Large deviations of the current in a two-dimensional diffusive system

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Abstract. In this notes we study the large deviations of the time-averaged current in the two-dimensional (2D) Kipnis-Marchioro-Presutti model of energy transport when subject to a boundary gradient. We use the tools of hydrodynamic fluctuation theory, supplemented with an appropriate generalization of the additivity principle. As compared to its one-dimensional counterpart, which amounts to assume that the optimal profiles responsible of a given current fluctuation are time-independent, the 2D additivity conjecture requires an extra assumption, i.e. that the optimal, divergence-free current vector field associated to a given fluctuation of the time-averaged current is in fact constant across the system. Within this context we show that the current distribution exhibits in general non-Gaussian tails. The ensuing optimal density profile can be either monotone for small current fluctuations, or non-monotone with a single maximum for large enough current deviations. Furthermore, this optimal profile remains invariant under arbitrary rotations of the current vector, providing a detailed example of the recently introduced Isometric Fluctuation Relation.

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INTRODUCTION

Large deviation functions measure the rate at which the empiric average of an observable converges toward its asymptotic value. Think for instance on the time-averaged current in a d-dimensional mesoscopic conductor of length L. As time increases, and provided that the system is ergodic, the time-averaged current $\mathbf{J} = \tau^{-1}\int_0^\tau \mathbf{j}(t)\,dt$ quickly converges toward its ensemble average $\langle\mathbf{J}\rangle$. For finite times, the measured $\mathbf{J}$ may fluctuate and the probability of a given output follows in general a large-deviation principle [1] for long times, $\mathrm{P}_L(\mathbf{J},\tau) \sim \exp[+\tau L^d G(\mathbf{J})]$. Here $G(\mathbf{J})$ is the current large deviation function (LDF), and measures the (exponential) rate at which $\mathbf{J} \to \langle\mathbf{J}\rangle$ as τ increases (notice that $G(\mathbf{J}) \leq 0$, with $G(\langle\mathbf{J}\rangle) = 0$).

Large deviation functions akin to $G(\mathbf{J})$ play an important role in statistical physics [2]. For instance, the LDF of the density profile in equilibrium systems can be simply related with the free-energy functional, a central object in the theory [1, 2, 3]. In a similar way, we can use LDFs in systems far from equilibrium to define the nonequilibrium analog of the free-energy functional. This top-down approach is specially appealing in a nonequilibrium context, as we don't know in this case the statistical weight of microscopic configurations, equivalent to the equilibrium Boltzmann-Gibbs measure, from which to build up a nonequilibrium *partition function*. In any case, LDFs out of equilibrium may be non-local and/or non-convex [3, 4], reflecting in this way the pathologies associated to nonequilibrium behavior.

Computing LDFs from scratch, starting from microscopic dynamics, is in general an extraordinary difficult task which has been successfully accomplished only for a handfull of simple stochastic lattice gases. However, in a series of recent works [4], Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim have introduced a hydrodynamic fluctuation theory (HFT) which describes large dynamic fluctuations in diffusive systems and the associated LDFs starting from a macroscopic rescaled description of the system of interest, where the only inputs are the system transport coefficients. This is a very general approach which leads however to a hard variational problem whose solution remains challenging in most cases. However, if supplemented with a recently-introduced conjecture named additivity principle [5], HFT can be readily applied to obtain explicit predictions, opening the door to a systematic way of computing LDFs in nonequilibrium systems.

In this paper we apply this program to study the statistics of current fluctuations in a simple but very general model of diffusive energy transport in two dimensions, namely the 2D Kipnis-Marchioro-Presutti (KMP) model [6].

HFT FOR CURRENT FLUCTUATIONS

We start from a rescaled ($\mathbf{r} \to \mathbf{r}/L$, $t \to t/L^2$) continuity equation

$$\partial_t \rho(\mathbf{r},t) = -\boldsymbol{\nabla} \cdot \Big(\mathbf{Q}[\rho(\mathbf{r},t)] + \boldsymbol{\xi}(\mathbf{r},t) \Big), \tag{1}$$

