

Exactly soluble Ising models with anisotropic interactions and arbitrary external magnetic field[†]

P L Garrido and J Marro

Facultad de Física, Universidad de Barcelona, Diagonal 647, 08028 Barcelona, Spain

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Abstract. We consider several Ising models with anisotropic interactions which can be solved exactly in the thermodynamic limit, for instance using a transfer matrix method. These include a two-dimensional system where each spin interacts directly with the two nearest neighbours along one of the principal axes of the lattice and has a coherent-field coupling with the spins along the second principal direction and a multidimensional system with coherent-field couplings along any of the principal axes.

1. Introduction

Many interesting equilibrium (macroscopic) phenomena, such as phase transitions, can be studied explicitly by applying Gibbs ensemble theory to simplified model systems with a given (microscopic) Hamiltonian. In practice, however, the number of mathematically well defined model systems with a physical relevance which can be solved exactly is very limited. Lattice model systems are in a sense the most interesting ones in physics; actually they capture many of the essential physical features of equilibrium cooperative phenomena and, in particular, they are now recognised to have great relevance for many phase transitions in nature (see, for instance, Thompson 1972). In any case, Ising-like lattice models have only exceptionally a simple exact solution; exceptions are the celebrated solution by Onsager (1944) of the two-dimensional Ising model with nearest-neighbour interactions for zero magnetic field, and some mean-field solutions such as those by Bragg and Williams (the so-called coherent field) or Bethe and Peierls (quasi-chemical) (see, for instance, Smart 1966, Pathria 1977, Ziman 1979). The latter, however, are known to fail to reproduce the correct behaviour near the critical temperature as well as the spin-wave behaviour at low temperatures, and yield only semi-quantitative agreement at best in other cases; this is due to the fact that they involve a defective treatment of the detailed spin interactions and, as a consequence, fail to take proper account of short-ranged correlations, symmetries and dimensionality of the systems. It thus seems interesting to consider further variations of the Ising model having an exact solution and more realistic interactions in some sense.

This paper is mainly devoted to the study of a two-dimensional Ising model with 'exact' nearest-neighbour interactions along one of the principal directions of the lattice and a mean (coherent) field coupling along the other. The model may then be solved

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exactly, even in the thermodynamic limit, when an arbitrary external magnetic field is present, for example, by a transfer matrix method (Kramers and Wannier 1941; see also, for instance, Thompson 1972, Pathria 1977). Thus it may serve to illustrate some interesting aspects such as the effects of anisotropy, both in the sense of different strengths of the main interactions and in the sense of a coexistence of ferromagnetic with antiferromagnetic interactions, or to clarify somewhat the range of validity of the classical universality class. As a matter of fact, our model is still characterised by classical critical exponents for any ratio λ between the interaction strengths. From a more practical point of view, the model is also interesting because it may show up a more realistic behaviour near the critical temperature than standard mean-field models when λ is used as a parameter. Moreover, it is very convenient to use as a reference state to analyse the properties of stationary non-equilibrium states in Ising-like models; such a study will be published elsewhere (Garrido and Marro 1987). There is also some hope that the present model may be of interest in describing some cooperative anisotropic surface phenomena, for instance. Finally, as an extension of the above model we also consider the case of a multidimensional Ising model with coherent-field couplings along each one of the principal directions of the lattice, a situation which seems, in principle, more interesting than the standard mean-field one, given that it allows the explicit consideration of the space dimension, a number of anisotropies, etc.

2. Nearest neighbours and coherent-field interactions

We first consider a system defined through the Hamiltonian

$$H = - \sum_{i,j=1}^{N,M} [J_x s_{ij} (s_{i-1,j} + s_{i+1,j}) + J_y s_{ij} m_i] \quad (2.1)$$

where it is assumed that there is a spin variable $s_{ij} = \pm 1$ at each lattice site ($i = 1, \dots, N; j = 1, \dots, M$), the indexes i and j describe respectively the \hat{x} and \hat{y} directions corresponding to the two principal axes of the lattice with exchange energies J_x and J_y , both being positive constants, and

$$m_i = M^{-1} \sum_{j=1}^M \langle s_{ij} \rangle \quad (2.2)$$

where the bracket represents a canonical average. The above amounts to considering a two-dimensional ferromagnetic Ising model with anisotropic interactions such that there is a nearest-neighbour coupling along \hat{x} and a coherent field (or Bragg-Williams mean field) along \hat{y} ; it seems convenient to consider separately the antiferromagnetic case, see § 3.

