{2n}$$

as $Q_n(z) + z^{n^*} Q_n^*(z)$ with $n^* = n + 1$ or $n^* = n$, respectively, and

$$Q_n(z) = \sum_{k=0}^{n} \alpha_k z^k, \qquad Q_n^*(z) \stackrel{def}{=} z^n \overline{Q_n(\overline{z}^{-1})} \equiv z^n Q_n(z^{-1}), \tag{B3}$$

with $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_n$ (with $\alpha_k = a_k$ for $k < n$ and $\alpha_n = a_n$ or $\alpha_n = \frac{1}{2} a_n$).

Hence $Q_n(z) = a_0 \prod (z - \frac{1}{z_k})$ with $|z_k| \leq 1$ by the above remark, and $Q_n^*(z) = a_0 z^n \prod (\frac{1}{z} - \frac{1}{\overline{z}_k})$.

If ζ is a zero for P_n it is $Q_n(\zeta) = -\zeta^{n^*} Q_n^*(\zeta)$ or

$$\frac{|\zeta|^{n^*} \prod |1 - \zeta \overline{z}_k^{-1}|}{\prod |\zeta - z_k^{-1}|} = \frac{|\zeta|^{n^*} \prod |\zeta - \overline{z}_k|}{\prod |1 - \zeta z_k|} = 1, \tag{B4}$$

possible only if $|\zeta| = 1$: because if $|a| < 1$ then it is $|\frac{z-a}{1-\overline{a}z}| > 1$ for $|z| > 1$ and <1 for $|z| < 1$ (and $n^* > 0$); or if $|z_k| \equiv 1$ Eq. (B4) means $|\zeta|^{n^*} = 1$.

APPENDIX C: TREE EXPANSION

To invert the relation $x = z\,\widehat{C}(z)^2$ let $f(z) \stackrel{def}{=} z\,\widehat{C}(z)^2 - z = \frac{1+r^2}{2} z^2 + \ldots \equiv \sum_{n=2}^{\infty} f_n z^n$, see Eq. (6.4), with $f_n = P_{n-1}$ a polynomial in r^2 of degree $n - 1$. Consider the equation

$$z = x - f(z), \text{ with } z = x + h(x), \quad \text{or} \quad h(x) = -f(x + h(x)) \tag{C1}$$

and look for a solution $h(x) = \sum_{k=2}^{\infty} h_k x^k$.

Let θ be a tree graph with a root and nodes v into each of which merge $s_v = 0, 1, 2, 3, 4, \ldots$ branches oriented towards the root; each node v carries a label $n_v = 2, \ldots$ with the restriction $\sum_v s_v = k$ unless v is an end node in which case $n_v = 1$ and $s_v = 0$.

Such labeled tree graphs will form a family $\Theta(k)$ of trees identified by the rule that two trees are regarded as identical if they can be overlapped by pivoting the branches avoiding that any two branches overlap in the course of the process.

Define the "value," $\mathrm{Val}(\theta)$, of θ as $\mathrm{Val}(\theta) = \sigma(\theta) \prod_{v \in \theta} \binom{n_v}{s_v} f_{n_v}$ with $\sigma(\theta) \stackrel{def}{=} \prod_{v, n_v > 1} (-1)$ and $f_1 = 1$. Notice that if he coefficients defining f_n, $n \geq 2$, are all *negative* then $\mathrm{Val}(\theta) \geq 0$ for all θ. Then

$$h(x) = \sum_{k=1}^{\infty} \sum_{\theta \in \Theta(k)} \mathrm{Val}(\theta) x^k \tag{C2}$$

as it is checked by induction (for instance).

In our case $f_n = P_{n-1}(r^2)$ and the contributions of the value of a tree θ to the coefficient of x^k is a product of symmetric polynomials $P_{n_i}(r^2) = \binom{2n_i}{n_i} \sum_{h=0}^{n_i} \binom{2h}{h} \binom{2(n_i - h)}{(n_i - h)} r^{2h}$ with symmetrically decreasing coefficients and of total degree $p - 1$ in r^2; consequently the polynomials in r^2 being sums of products of polynomials enjoying the $z \longleftrightarrow z^{-1}$ symmetry property do enjoy it as well. However positivity and monotony is in general lost.

Notice that Eqs. (6.7) and (8.2) suggest that the coefficients of x^n with $n > 1$ are negative polynomials $p_n(r^2)$: in fact $-p_n(x)$ are either squares of $(1 - r^2)$ times polynomials with positive coefficients for n odd or, for n even, $(1 - r^2)^2$ times $(1 + r^2)\widetilde{p}_n(r^2)$ which after performing the multiplication become a polynomial with positive coefficients. This property, however, does not follow from the above tree expansion and we wonder whether it continues to be true at the higher orders.

[1] C. Jacobi, "Sur la rotation d'un corps," Gesammelte Werke **2**, 291–352 (1849).
[2] E. T. Whittaker, *A Treatise on the Analytic Dynamics of Particles & Rigid Bodies* (Cambridge University Press, Cambridge, 1917).
[3] V. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer, Berlin, 1989).
[4] Yu. Sadov, "The action-angles variables in the Euler-Poinsot problem," J. Appl. Math. Mech. **34**, 922–925 (1970).
[5] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).
[6] G. Gallavotti, *The Elements of Mechanics* (Springer-Verlag, Berlin, 1983).
[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).
[8] J. P. Françoise, P. Garrido, and G. Gallavotti, "Pendulum, elliptic functions and relative cohomology classes," J. Math. Phys. **51**, 032901 (2010).
[9] J. P. Françoise, "Successive derivatives of a first return map, application to the study of quadratic vector fields," Ergod. Theory Dyn. Syst. **16**, 87–96 (1996).
[10] J. P. Françoise, "The successive derivatives of the period function of a plane vector field," J. Differ. Equations **146**, 320–335 (1998).
[11] G. Gallavotti and C. Marchioro, "On the calculation of an integral," J. Math. Anal. Appl. **44**, 661–675 (1973).
[12] J. P. Françoise, "Canonical partition functions of Hamiltonian systems and the stationary phase formula," Commun. Math. Phys. **117**, 37–47 (1988).
[13] J. P. Françoise, "Modèle local simultané d'une fonction et d'une forme de volume," Asterisque **59-60**, 119–130 (1978).
[14] V. Guillemin, "Band asymptotics in two dimensions," Adv. Math. **42**, 248–282 (1981).
[15] J. P. Françoise, "Relative cohomology and volume forms," Banach Cent. Publ. **20**, 207–222 (1988).
[16] K. Chen, "On the polynomials with all their zeros on the unit circle," J. Math. Anal. Appl. **190**, 714–724 (1995).
[17] D. Ruelle, "Extension of the Lee-Yang theorem," Phys. Rev. Lett. **26**, 303–304 (1971).
[18] G. Polya and G. Szegö, *Problems and Theorems in Analysis, I*, Classics in Mathematics (Springer, Berlin, 1978).