which describes a wide class of d-dimensional systems characterized by a single locally-conserved field $\rho(\mathbf{r},t)$, representing a density of e.g. energy, particles, momentum, charge, etc. Here $\mathbf{j}(\mathbf{r},t) \equiv -D[\rho]\boldsymbol{\nabla}[\rho(\mathbf{r},t)] + \boldsymbol{\xi}(\mathbf{r},t)$ is a fluctuating current, with local average $\mathbf{Q}[\rho(\mathbf{r},t)]$, and $\mathbf{r} \in \{0,1\}^d$. In particular, we focus in this paper on diffusive systems, for which the current obeys Fick's (or equivalently Fourier's) law, $\mathbf{Q}[\rho(\mathbf{r},t)] = -D[\rho]\boldsymbol{\nabla}\rho(\mathbf{r},t)$, where $D[\rho]$ is the diffusivity (a functional of the density profile in general). The (conserved) noise vector term $\boldsymbol{\xi}(\mathbf{r},t)$, which accounts for microscopic random fluctuations at the macroscopic level, is Gaussian and white with $\langle \boldsymbol{\xi}(\mathbf{r},t) \rangle = 0$ and $\langle \xi_\alpha(\mathbf{r},t)\xi_\beta(\mathbf{r}',t') \rangle = 2L^{-d}\sigma[\rho]\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, being $\sigma[\rho]$ the mobility functional and L the system linear size. This noise source represents the many fast microscopic degrees of freedom which are averaged out in the coarse-graining procedure resulting in eq. (1), and whose net effect on the macroscopic evolution amounts to a Gaussian random perturbation according to the central limit theorem. The above equation must be supplemented with appropriate boundary conditions, which in this work we choose to be gradient-like in the $\hat{x}$-direction, with fixed densities ρ_L and ρ_R at the left and right reservoirs, respectively, and periodic boundary conditions in all other directions.

We are interested in the probability $\mathrm{P}_L(\mathbf{J},\tau)$ of observing a space and time averaged current $\mathbf{J} = \tau^{-1}\int_0^\tau dt \int d\mathbf{r}\, \mathbf{j}(\mathbf{r},t)$. For long times [1, 2, 3, 4] $\mathrm{P}_L(\mathbf{J},\tau) \sim \exp[+\tau L^d G(\mathbf{J})]$, and we aim at computing the current LDF $G(\mathbf{J})$ starting from eq. (1) within a path integral formalism. In particular, the probability of observing a history $\{\rho(\mathbf{r},t),\mathbf{j}(\mathbf{r},t)\}_0^\tau$ of duration τ for the density and current fields, starting from a given initial state, can be written as a path integral over all possible noise realizations $\{\boldsymbol{\xi}(\mathbf{r},t)\}_0^\tau$

$$\mathrm{P}(\{\rho,\mathbf{j}\}_0^\tau) = \int \mathscr{D}\boldsymbol{\xi}\, \exp\left[-L^d \int_0^\tau dt \int d\mathbf{r} \frac{\boldsymbol{\xi}^2}{2\sigma[\rho]}\right] \prod_t \prod_{\mathbf{r}} \delta\left[\boldsymbol{\xi} - (\mathbf{j} + D[\rho]\boldsymbol{\nabla}\rho)\right], \quad (2)$$

with $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ coupled via the continuity equation, $\partial_t \rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0$. Notice that this coupling does not determine univocally the relation between ρ and $\mathbf{j}$. For instance, the fields $\rho'(\mathbf{r},t) = \rho(\mathbf{r},t) + \chi(\mathbf{r})$ and $\mathbf{j}'(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) + \mathbf{g}(\mathbf{r},t)$, with $\chi(\mathbf{r})$ arbitrary and $\mathbf{g}(\mathbf{r},t)$ divergenceless, satisfy the same continuity equation. This freedom can be traced back to the loss of information during the coarse-graining from the microscale to the macroscale [4]. Eq. (2) naturally leads to $\mathrm{P}\left(\{\rho,\mathbf{j}\}_0^\tau\right) \sim \exp\left(+L^d I_\tau[\rho,\mathbf{j}]\right)$, with

$$I_\tau[\rho,\mathbf{j}] = -\int_0^\tau dt \int d\mathbf{r} \frac{(\mathbf{j}(\mathbf{r},t) + D[\rho]\boldsymbol{\nabla}\rho(\mathbf{r},t))^2}{2\sigma[\rho]}. \quad (3)$$