The canonical partition function, as a consequence of the unrestricted sum in the Hamiltonian, factorises

$$Z = \prod_j Z_j \quad (2.3)$$

with

$$Z_j = \sum' L(s_{1j}, s_{2j}) L(s_{2j}, s_{3j}) \dots L(s_{Nj}, s_{1j}) \quad (2.4)$$

and

$$L(\sigma, \sigma') = \exp[2\beta J_x \sigma \sigma' + (\beta J_y / 2) m (\sigma + \sigma')] \quad (2.5)$$

for $\sigma \equiv s_{ij}$, $\sigma' = s_{i+1,j}$ ($\beta = 1/kT$); also, the restricted sum in (2.4) only extends to (s_{ij} ; $i = 1, \dots, N$; j fixed) and periodic boundary conditions are implied, e.g. $s_{N+1,j} = s_{1,j}$ for all j ; also $m_{N+1} = m_1$ but this turns out to be irrelevant because homogeneity is assumed in the sense that $m_i = m$ for all i . Thus one may define the *transfer matrix* (Kramers and Wannier 1941) L with elements (2.5):

$$L = \begin{pmatrix} \exp(2\beta J_x + \beta J_y m) & \exp(-2\beta J_x) \\ \exp(-2\beta J_x) & \exp(2\beta J_x - \beta J_y m) \end{pmatrix} \quad (2.6)$$

and one has

$$Z_j = \text{Tr}(L^N) = \lambda_1^N + \lambda_2^N \quad (2.7)$$

where λ_1 and λ_2 represent the eigenvalues of L , $\lambda_1 > \lambda_2$. It follows

$$Z = \prod_j \lambda_1^N [1 + (\lambda_2/\lambda_1)^N] \quad (2.8)$$

and, consequently, the free energy density

$$a = -\beta^{-1} \ln \lambda_1 \quad (2.9)$$

when $N, M \rightarrow \infty$, i.e. only the largest eigenvalue is relevant for thermodynamics. λ_1 follows from the eigenvalue equation $\det(L - \lambda) = 0$ implying

$$\lambda_1 = \exp(2\beta J_x) \cosh(\beta J_y m) + [\exp(4\beta J_x) \sinh^2(\beta J_y m) + \exp(-4\beta J_x)]^{1/2}. \quad (2.10)$$

The action of an external field h is represented by an extra term $-h \sum_{ij} s_{ij}$ in the Hamiltonian (2.1), i.e. it just amounts to the substitution $J_y m \rightarrow J_y m + h$ in (2.10), namely

$$\lambda_1 = \exp(2\beta J_x) \cosh[\beta(J_y m + h)] + \{\exp(4\beta J_x) \sinh^2[\beta(J_y m + h)] + \exp(-4\beta J_x)\}^{1/2}. \quad (2.11)$$

This solves the general problem; for instance, the magnetisation per spin is given by

$$\begin{aligned} m(h, T) &= -\partial a / \partial h = \beta^{-1} (\partial \ln \lambda_1 / \partial h) \\ &= \alpha(h, T) [\alpha(h, T)^2 + \exp(-4\beta J_x)]^{-1/2} \end{aligned} \quad (2.12a)$$

with

$$\alpha(h, T) = \exp(2\beta J_x) \sinh[\beta(J_y m + h)]. \quad (2.12b)$$

The case $J_y = 0$ ($J_x \neq 0$) corresponds to the familiar one-dimensional Ising model with nearest-neighbour interactions (see, for instance, Thompson 1972); in zero field, $h = 0$, the (spontaneous) magnetisation is zero for all non-zero finite temperatures. On the other hand, the case $J_x = 0$ ($J_y \neq 0$) reduces to the usual one-dimensional Ising model under a mean-field hypothesis (Thompson 1972). More interesting is the case $J_x \neq 0$, $J_y \neq 0$. The corresponding spontaneous magnetisation per spin, m_0 , which is the solution of (2.12) for $h = 0$, can be proved to have the limiting values $m_0 \rightarrow 1$ as $T \rightarrow 0$, $m_0 \rightarrow 0$ as $T \rightarrow \infty$, and there is a phase transition for all finite values of J_y/J_x ; figures 1-4 show up the behaviour of m_0 and m as a function of T , h and J_y/J_x . Figure 3 depicts the influence of the anisotropy parameter $\lambda \equiv J_y/J_x$ on the spontaneous magnetisation. This suggests in particular the use of λ as an adjustable parameter, for instance (figure 4) $\lambda \rightarrow \infty$ makes m_0 to tend towards the Bragg-Williams behaviour, while smaller values of λ may approximate near T_c to the behaviour of our model to