The probability $\mathrm{P}_L(\mathbf{J},\tau)$ of observing an averaged current $\mathbf{J}$ can be written now as

$$\mathrm{P}_L(\mathbf{J},\tau) \sim \int^* \mathscr{D}\rho \mathscr{D}\mathbf{j}\, \mathrm{P}(\{\rho,\mathbf{j}\}_0^\tau)\, \delta\left(\mathbf{J} - \tau^{-1}\int_0^\tau dt \int d\mathbf{r}\,\mathbf{j}(\mathbf{r},t)\right),$$

where the asterisk means that this path integral is restricted to histories $\{\rho,\mathbf{j}\}_0^\tau$ coupled via the continuity equation. As the exponent of $\mathrm{P}\left(\{\rho,\mathbf{j}\}_0^\tau\right)$ is extensive in both τ and L^d, see above, for long times and large system sizes the above path integral is dominated by the associated saddle point, resulting in the following current LDF

$$G(\mathbf{J}) = \tau^{-1} \max_{\rho,\mathbf{j}} I_\tau[\rho,\mathbf{j}], \quad (4)$$

with the constraints $\mathbf{J} = \tau^{-1}\int_0^\tau dt \int d\mathbf{r}\,\mathbf{j}(\mathbf{r},t)$ and $\partial_t\rho + \boldsymbol{\nabla}\cdot\mathbf{j} = 0$. The optimal density and current fields solution of this variational problem, denoted here as $\rho_{\mathbf{J}}(\mathbf{r},t)$ and $\mathbf{j}_{\mathbf{J}}(\mathbf{r},t)$, can be interpreted as the optimal path the system follows in order to sustain a long-time current fluctuation $\mathbf{J}$. It is worth emphasizing here that the existence of an optimal path rests on the presence of a selection principle at play, namely a long time, large size limit which selects, among all possible paths compatible with a given fluctuation, an optimal one via a saddle point mechanism.

The variational problem posed in eq. (4) is a complex spatiotemporal problem whose solution remains challenging in most cases. In order to proceed, we now make the following hypotheses:

1. We assume that the optimal profiles responsible of a given current fluctuation are time-independent, $\rho_{\mathbf{J}}(\mathbf{r})$ and $\mathbf{j}_{\mathbf{J}}(\mathbf{r})$. This, together with the continuity equation, implies that the optimal current vector field is divergence-free, $\boldsymbol{\nabla}\cdot\mathbf{j}_{\mathbf{J}}(\mathbf{r}) = 0$.
2. A further simplification consists in assuming that this optimal current field is in fact constant across space, so $\mathbf{j}_{\mathbf{J}}(\mathbf{r}) = \mathbf{J}$.

Provided that these hypotheses hold, the current LDF can be written as

$$G(\mathbf{J}) = -\min_{\rho(\mathbf{r})} \int \frac{(\mathbf{J} + D[\rho(\mathbf{r})]\boldsymbol{\nabla}\rho(\mathbf{r}))^2}{2\sigma[\rho(\mathbf{r})]} d\mathbf{r}, \quad (5)$$

which expresses the *locally*-Gaussian nature of fluctuations [3, 11]. In this way the probability $\mathrm{P}_L(\mathbf{J},\tau)$ is simply the Gaussian weight associated to the optimal density profile responsible of such fluctuation. Note however that the minimization procedure gives rise to a nonlinear problem which results in general in a current distribution with non-Gaussian tails [3, 4, 7]. As opposed to the general problem in eq. (4), its simplified version, eq. (5), can be readily used to obtain quantitative predictions for the current statistics in a large variety of nonequilibrium systems.

The minimization of the functional in eq. (5) leads to the following differential equation for the optimal profile $\rho_{\mathbf{J}}(\mathbf{r})$

$$D^2[\rho_{\mathbf{J}}]\big(\boldsymbol{\nabla}\rho_{\mathbf{J}}\big)^2 = \mathbf{J}^2\big\{1+2\sigma[\rho_{\mathbf{J}}]K(\mathbf{J}^2)\big\}\,, \tag{6}$$

where $K(\mathbf{J}^2)$ is a constant which guarantees the correct boundary conditions for $\rho_{\mathbf{J}}(\mathbf{r})$. Remarkably, the optimal profile solution of eq. (6) depends exclusively on the magnitude of the current vector, via $\mathbf{J}^2$, not on its orientation, i.e. $\rho_{\mathbf{J}}(\mathbf{r}) = \rho_{|\mathbf{J}|}(\mathbf{r})$, as demanded for time-reversible systems by the recently introduced Isometric Fluctuation Relation (IFR) [11].