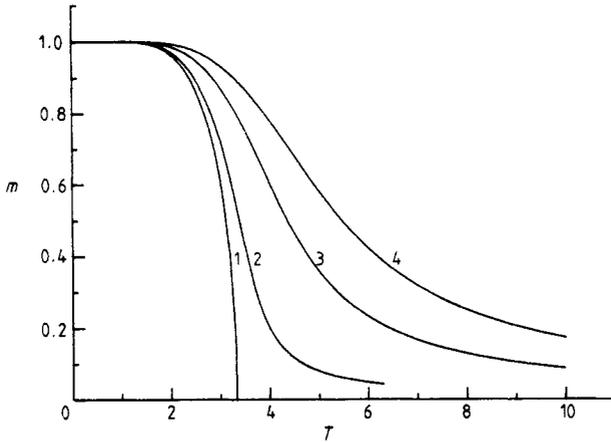


Figure 1. The magnetisation, as given by (2.12), plotted against temperature for $J_y = J_x$ and different values of the field $h = 0$ (curve 1), 0.1 (curve 2), 0.5 (curve 3) and 1 (curve 4).

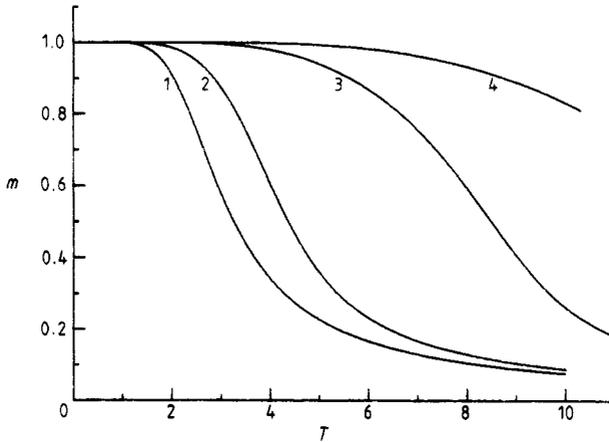


Figure 2. As figure 1 for $h = 0.5$ and different values of J_y/J_x : $J_y/J_x = 0.1$ (curve 1), 1 (curve 2), 5 (curve 3) and 10 (curve 4).

the Onsager solution of the isotropic two-dimensional Ising model. The influence of the external magnetic field on m is depicted by figures 1 and 2.

The critical temperature follows from (2.12) for $h = 0$ as $m_0 \rightarrow 0$; one has

$$kT_c = J_y \exp(4J_x/kT_c). \tag{2.13}$$

The behaviour of $T_c = T_c(\lambda)$ is depicted by figure 5 showing practically a linear dependence for $\lambda > 3$.

The spontaneous magnetisation critical exponent β which is defined as

$$m_0 \sim B_0(-\varepsilon)^\beta (1 + B(-\varepsilon)^\theta + \dots) \quad T \rightarrow T_c^- \tag{2.14}$$

$\theta > 0$, $\varepsilon \equiv T/T_c - 1$, follows after some algebra as $\beta = \frac{1}{2}$, and

$$B_0^2 = 2(1 + 4J_x\beta_c) \{ \beta_c^2 J_y^2 [\exp(8\beta_c J_x) - \frac{1}{3}] \}^{-1}. \tag{2.15}$$

This suggests a classical critical behaviour of the model (a fact which is already obvious from figure 4 for large λ) for all finite values of λ ; one also has $\theta = 1$, independently

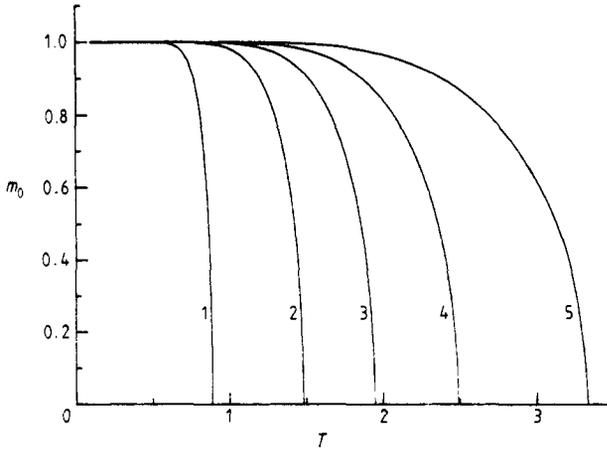


Figure 3. The spontaneous magnetisation, (2.12), for $h=0$ plotted against temperature for decreasing values of the anisotropy parameter: $J_y/J_x = 0.01$ (curve 1), 0.1 (curve 2), 0.25 (curve 3), 0.5 (curve 4) and 1 (curve 5).

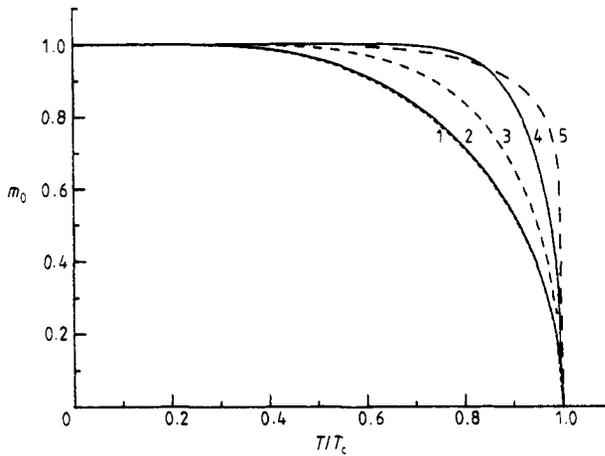


Figure 4. The spontaneous magnetisation plotted against temperature, normalised to the corresponding critical temperature, for different models: curve 1 (Bragg-Williams), curve 2 (the model in § 2 for $J_y/J_x = 20$), curve 3 (Bethe-Peierls), curve 4 (the model in § 2 for $J_y/J_x = 0.01$), curve 5 (Onsager). Notice that curves 1 (broken curve) and 2 (full curves) are almost indistinguishable from each other.

of λ . That classical behaviour is confirmed by computing the other critical exponents, for instance, the critical isotherm exponent:

$$h \sim D|m|^\delta \quad T = T_c \tag{2.16}$$

follows after some algebra as $\delta = 3$ and

$$D = J_y(3 + \beta_c^2 J_y^2)/6. \tag{2.17}$$

Notice that there is a hidden dependence here on J_x through β_c .

The energy density, on the other hand, is given by

$$\langle e \rangle = -\partial \ln \lambda_1 / \partial \beta. \tag{2.18}$$

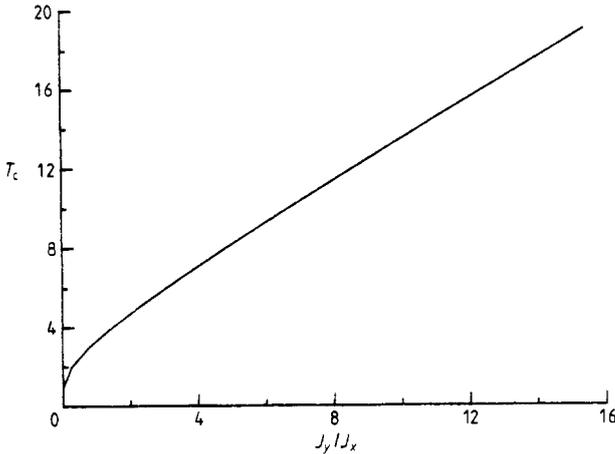


Figure 5. The dependence of the critical temperature T_c on the ratio J_y/J_x between the strength of the interactions along the two principal directions, as given by (2.13).

The simplest case occurs for $T > T_c$ and $h = 0$:

$$\langle e \rangle = -2J_x \tanh(2\beta J_x) \quad T > T_c \tag{2.19}$$

and

$$C_v = \partial \langle e \rangle / \partial T = (4J_x^2 / kT^2) \operatorname{sech}^2(2\beta J_x) \quad T > T_c. \tag{2.20}$$