The assumption of time-independent optimal profiles has been shown [4] to be equivalent to the additivity principle recently introduced by Bodineau and Derrida for one-dimensional (1D) diffusive systems [5]. Let $\mathrm{P}_L(J,\rho_L,\rho_R,\tau)$ be the probability of observing a time-averaged current J during a long time τ in a *one-dimensional* system of size L in contact with boundary reservoirs at densities ρ_L and ρ_R. The additivity principle relates this probability with the probabilities of sustaining the same current in subsystems of lengths $L-\ell$ and ℓ, i.e, $P_L(J,\rho_L,\rho_R,\tau) = \max_\rho[P_{L-\ell}(J,\rho_L,\rho,\tau)\times P_\ell(J,\rho,\rho_R,\tau)]$. The maximization over the contact density ρ can be rationalized by writing this probability as an integral over ρ of the product of probabilities for subsystems and noticing that these should obey also a large deviation principle. Hence a saddle-point calculation in the long-τ limit leads to the above expression. The additivity principle can be rewritten for the current LDF as $LG(J,\rho_L,\rho_R) = \max_\rho[(L-\ell)G(J,\rho_L,\rho)+\ell G(J,\rho,\rho_R)$. Slicing iteratively the 1D system of length N into smaller and smaller segments, and assuming locally-Gaussian current fluctuations, it is easy to show that in the continuum limit a variational form for $G(J,\rho_L,\rho_R)$ is obtained which is just the 1D counterpart of eq. (5). Interestingly, for 1D systems the conjecture of time-independent optimal profiles implies that the optimal current profile must be constant. This is no longer true in higher dimensions, as any divergence-free current field with spatial integral equal to $\mathbf{J}$ is compatible with the equations. This gives rise to a variational problem with respect to the (time-independent) density and current fields which still poses many technical difficulties. Therefore an additional assumption is needed, namely the constancy of the optimal current vector field across space. These two hypotheses are equivalent to the iterative procedure of the additivity principle in higher dimensions.

The validity of the additivity principle has been recently confirmed for a broad range of current fluctuations in extensive numerical simulations of the 1D Kipnis-Marchioro-Presutti model of energy transport. However, this conjecture is known to break down in some special cases for extreme current fluctuations, where time-dependent optimal profiles in the form of traveling waves propagating along the current direction may emerge. Even in these cases the additivity principle correctly predicts the current distribution in a very large current interval. As for higher-dimensional systems, the range of applicability of the generalized additivity hypothesis here proposed is an open issue [12].

In what follows we derive explicit predictions for the current LDF in the 2D KMP model of heat conduction based on the above generalization of the additivity principle.

RESULTS FOR THE 2D-KMP MODEL

The 2D-KMP model is a microscopic stochastic lattice model of energy transport in which Fourier's law holds. Each site on the lattice models an harmonic oscillator which is mechanically uncoupled from its nearest neighbors but interacts with them through a random process which redistributes energy locally. The system is coupled to boundary heat baths along the x-direction at *temperatures* ρ_L and ρ_R, whereas periodic boundary conditions hold in the y-direction. For $\rho_L \neq \rho_R$ the system reaches a nonequilibrium steady state with a nonzero rescaled average current $\langle \mathbf{J} \rangle = \hat{x}(\rho_L - \rho_R)/2$ and a stationary profile $\rho_{\text{st}}(\mathbf{r}) = \rho_L + x(\rho_R - \rho_L)$. At the macroscopic level the KMP model is characterized by a diffusivity $D[\rho] = \frac{1}{2}$, and a mobility $\sigma[\rho] = \rho^2$ which characterizes the variance of energy current fluctuations in equilibrium ($\rho_L = \rho_R$).

To study the statistics of the averaged current, first notice that the symmetry of the problem suggests that the optimal density profile associated to a given current fluctuation depends exclusively on x, with no structure in the y-direction, i.e. $\rho_{|\mathbf{J}|}(\mathbf{r}) = \rho_{|\mathbf{J}|}(x)$, compatible with the presence of an external gradient along the x-direction. Under these considerations, and denoting $J = |\mathbf{J}|$, eq. (6) becomes

$$\left(\frac{d\rho_J(x)}{dx}\right)^2 = 4J^2\left(1 + 2K(J)\rho_J^2(x)\right) \tag{7}$$

Here two different scenarios appear. On one hand, for large enough $K(J)$ the rhs of eq. (7) does not vanish $\forall x \in [0,1]$ and the resulting profile is monotone. In this case, and assuming $\rho_L > \rho_R$ henceforth without loss of generality,

$$\frac{d\rho_J(x)}{dx} = -2J\sqrt{1 + 2\rho_J^2(x)K(J)}\,. \tag{8}$$

On the other hand, for $K(J) < 0$ the rhs of eq. (7) may vanish at some points, resulting in a $\rho_J(x)$ that is non-monotone and takes an unique value $\rho_J^* \equiv \sqrt{-1/2K(J)}$ in the extrema. Notice that the rhs of eq. (7) may be written in this case as $4J^2[1 - (\rho_J(x)/\rho_J^*)^2]$. It is then clear that, if non-monotone, the profile $\rho_J(x)$ can only have a single maximum because: (i) $\rho_J(x) \leq \rho_J^*$ $\forall x \in [0,1]$ for the profile to be a real function, and (ii) several maxima are not possible because they should be separated by a minimum, which is not allowed because of (i). Hence for the non-monotone case (recall $\rho_L > \rho_R$)