These expressions may be understood by noticing that the Hamiltonian (2.1) practically reduces for $T > T_c$ ($m_0 = 0$) to the one-dimensional one. On the contrary, the corresponding expressions for $T < T_c$ are very interesting. One has for $T < T_c$

$$\langle e \rangle = -2J_x m \coth(\beta J_y m) - J_y m^2 + 4J_x m^2 \cosh(4\beta J_x) \exp(-4\beta J_x) \times \{ \sinh(\beta J_y m) [m \cosh(\beta J_y m) + \sinh(\beta J_y m)] \}^{-1} \tag{2.21}$$

which goes continuously to the value (2.19) as one crosses T_c . Figures 6 and 7 illustrate the behaviour of the energy and specific heat respectively. Again, one observes that

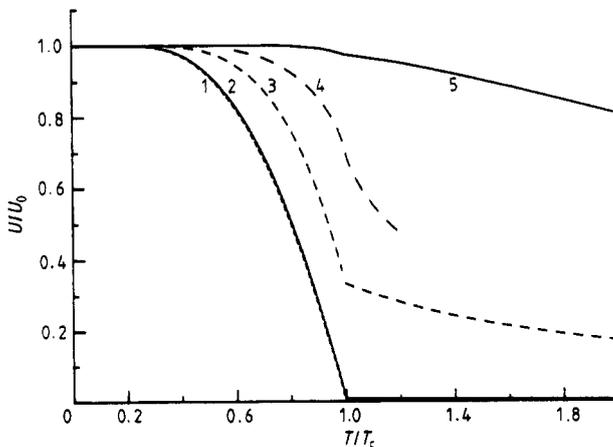


Figure 6. The energy per spin, normalised to the zero-temperature value, plotted against temperature, normalised to the corresponding critical temperature, for the same models as in figure 4.

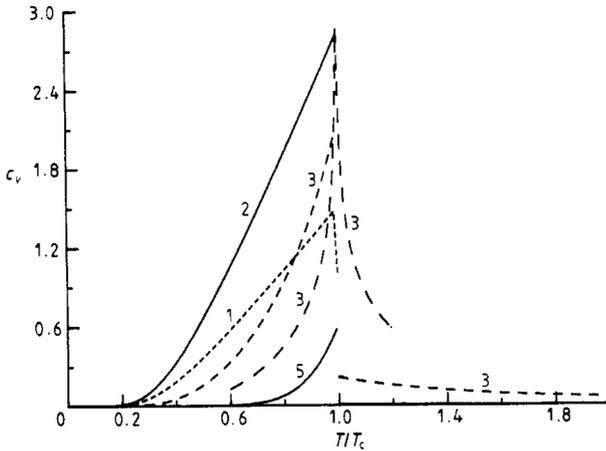


Figure 7. The specific heat at zero field plotted against temperature, normalised to the corresponding critical temperature, for the same models as in figure 4. Notice that the vertical axis is in absolute units.

using λ as a parameter one may approximate the behaviour of the more conventional models; large values of λ lead to a Bragg-Williams behaviour, while small and intermediate values of λ may reproduce approximately the Onsager solution and the Bethe-Peierls model, respectively. Of course, a given value of λ affects each quantity differently (cf the cases $\lambda = 0.01$ in figures 4 and 6).

The magnetic susceptibility readily follows as

$$\chi = \beta m(1 - m^2) \{ \tanh[\beta(J_y m + h)] - \beta J_y m \}^{-1} \tag{2.22}$$

whose critical behaviour is characterised by $\gamma = 1$.

3. Antiferromagnetic interactions

The model defined in § 2 may also be considered in the case $J_x < 0$. However, the local order parameter m_i needs then a specific treatment. We thus write instead of (2.4) and (2.5):

$$Z_j = \sum' L^{(1)}(s_{1j}, s_{2j}) L^{(2)}(s_{2j}, s_{3j}) \dots L^{(N)}(s_{Nj}, s_{1j}) \tag{3.1}$$

and

$$L^{(i)}(\sigma, \sigma') = \exp[2\beta J_x \sigma \sigma' + (\beta J_y / 2)(\sigma m_i + \sigma' m_{i+1})] \tag{3.2}$$

where $\sigma \equiv s_{ij}$ and $\sigma' \equiv s_{i+1,j}$ as before, and $m_i = m_o$ for all odd i , $m_i = m_e$ for all even i . That is, one has now two transfer matrices:

$$L_{o(e)} = \begin{pmatrix} \exp(2\beta J_x + M^+) & \exp(-2\beta J_x + (-)M^-) \\ \exp(-2\beta J_x - (+)M^-) & \exp(2\beta J_x - M^+) \end{pmatrix} \tag{3.3}$$

where $M^+ \equiv (\beta J_y / 2)(m_o + m_e)$, $M^- \equiv (\beta J_y / 2)(m_o - m_e)$ and

$$Z_j = \text{Tr}[(L_o L_e)^{N/2}]. \tag{3.4}$$

The free energy density in the thermodynamic limit follows as

$$a = -(2\beta)^{-1} \ln \lambda'_1 \quad (3.5)$$

where λ'_1 represents the corresponding largest eigenvalue given by

$$\lambda'_1 = \exp(4\beta J_x) \cosh[\beta J_y(m_o + m_e)] + \exp(-4\beta J_x) \cosh[\beta J_y(m_o - m_e)] + g(J_x, J_y, m_o, m_e)^{1/2} \quad (3.6)$$

where

$$g(J_x, J_y, m_o, m_e) = \exp(8\beta J_x) \sinh^2[\beta J_y(m_o + m_e)] + \exp(-8\beta J_x) \sinh^2[\beta J_y(m_o - m_e)] + \cosh(2\beta J_y m_o) + \cosh(2\beta J_y m_e) + 2. \quad (3.7)$$

One may convince oneself that the more general expression (3.5) reduces to (2.9) when $m_o = m_e$ (the case considered in § 2).

However, we shall assume in the rest of this section that $J_x < 0$, $J_x = -|J_x|$; the system behaviour may then be described by using the odd and even order parameters at zero field:

$$m_{o(e)} = (\beta J_y)^{-1} (\partial \ln \lambda'_1 / \partial m_{o(e)}) \quad (3.8)$$

respectively. These are given from (3.6) as

$$m_{o(e)} = \{ \exp(-4\beta |J_x|) \sinh[\beta J_y(m_o + m_e)] + \exp(4\beta |J_x|) \sinh[\beta J_y(m_o - m_e)] \} g(-|J_x|, J_y, m_o, m_e)^{-1/2}. \quad (3.9)$$

As expected, one has the same critical temperature T_c as in (2.13) and the limiting behaviours $m_o \rightarrow 1$ and $m_e \rightarrow -1$ as $T \rightarrow 0$ for finite λ , and $m_o \rightarrow 0$, $m_e \rightarrow 0$ as $\lambda \rightarrow 0$. It also follows easily from (3.9) that $m_o = m_e$ in general corresponding to a kind of antiferromagnetic behaviour, namely consecutive vertical rows tend to present a different order.

4. Multidimensional system in the coherent-field approximation

A simple case of the model in § 2 corresponds to the Hamiltonian:

$$H = - \sum_{ij} (J_x s_{ij} \tilde{m}_j + J_y s_{ij} m_i) \quad (4.1)$$

with

$$m_i = N^{-1} \sum_j \langle s_{ij} \rangle \quad \tilde{m}_j = N^{-1} \sum_i \langle s_{ij} \rangle. \quad (4.2)$$

This amounts to assuming a coherent-field coupling along both \hat{x} and \hat{y} . One has immediately the conditions

$$m_i = (N\beta J_y)^{-1} (\partial \ln Z / \partial m_i) \quad \tilde{m}_j = (N\beta J_x)^{-1} (\partial \ln Z / \partial \tilde{m}_j) \quad (4.3)$$

the partition function

$$Z = \prod_{ij} \{ 2 \cosh[\beta (J_x \tilde{m}_j + J_y m_i)] \} \quad (4.4)$$

and, consequently,

$$m_i = N^{-1} \sum_j \tanh[\beta(J_x \tilde{m}_j + J_y m_i)] \quad (4.5)$$

$$m_j = N^{-1} \sum_i \tanh[\beta(J_x \tilde{m}_j + J_y m_i)] \quad (4.6)$$

and the free energy density

$$\begin{aligned} a &= -(\beta N^2)^{-1} \ln Z \\ &= -(\beta N^2)^{-1} \sum_{ij} \ln\{2 \cosh[\beta(J_x \tilde{m}_j + J_y m_i)]\}. \end{aligned} \quad (4.7)$$

As expected, when one assumes $\lambda = 0$ here the system reduces to a collection of one-dimensional systems with

$$\tilde{m}_j = \tanh(\beta J_x \tilde{m}_j) \quad (4.8)$$

and

$$m_i = N^{-1} \sum_j \tilde{m}_j \quad \forall i \quad (4.9)$$

while assuming homogeneity in the sense that $m_i = m$ and $\tilde{m}_j = \tilde{m}$ implies $\tilde{m} = m$ and

$$m = \tanh[\beta(J_x + J_y)m] \quad (4.10)$$

which is the familiar mean-field result.