$$\frac{d\rho_J(x)}{dx} = \begin{cases} +2J\sqrt{1-\left(\frac{\rho_J(x)}{\rho_J^*}\right)^2}, & x < x^* \\ \\ -2J\sqrt{1-\left(\frac{\rho_J(x)}{\rho_J^*}\right)^2}, & x > x^* \end{cases} \tag{9}$$

where x^* locates the profile maximum. This leaves us with two separated regimes for current fluctuations, with the crossover happening for $J = \frac{\rho_L}{2}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\rho_R}{\rho_L}\right)\right] \equiv J_c$. This crossover current can be obtained from eq. (15) below by letting $\rho_J^* \to \rho_L$

Region I: Monotonous Regime ($J < J_c$)

In this case, using eq. (8) to change variables in eq. (5) we have

$$G(\mathbf{J}) = \int_{\rho_L}^{\rho_R} d\rho_J \frac{1}{4J\rho_J^2\sqrt{1+2K(J)\rho_J^2}}\left[\left(J_x - J\sqrt{1+2K(J)\rho_J^2}\right)^2 + J_y^2\right], \tag{10}$$

with J_α the α component of vector $\mathbf{J}$. This results in

$$G(\mathbf{J}) = \frac{J_x}{2}\left(\frac{1}{\rho_R} - \frac{1}{\rho_L}\right) - J^2K(J) + \frac{J}{2}\left[\frac{\sqrt{1+2K(J)\rho_L^2}}{\rho_L} - \frac{\sqrt{1+2K(J)\rho_R^2}}{\rho_R}\right] \tag{11}$$

Notice that, for $\rho_J(x)$ to be monotone, $1+2K(J)\rho_J^2 > 0$. Thus, $K(J) > -(2\rho_L^2)^{-1}$. Integrating now eq. (8) we obtain the following implicit equation for $\rho_J(x)$ in this regime

$$2xJ = \begin{cases} \frac{1}{\sqrt{2K(J)}} \ln\left[\frac{\rho_L + \sqrt{\rho_L^2 + \frac{1}{2K(J)}}}{\rho_J(x) + \sqrt{\rho_J(x)^2 + \frac{1}{2K(J)}}}\right], & K(J) > 0 \\ \\ \frac{\sin^{-1}\left[\rho_L\sqrt{-2K(J)}\right] - \sin^{-1}\left[\rho_J(x)\sqrt{-2K(J)}\right]}{\sqrt{-2K(J)}}, & -\frac{1}{2\rho_L^2} < K(J) < 0 \end{cases} \tag{12}$$

Making $x = 1$ and $\rho_J(x=1) = \rho_R$ in the previous equation, we obtain the implicit expression for the constant $K(J)$. To get a feeling on how it depends on J, note that in the limit $K(J) \to (-1/2\rho_L^2)$, the current $J \to J_c$, while for $K(J) \to \infty$ one gets $J \to 0$. In addition, from eq. (12) we see that for $K(J) \to 0$ we find $J = (\rho_L - \rho_R)/2 = \langle \mathbf{J} \rangle$.

Sometimes it is interesting to work with the Legendre transform of the current LDF [3, 4, 7], $\mu(\boldsymbol{\lambda}) = \max_{\mathbf{J}}[G(\mathbf{J}) + \boldsymbol{\lambda}\cdot\mathbf{J}]$, where $\boldsymbol{\lambda}$ is a vector parameter conjugate to the current. Using the previous results for $G(\mathbf{J})$ is is easy to show [7] that $\mu(\boldsymbol{\lambda}) = -K(\boldsymbol{\lambda})\mathbf{J}^*(\boldsymbol{\lambda})^2$, where $\mathbf{J}^*(\boldsymbol{\lambda})$ is the current associated to a given $\boldsymbol{\lambda}$, and the constant $K(\boldsymbol{\lambda}) = K(|\mathbf{J}^*(\boldsymbol{\lambda})|)$. The expression for $\mathbf{J}^*(\boldsymbol{\lambda})$ can be obtained from eq. (12) above in the limit $x \to 1$. Finally, the optimal profile for a given $\boldsymbol{\lambda}$ is just $\rho_{\boldsymbol{\lambda}}(x) = \rho_{|\mathbf{J}^*(\boldsymbol{\lambda})|}(x)$.

Region II: Non-Monotonous Regime ($J > J_c$)

In this case the optimal profile has a single maximum $\rho_J^* \equiv \rho_J(x = x^*)$ with $\rho_J^* = 1/\sqrt{-2K(J)}$ and $-1/2\rho_L^2 < K(J) < 0$. Splitting the integral in eq. (5) at x^*, and using now eq. (9) to change variables, we arrive at

$$\begin{aligned} G(\mathbf{J}) &= \frac{J_x}{2}\left(\frac{1}{\rho_R} - \frac{1}{\rho_L}\right) - \frac{J}{2}\left[\frac{1}{\rho_L}\sqrt{1-\left(\frac{\rho_L}{\rho_J^*}\right)^2} + \frac{1}{\rho_R}\sqrt{1-\left(\frac{\rho_R}{\rho_J^*}\right)^2}\right. \\ &- \left.\frac{1}{2\rho_J^*}\left(\pi - \sin^{-1}\left(\frac{\rho_L}{\rho_J^*}\right) - \sin^{-1}\left(\frac{\rho_R}{\rho_J^*}\right)\right)\right]. \end{aligned} \tag{13}$$

Integrating eq. (9) one gets an implicit equation for the non-monotone optimal profile

$$2xJ = \begin{cases} \rho_J^*\left[\sin^{-1}\left(\frac{\rho_J(x)}{\rho_J^*}\right) - \sin^{-1}\left(\frac{\rho_L}{\rho_J^*}\right)\right] & \text{for } 0 \le x < x^* \\ 2J + \rho_J^*\left[\sin^{-1}\left(\frac{\rho_R}{\rho_J^*}\right) - \sin^{-1}\left(\frac{\rho_J(x)}{\rho_J^*}\right)\right] & \text{for } x^* < x \le 1 \end{cases} \tag{14}$$

At $x = x^*$ both branches of the above equation must coincide, and this condition provides simple equations for both x^* and ρ_J^*

$$J = \frac{\rho_J^*}{2}\left[\pi - \sin^{-1}\left(\frac{\rho_L}{\rho_J^*}\right) - \sin^{-1}\left(\frac{\rho_R}{\rho_J^*}\right)\right] \; ; \; x^* = \frac{\frac{\pi}{2} - \sin^{-1}\left(\frac{\rho_L}{\rho_J^*}\right)}{\pi - \sin^{-1}\left(\frac{\rho_L}{\rho_J^*}\right) - \sin^{-1}\left(\frac{\rho_R}{\rho_J^*}\right)}. \tag{15}$$

Finally, as in the monotone regime, we can compute the Legendre transform of the current LDF, obtaining as before $\mu(\boldsymbol{\lambda}) = -K(\boldsymbol{\lambda})\mathbf{J}^*(\boldsymbol{\lambda})^2$, where $\mathbf{J}^*(\boldsymbol{\lambda})$ and the constant $K(\boldsymbol{\lambda}) = K(|\mathbf{J}^*(\boldsymbol{\lambda})|)$ can be easily computed from the previous expressions [7]. Note that, in $\boldsymbol{\lambda}$-space, monotone profiles are expected for $|\boldsymbol{\lambda} + \boldsymbol{\varepsilon}| \le \frac{1}{2\rho_R}\sqrt{1-\left(\frac{\rho_R}{\rho_L}\right)^2}$, where $\boldsymbol{\varepsilon} = (\frac{1}{2}\left(\rho_L^{-1} - \rho_R^{-1}\right), 0)$ is a constant vector directly related with the rate of entropy production in the system, while non-monotone profiles appear for $\frac{1}{2\rho_R}\sqrt{1-\left(\frac{\rho_R}{\rho_L}\right)^2} \le |\boldsymbol{\lambda} + \boldsymbol{\varepsilon}| \le \frac{1}{2}\left(\frac{1}{\rho_L} + \frac{1}{\rho_R}\right)$.