A more interesting situation corresponds to a solution such that

$$m^{(1)} = m_1 = m_3 = \dots \quad m^{(2)} = m_2 = m_4 = \dots \quad (4.11)$$

Assuming further that $\tilde{m}_j = \tilde{m}$ for all j , one has

$$\tilde{m} = (m^{(1)} + m^{(2)})/2 \quad (4.12)$$

$$m^{(1)} = \tanh[\beta(J_x \tilde{m} + J_y m^{(1)})] \quad m^{(2)} = \tanh[\beta(J_x \tilde{m} + J_y m^{(2)})]. \quad (4.13)$$

The case $\tilde{m} = 0$ then corresponds to a certain antiferromagnetic structure $m^{(1)} = -m^{(2)}$ and $\tilde{m} = m$ leads to $m^{(1)} = m^{(2)} = m$. The corresponding internal energy, which is

$$\langle e \rangle = -J_x N^{-1} \sum_j \tilde{m}_j^2 - J_y N^{-1} \sum_i m_i^2 \quad (4.14)$$

in general becomes $\langle e \rangle = -J_y m^{(1)2}$ and $\langle e \rangle = -(J_x + J_y)m^2$ for those two cases respectively. Also, one may show that the free energy density is larger for that 'antiferromagnetic structure' than for the familiar ferromagnetic one so that the former is not necessarily stable.

The model also presents solutions corresponding to the conditions

$$\begin{aligned} m^{(1)} &= m_i = m_{i+r} = m_{i+2r} = \dots \\ m^{(2)} &= m_{i+1} = m_{i+r+1} = \dots \\ &\vdots \\ m^{(r)} &= m_{i+r-1} = \dots \end{aligned} \quad (4.15)$$

One has

$$m^{(i)} = N^{-1} \sum_j \tanh[\beta(J_x \tilde{m}_j + J_y m^{(i)})] \quad (4.16)$$

and

$$\tilde{m}_j = r^{-1} \sum_{k=1}^r m^{(k)} \tag{4.17}$$

Then $\tilde{m} = 0$ implies $\sum_k m^{(k)} = 0$ and $m^{(1)} = 0$ is the only solution when r is odd, while for even r one may have some ‘disordered antiferromagnetic structures’ with periodicity r such that one has $r/2$ of a kind. The corresponding free energy density:

$$a = -kT \ln 2 + r^{-1} \sum_{r'=1}^r \ln\{\cosh[\beta(J_x \tilde{m} + J_y m^{(r')})]\} \tag{4.18}$$

is such, in particular, that $\partial a / \partial \tilde{m} = 0$ and $\partial^2 a / \partial \tilde{m}^2 > 0$.

Finally, it seems interesting to notice that the model defined through the Hamiltonian (4.1) may easily be generalised to d -dimensional spaces, i.e. the generalised Hamiltonian is

$$H = - \sum_{\alpha=1}^d \sum_{i_1=1}^N \dots \sum_{i_d=1}^N J_{\alpha} s_{i_1 \dots i_d} m_{i_{\alpha}} \tag{4.19}$$

with

$$m_{i_{\alpha}} = N^{-(d-1)} \sum_{i_1} \dots \sum_{i_{\alpha-1}} \sum_{i_{\alpha+1}} \dots \sum_{i_d} \langle s_{i_1 \dots i_d} \rangle \tag{4.20}$$

The internal energy then follows as

$$\langle e \rangle = -N^{-1} \sum_{\alpha=1}^d J_{\alpha} \sum_{i_{\alpha}} m_{i_{\alpha}}^2 \tag{4.21}$$

the partition function as

$$Z = \prod_{i_1} \dots \prod_{i_d} \left[2 \cosh \left(\beta \sum_{\alpha=1}^d J_{\alpha} m_{i_{\alpha}} \right) \right] \tag{4.22}$$

and the local magnetisations as

$$m_{i_{\alpha'}} = N^{-(d-1)} \sum_{i_1} \dots \sum_{i_d} \tanh \left(\beta \sum_{\alpha=1}^d J_{\alpha} m_{i_{\alpha}} \right) \tag{4.23}$$

where the sum over $i_{\alpha'}$ is lacking, and $\alpha' = 1, \dots, d$. The specific solutions in this case may be investigated as before.

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