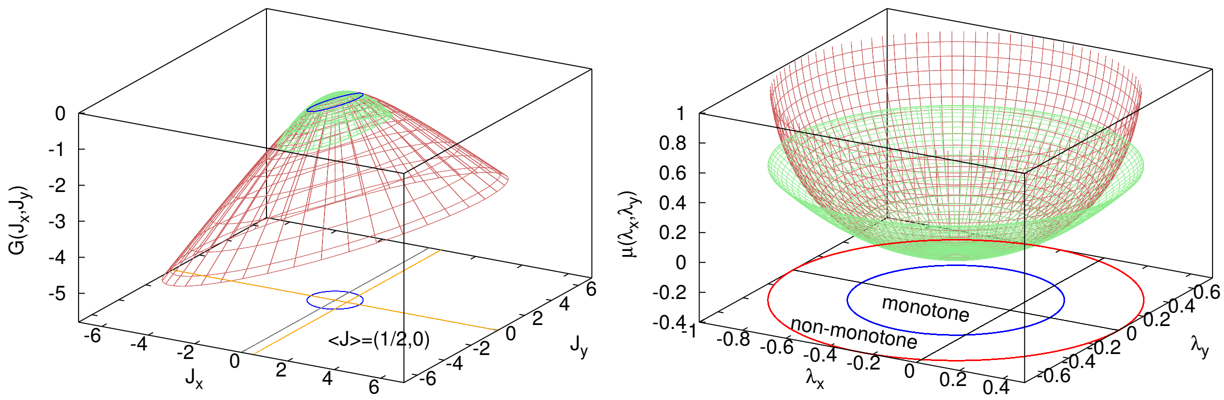

FIGURE 1. Left panel: $G(\mathbf{J})$ for the 2D-KMP model for $\rho_L = 2$ and $\rho_R = 1$. The blue circle signals the crossover from monotone ($J < J_c \equiv \pi/3$) to non-monotone ($J > \pi/3$) optimal profiles. The green surface corresponds to the Gaussian approximation for small current fluctuations. Right panel: Legendre transform of both $G(\mathbf{J})$ (brown) and $G(\mathbf{J} \to \langle \mathbf{J} \rangle)$ (green), together with the projection in $\boldsymbol{\lambda}$-space of the crossover between monotonous and non-monotonous regime.

Fig. 1 shows the predicted $G(\mathbf{J})$ (left panel) and its Legendre transform (right panel) for the 2D-KMP model. Notice that the LDF is zero for $\mathbf{J} = \langle \mathbf{J} \rangle = ((\rho_L - \rho_R)/2, 0)$ and negative elsewhere. For small current fluctuations, $\mathbf{J} \approx \langle \mathbf{J} \rangle$, $G(\mathbf{J})$ obeys the following quadratic form

$$G(\mathbf{J}) \approx -\frac{1}{2} \left(\frac{(J_x - (\rho_L - \rho_R)/2)^2}{\sigma_x^2} + \frac{J_y^2}{\sigma_y^2} \right), \tag{16}$$

with $\sigma_x^2 = (\rho_L^2 + \rho_L \rho_R + \rho_R^2)/3$ and $\sigma_y^2 = \rho_L \rho_R$, resulting in Gaussian statistics for currents near the average as expected from the central limit theorem. A similar expansion for the Legendre transform yields

$$\mu(\boldsymbol{\lambda}) \approx \frac{\lambda_x}{2} \left[(\rho_L - \rho_R) + \sigma_x^2 \lambda_x \right] + \frac{\sigma_y^2}{2} \lambda_y^2. \tag{17}$$

Notice that beyond this restricted Gaussian regime, current statistics is in general non-Gaussian. In particular, for large enough current deviations, $G(\mathbf{J})$ decays linearly, meaning that the probability of such fluctuations is *exponentially* small in J (rather than J^2). Fig. (2) shows the x-dependence of optimal density profiles for different values of J, including both the monotone and non-monotone regimes.

Despite the complex structure of $G(\mathbf{J})$ in both regime I and II above, it can be easily checked that for any pair of isometric current vectors $\mathbf{J}$ and $\mathbf{J}'$, such that $|\mathbf{J}| = |\mathbf{J}'|$, the current LDF obeys

$$G(\mathbf{J}) - G(\mathbf{J}') = \boldsymbol{\varepsilon} \cdot (\mathbf{J} - \mathbf{J}'), \tag{18}$$

where $\boldsymbol{\varepsilon} = (\frac{1}{2}(\rho_L^{-1} - \rho_R^{-1}), 0)$ is the constant vector defined above, linked to the rate of entropy production in the system. The above equation is known as Isometric Fluctuation

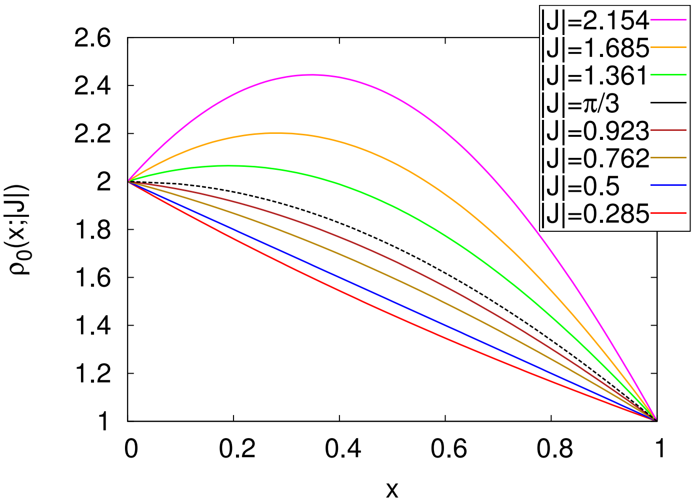

FIGURE 2. Optimal $\rho_J(x)$ for $\rho_L = 2$ and $\rho_R = 1$ and different J. The dash line ($J = J_c = \pi/3$) corresponds to the crossover between the monotone and non-monotone regimes.

Relation (IFR) [11], and is a general results for time-reversible systems described at the macroscopic level by a continuity equation similar to eq. (1), for which time-reversibility implies the invariance of optimal profiles under arbitrary rotations of the current vector. The IFR, which has been confirmed in extensive simulations [11], links in a strikingly simple manner the probability of any pair of isometric current fluctuations, and includes as a particular case the Gallavotti-Cohen Fluctuation Theorem in this context. However, the IFR adds a completely new perspective on the high level of symmetry imposed by time-reversibility on the statistics of nonequilibrium fluctuations.

CONCLUSIONS

We have derived explicit predictions for the statistics of current fluctuations in a simple but very general model of diffusive energy transport in two dimensions, the KMP model [6]. For that, we used the hydrodynamic fluctuation theory recently introduced by Bertini and coworkers [4], supplemented with a reasonable set of simplifying hypotheses, namely:

(i) The optimal profiles responsible of a given current fluctuation are time-independent.
(ii) The resulting divergence-free optimal current profile is in fact constant across space.
(iii) The ensuing optimal density profile has structure only along the gradient direction.

While assumption *(i)* is known to break down for extreme current fluctuations in some particular cases [4, 9, 10], it would be interesting to explore the range of validity of conjectures *(ii)-(iii)* in the time-independent regime. This could be achieved using a local stability analysis in the spirit of the results in [9]. Moreover, the emergence of time-dependent optimal profiles (probably traveling waves) in high-dimensional systems is an open and interesting problem which deserves further study.

Provided that hypotheses *(i)-(iii)* hold, we have obtained explicitly the current distribution for this model, which exhibits in general non-Gaussian tails. The optimal density profile which facilitates a given fluctuation can be either monotone for small current fluctuations, or non-monotone with a single maximum for large enough deviations. Furthermore, this optimal profile remains invariant under arbitrary rotations of the current vector, providing a detailed example of the recently introduced Isometric Fluctuation Relation [11].

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REFERENCES

1. R.S. Ellis, *Entropy, Large Deviations and Statistical Mechanics*, Springer, New York, 1985.
2. H. Touchette, *Phys. Rep.* **478**, 1 (2009).
3. B. Derrida, *J. Stat. Mech.* P07023 (2007); T. Bodineau, and B. Derrida, *C. R. Physique* **8**, 540 (2007).
4. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, *Phys. Rev. Lett.* **87**, 040601 (2001); *Phys. Rev. Lett.* **94**, 030601 (2005); *J. Stat. Mech.* P07014 (2007); *J. Stat. Phys.* **135**, 857 (2009).
5. T. Bodineau, and B. Derrida, *Phys. Rev. Lett.* **92**, 180601 (2004).
6. C. Kipnis, C. Marchioro, and E. Presutti, *J. Stat. Phys.* **27**, 65 (1982).
7. P. I. Hurtado, and P. L. Garrido, *Phys. Rev. Lett.* **102**, 250601 (2009); *Phys. Rev. E* **81**, 041102 (2010).
8. P. I. Hurtado, and P. L. Garrido, *J. Stat. Mech.* (2009) P02032.
9. T. Bodineau and B. Derrida, *Phys. Rev. E* **72**, 066110 (2005).
10. P. I. Hurtado, and P. L. Garrido, to appear.
11. P. I. Hurtado, C. Pérez-Espigares, J. J. del Pozo, and P. L. Garrido, arXiv:1009.1243
12. C. Pérez-Espigares, P. L. Garrido, and P. I. Hurtado, to appear.